Results and problems on longest paths in graphs

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A longest path in a graph is called a detour.

We consider the minimum number of detours in classes of connected graphs.

The following result must be well-known and it is easy to prove.

Theorem 1. The minimum number of detours in a connected graph of minimum degree at least 2 and order at least 4 is 4.

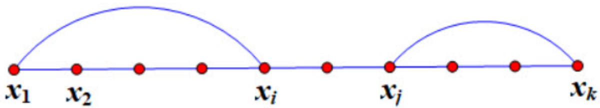

Fig. 1. The case $i \leq j$

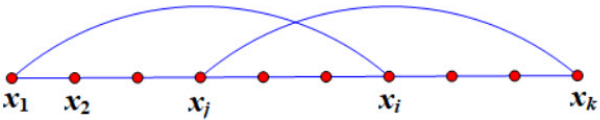

Fig. 2. The case $i > j$

Notation. $f(G)$ denotes the number of detours in a graph G.

Theorem 2. Let G be a connected graph of minimum degree at least 2 and order at least 9. If $f(G)$ is an odd number, then $f(G) \geq 9$. Furthermore, the lower bound 9 can be attained for every order by both graphs of connectivity 1 and graphs of connectivity 2.

Thus, as the number of detours in a connected graph of minimum degree at least 2 and order at least 9, the possibilities 3, 5 and 7 are excluded.
The reason for this interesting phenomenon does not seem obvious, in view of the fact that the numbers 4, 6, 8 and 9 can be attained.

For every integer $n \geq 9$ we construct a graph H_n of order n and connectivity 1 which contains exactly 9 detours. Every H_n is traceable.

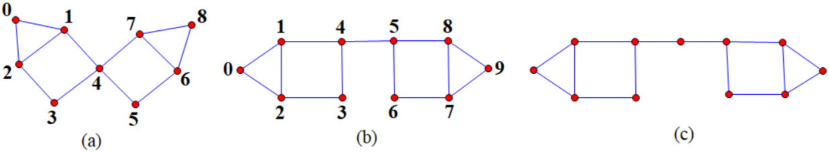

Fig. 4. The graphs H_9, H_{10} and H_{11}

For $n \geq 11$, H_n is obtained from H_{10} by subdividing the edge $(4, 5)$ $n - 10$ times.

For every integer $n \geq 9$ we construct a graph M_n of order n and connectivity 2 which contains exactly 9 detours. Every M_n is traceable.

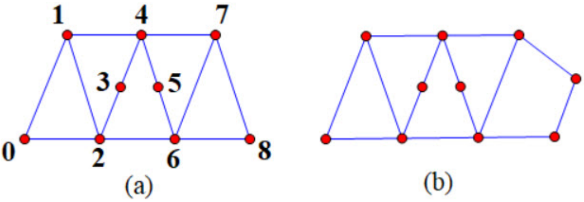

Fig. 5. The graphs M_9 and M_{10}

For $n \geq 10$, M_n is obtained from M_9 by subdividing the edge $(7, 8)$ $n - 9$ times.

Theorem 3. For every integer $n \geq 4$, there exists a 2-connected graph of order n with exactly 6 detours, and for every integer $n \geq 6$, there exists a 2-connected graph of order n with exactly 8 detours.

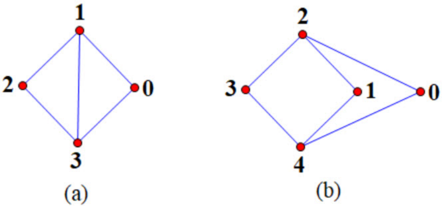

Fig. 6. The graphs D_4 and D_5

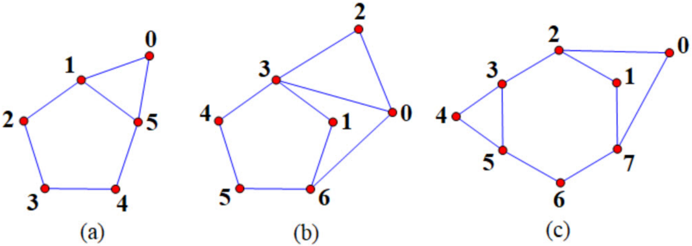

Fig. 7. The graphs F_6, F_7 and F_8

We pose two problems. Recall that $f(G)$ denotes the number of detours in a graph G.

Problem 1. Let k and n be integers with $3 \leq k \leq n-2$. Denote by $\Gamma(k,n)$ the set of connected graphs with minimum degree k and order n. Define

$$a(k,n) = \min\{f(G) | \; G \in \Gamma(k,n)\}.$$

Determine $a(k,n)$.

Problem 2. Let k, n and $\Gamma(k,n)$ be as in Problem 1. Define

$$b(k,n) = \min\{f(G) | \; G \in \Gamma(k,n) \text{ and } f(G) \text{ is odd}\}.$$

Determine $b(k,n)$.

Perhaps for sufficiently large orders n, $a(k,n)$ and $b(k,n)$ are independent of n. We may also ask the two corresponding problems by replacing "with minimum degree k" in Problems 1 and 2 above by "with connectivity k".

One of the most important unsolved problems in graph theory is the following conjecture.

Conjecture 1 (Thomassen's chord conjecture, 1976). Every longest cycle in a 3-connected graph has a chord.

Since a 3-connected graph has minimum degree at least 3, Conjecture 1 is implied by the following conjecture.

Conjecture 2 (Harvey, 2017). Every longest cycle in a 2-connected graph with minimum degree at least 3 has a chord.

Definition. Let H be a subgraph of a graph G. A vertex v of H is said to be H-bound if all the neighbors of v in G lie in H; i.e., $N_G(v) \subseteq V(H)$.

We pose the following

Conjecture 3 (July 2022). Let G be a k-connected graph with $k \geq 2$ and let x, y be two distinct vertices of G. If P is a longest (x, y)-path in G, then P contains $k - 1$ internal P-bound vertices.

A computer search shows that Conjecture 3 holds for all graphs of order ≤ 10, for cubic graphs of order ≤ 18, for 4-regular graphs of order ≤ 14, for triangle-free graphs of order ≤ 12 and for C_4-free graphs of order ≤ 13.

The case $k = 2$ of Conjecture 3 without the word "internal" (weaker version) has the following form:

Conjecture 4. Let G be a 2-connected graph and let x, y be two distinct vertices of G. If P is a longest (x, y)-path in G, then P contains a P-bound vertex.

Conjecture 4 implies Conjecture 2, and hence Conjecture 1. It has the following equivalent form:

Conjecture 5 (The ST conjecture). Suppose that the vertex set of a graph G consists of two disjoint sets S and T such that
(1) $G[S]$ is an (x, y)-path P and T is an independent set;
(2) every vertex in S has at least one neighbor in T;
(3) every vertex in T has at least two neighbors in S.
Then P is not a longest (x, y)-path in G.

References

1, B.R. Alspach and C.D. Godsil (Eds.), Cycles in Graphs, Ann. Discrete Math., 27(1985), 461-468.

2, D.J. Harvey, A cycle of maximum order in a graph of high minimum degree has a chord, Electron. J. Combin., 24(2017), no.4, paper No.4.33.

3, X. Zhan, The minimum number of detours in graphs, Australas. J. Combin., 90(2024), 85-93.

4, X. Zhan, A conjecture generalizing Thomassen's chord conjecture in graph theory, Bull. Iranian Math. Soc., 50(2024), no.5, Paper No.69.

THANK YOU