A sufficient condition for pancyclic graphs

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Background and the Main Result

The order of a graph is its number of vertices, and the size is its number of edges. A k-cycle is a cycle of length k.

In 1971 Bondy introduced the concept of a pancyclic graph.

A graph G of order n is called pancyclic if for every integer k with $3 \leq k \leq n$, G contains a k-cycle.

Let s and t be given integers. A graph G is called an $[s, t]$-graph if any induced subgraph of G of order s has size at least t.

Theorem 1. (Liu-Wang, 刘春房-王江鲁， 2005) Every 2-connected $[4, 2]$-graph of order at least 6 is hamiltonian.

Theorem 2. (Liu-Wang-Gao, 刘晓妍-王江鲁-高国成，2007) Let G be a 2-connected $[4, 2]$-graph of order at least 7. If G contains a k-cycle with $k < n$, then G contains a $(k + 1)$-cycle.

Theorem 3. Every 2-connected $[4, 2]$-graph of order at least 7 is pancyclic.

Lemmas

Lemma 4. Given positive integers $n \geq k \geq 2$, let $x_1, x_2, \ldots, x_k$ be positive integers such that $\sum_{i=1}^{k} x_i = n$. Then

$$n - 1 \leq \sum_{i=1}^{k-1} x_i x_{i+1} \leq \begin{cases} \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil & \text{if } \ k = 2,\, 3 \\ ab + k - 5 & \text{if } \ k \geq 4 \end{cases}$$

where $a = \lfloor (n-k+4)/2 \rfloor$ and $b = \lceil (n-k+4)/2 \rceil$. For any n and k, the lower and upper bounds can be attained.

Lemma 5. (Z., 2007) Given positive integers $n \geq k \geq 2$, let $x_1, x_2, \ldots, x_k$ be positive integers such that $\sum_{i=1}^{k} x_i = n$. Then

$$2n - k \leq \sum_{i=1}^{k} x_i x_{i+1} \leq \begin{cases} \lfloor n^2/k \rfloor & \text{if } \ k \leq 4 \\ 2n - k + \lfloor (n-k)^2/4 \rfloor & \text{if } \ k \geq 5. \end{cases}$$

where $x_{k+1} \triangleq x_1$. For any n and k, the lower and upper bounds can be attained.

Definition. Given a graph H and a positive integer k, the k-blow-up of H, denoted by $H^{(k)}$, is the graph obtained by replacing every vertex of H with k different vertices where a copy of u is adjacent to a copy of v in the blow-up graph if and only if u is adjacent to v in H.

For example, $C_5^{(2)}$ is as follows.

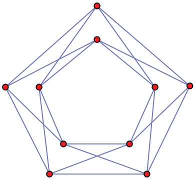

图: The 2-blow-up of C_5

Lemma 6. Let G be a $[p+2,p]$-graph of order n with $\delta(G) \geq p \geq 2$ and $n \geq 2p+3$. Then G is triangle-free if and only if p is even, $p \geq 6$ and $G = C_5^{(p/2)}$.

Proof.

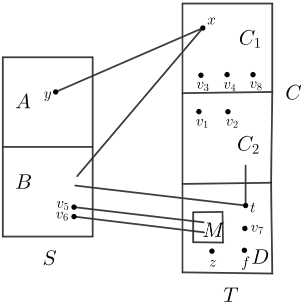

图: The structure of G

Comparison

Theorem 7. (Chvátal-Erdős, 1972) $\kappa(G) \geq \alpha(G) \Rightarrow G$ is hamiltonian.

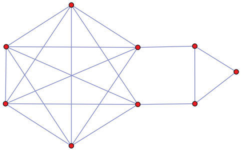

图: The graph Z_9

Z_n is a 2-connected [4, 2]-graph of order n, but $2 = \kappa(Z_n) < \alpha(Z_n) = 3$.

A conjecture

Conjecture. Let $p \in \{3, 4, 5\}$. Then every 2-connected $[p + 2, p]$-graph of order at least $2p + 3$ and minimum degree at least p is pancyclic.

References

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T H A N K Y O U