Homogeneously traceable graphs and minimally hamiltonian-connected graphs

Xingzhi Zhan (詹兴致)

zhan@math.ecnu.edu.cn

East China Normal University

Part 1, Homogeneously traceable graphs

Joint work with Yanan Hu (胡亚楠)

A Hamilton path in a graph G is a path containing all the vertices of G.

A graph is called traceable if it contains a Hamilton path.

A graph is called homogeneously traceable if every vertex is an endpoint of a Hamilton path.

In 1979 Chartrand, Gould and Kapoor proved that
for every integer $n \geq 9$, there exists a homogeneously traceable nonhamiltonian graph of order n.

The graphs they constructed are irregular. Thus it is natural to consider the existence problem of regular homogeneously traceable nonhamiltonian graphs.

Theorem 1 For every even integer $n \geq 10$, there exists a cubic homogeneously traceable nonhamiltonian graph of order n.

For every integer $p \geq 18$, there exists a 4-regular homogeneously traceable graph of order p and circumference $p - 4$.

Main ideas in the proof

Definition. Let v be a vertex of degree d in a graph.
Blowing up v into the complete graph K_d is the operation of replacing v by K_d and adding d edges joining the vertices of K_d to the vertices in $N(v)$ such that the new edges form a matching.

The operation of blowing up a vertex of degree 4 into K_4 is depicted in Figure 1.

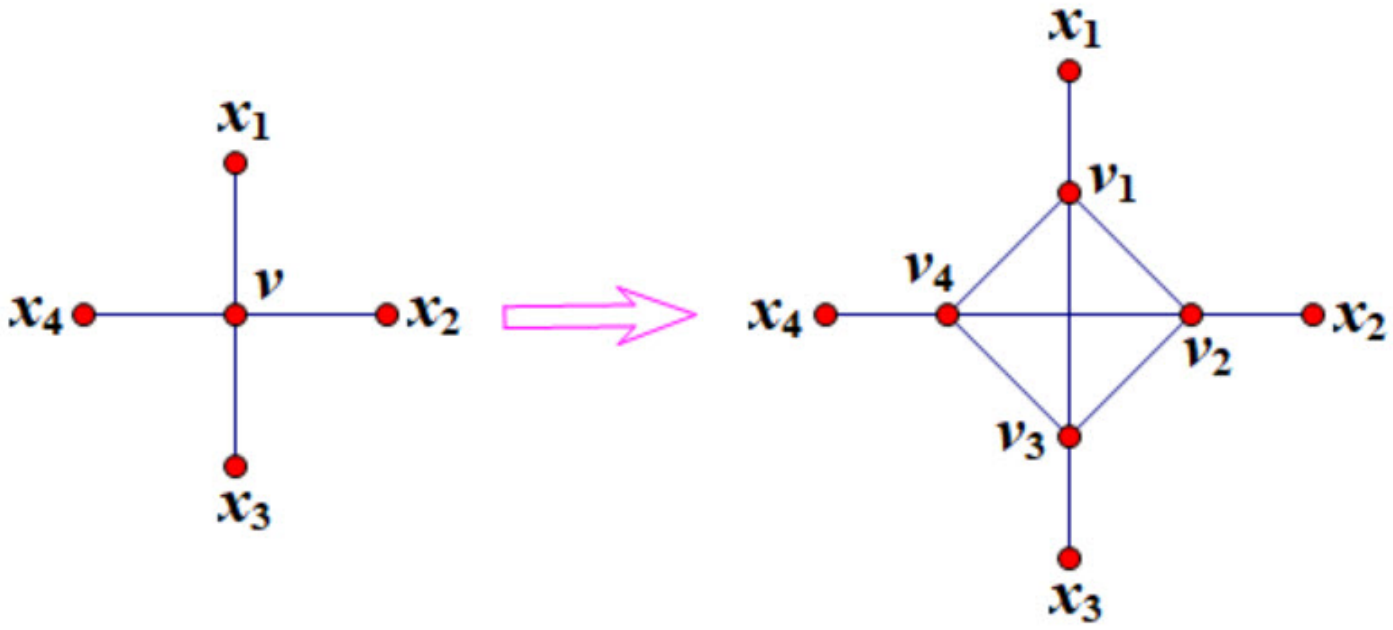

Fig. 1. Blowing up v into K_4

Definition. A graph G is called doubly homogeneously traceable if for any vertex v of G, there are two Hamilton v-paths P and Q such that the two edges incident to v on P and Q are distinct.

