Unsolved problems in graph theory

Xingzhi Zhan (詹兴致)

zhan@math.ecnu.edu.cn

East China Normal University

1, Bound vertices in longest paths

We consider finite simple graphs.

Let H be a subgraph of a graph G. A vertex v of H is said to be H-bound if all the neighbors of v in G lie in H; i.e.,

$$N_G(v) \subseteq V(H).$$

Conjecture 1 (Z., 2024). Let G be a k-connected graph with $k \geq 2$ and let x, y be two distinct vertices of G. If P is a longest (x, y)-path in G, then P contains $k-1$ internal P-bound vertices.

The case $k = 2$ of Conjecture 1 implies the following famous

Conjecture 2 (Thomassen, 1976). Every longest cycle in a 3-connected graph has a chord.

Conjecture 1 has been proved to be true in the following two cases:

(1) 2-connected cubic graphs (Li-Liu, 2024);

(2) induced-fork free graphs; in particular, claw-free graphs (Li-Liu, 2025).

2, The minimum number of detours

A longest path in a graph G is called a detour of G.

$f(G)$ denotes the number of detours in a graph G.

Problem 3 (Z., 2024). Let k and n be integers with $3 \leq k \leq n-2$. Denote by $\Gamma(k, n)$ the set of connected graphs with minimum degree k and order n. Define

$$a(k, n) = \min\{f(G) | \ G \in \Gamma(k, n)\}.$$

Determine $a(k, n)$.

Problem 4 (Z., 2024). Let k, n and $\Gamma(k, n)$ be as in Problem 3. Define

$$b(k, n) = \min\{f(G) | \ G \in \Gamma(k, n) \text{ and } f(G) \text{ is odd}\}.$$

Determine $b(k, n)$.

Perhaps for sufficiently large orders n, $a(k,n)$ and $b(k,n)$ are independent of n. We may also ask the two corresponding problems by replacing "with minimum degree k" in Problems 3 and 4 above by "with connectivity k".

It has been proved that $a(2,n) = 4$ for $n \geq 4$, $b(2,n) = 9$ for $n \geq 9$ and $a(3,n) = 36$ for $n \geq 18$.

3, Minimally hamiltonian-connected graphs

A graph is called hamiltonian-connected if between any two distinct vertices there is a Hamilton path.
A hamiltonian-connected graph G is said to be minimally hamiltonian-connected if for every edge e of G, the graph $G - e$ is not hamiltonian-connected.

Problem 5 (Z., 2022). What are the possible values of the minimum degree of a minimally hamiltonian-connected graph of order n?

A computer search shows that every minimally hamiltonian-connected graph of order n with $4 \leq n \leq 10$ has minimum degree 3.

Problem 6 (Z., 2022). Does there exist a minimally hamiltonian-connected graph with minimum degree at least 4?

4, Homogeneously traceable graphs

A graph G is said to be homogeneously traceable if every vertex of G is an endpoint of a Hamilton path.

Hamiltonian graphs and hypohamiltonian graphs are homogeneously traceable.

In 1979, Chartrand, Gould and Kapoor proved that for every integer n with $n \geq 9$, there exists a homogeneously traceable nonhamiltonian graph of order n.

Conjecture 7 (Hu-Z., 2022). The minimum circumference of a homogeneously traceable graph of order n is $\lceil 2n/3 \rceil + 2$.

5, Edge-pancyclic graphs

A graph G of order n is called *edge-pancyclic* if for every integer k with $3 \le k \le n$, every edge of G lies in a k-cycle.

Problem 8 (Li-Liu-Z., 2025). Determine the minimum size of an edge-pancyclic graph of order n.

A computer search shows that the minimum size is $2n-2$ for $4 \le n \le 12$, but we have the following

Theorem (Li-Liu-Z., 2025). Given any integer $k \ge 3$, let $n = 6k^2 - 5k$. Then there exists an edge-pancyclic graph of order n and size $2n - k$.

6, The Turán number

Given a graph H and a positive integer n, the Turán number of H for the order n, denoted $\mathrm{ex}(n, H)$, is the maximum size of a simple graph of order n not containing H as a subgraph.

In 1955，Rademacher proved that every graph of order n and size $\mathrm{ex}(n, K_3) + 1$ contains at least $\lfloor n/2 \rfloor$ triangles. A similar result for a general K_p was proved by Moon in 1965

In 1990, Erdős posed the following

Problem. For which graphs H is it true that every graph on n vertices and $\mathrm{ex}(n, H) + 1$ edges contains at least two Hs? Perhaps this is always true.

This problem was solved negatively in 2022.

Problem 9 (Qiao-Z., 2022). Determine all the pairs (H, n), where H is a graph and n is a positive integer, such that there exists a graph of order n and size $\mathrm{ex}(n, H) + 1$ which contains exactly one copy of H.

7, The periphery of a graph

In 2021, Hu and Z. determined which cardinalities are possible for the center of a graph with given order and radius. For example, there exists a graph of order 14 and radius 6 whose center has cardinality s if and only if

$$s \in \{1, 2, 3, 4, 9, 10, 11, 12, 14\}.$$

Problem 10 (Z., 2019) Determine which cardinalities are possible for the periphery of a graph with given order and diameter.

8, Cubic hamiltonian graphs

Problem 11 (Z., 2023) Is it true that every cubic hamiltonian graph of order n contains an $(n-2)$-cycle?

A computer search shows that the answer is “yes” for $n \leq 26$.

I conjecture the answer is “no” for sufficiently large orders n.

9, Maximally nonhamiltonian graphs

A nonhamiltonian graph is called maximally nonhamiltonian if adding any new edge results in a hamiltonian graph.

Problem 12 (Horak-Siran, 1986). Is it true that for every positive integer k, there exists a maximally nonhamiltonian graph of girth $> k$?

The Coxeter graph is a maximally nonhamiltonian graph of girth 7. No maximally nonhamiltonian graph of girth greater than 7 is known.

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