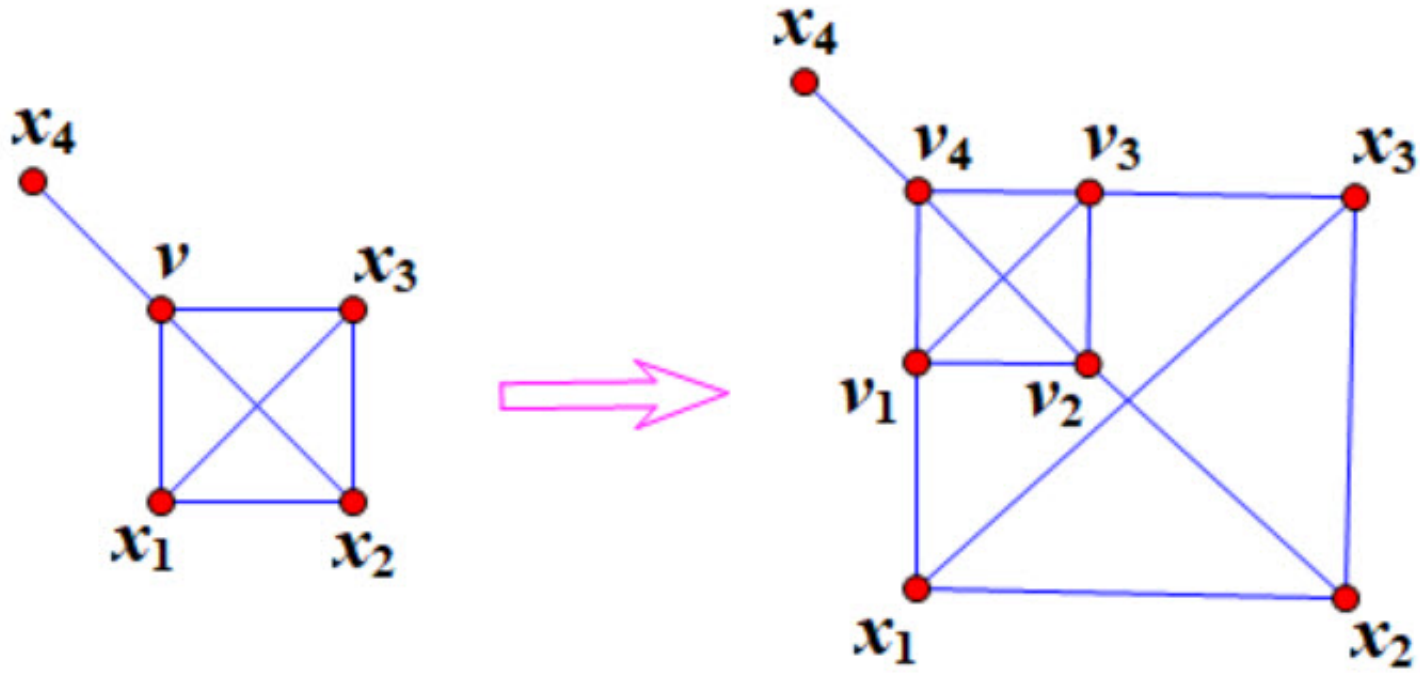

Fig. 2. Local changes

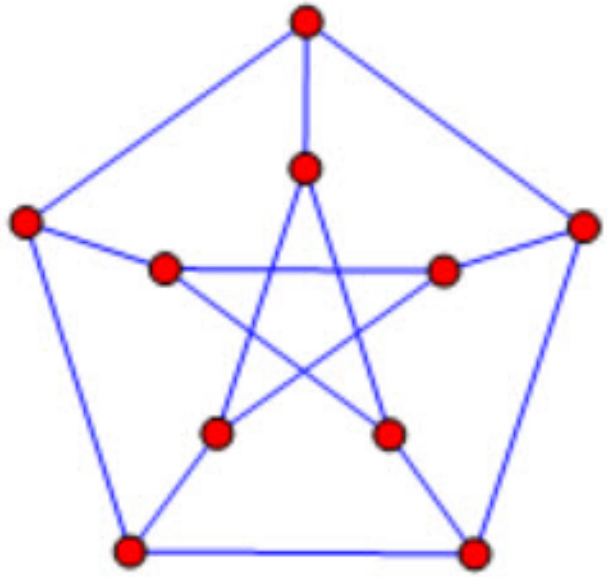

Fig. 3. The Petersen graph

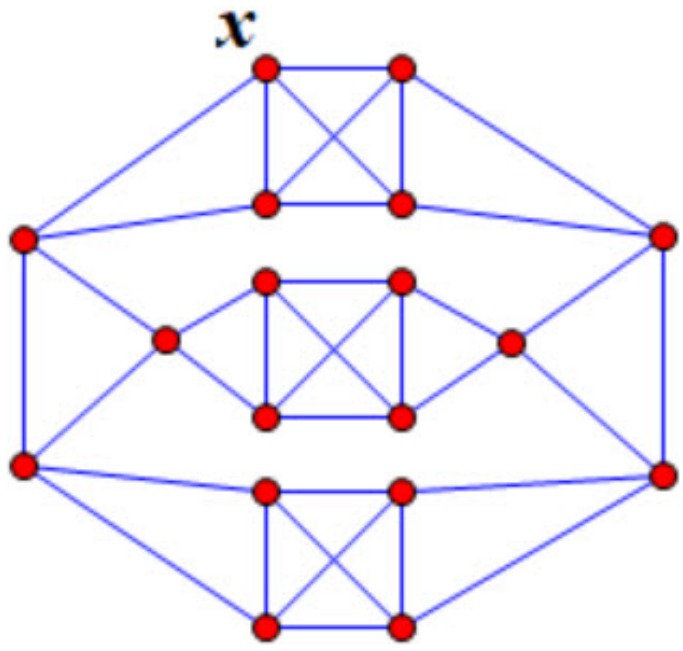

Fig. 4. The 4-regular base graph of order 18

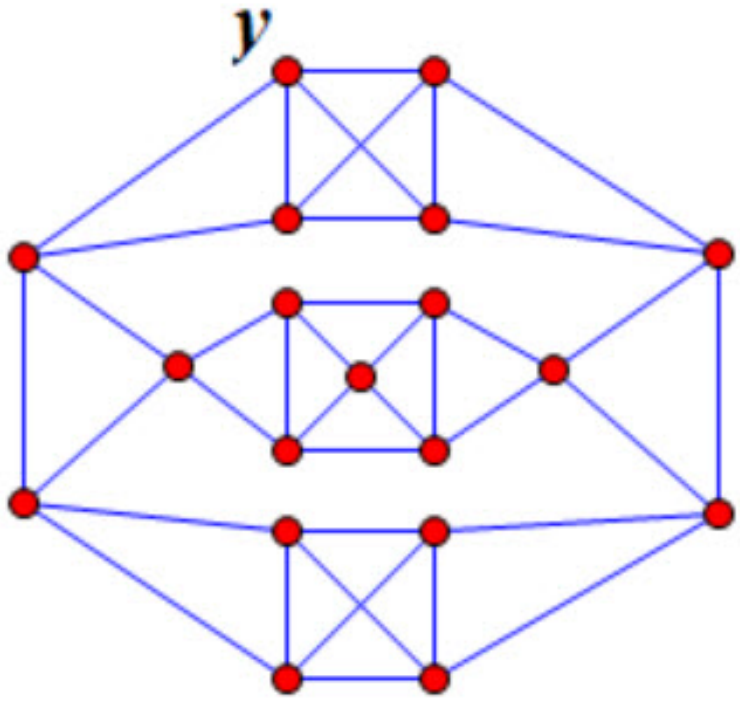

Fig. 5. The 4-regular base graph of order 19

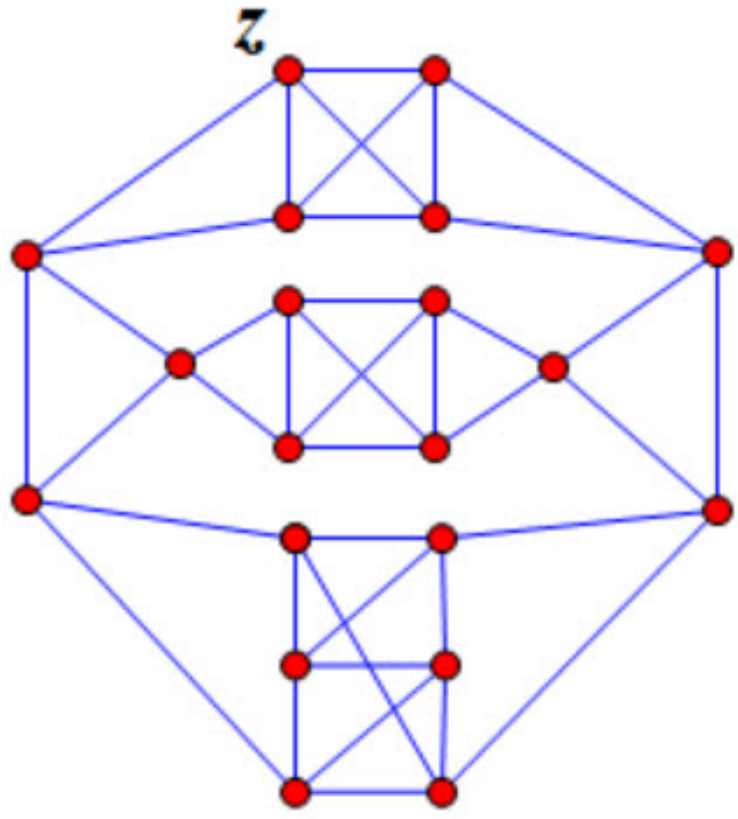

Fig. 6. The 4-regular base graph of order 20

Problem 2 Determine those integer pairs (k, n) such that there exists a k-regular homogeneously traceable nonhamiltonian graph of order n.

Conjecture 3 The minimum circumference of a homogeneously traceable graph of order $n \geq 9$ is $\lceil 2n/3 \rceil + 2$.

The circumference $\lceil 2n/3 \rceil + 2$ in Conjecture 3 is attained by the following graph

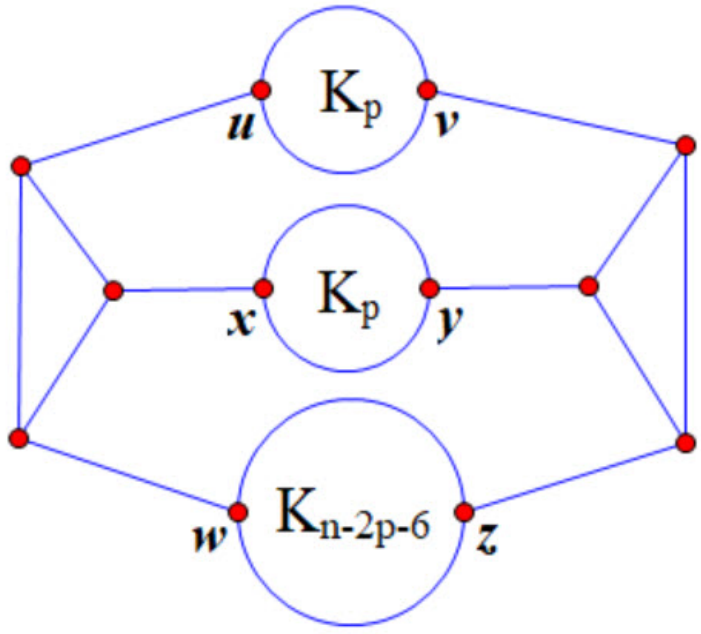

Fig. 7. A homogeneously traceable graph of a small circumference

where $p = \lfloor (n-6)/3 \rfloor$ and when $p \geq 2$ the vertices u and v are distinct, x and y are distinct and w and z are distinct.

Part 2, The max and min degrees of minimally hamiltonian-connected graphs

A graph is called hamiltonian-connected if between any two distinct vertices there is a Hamilton path.

A hamiltonian-connected graph G is said to be minimally hamiltonian-connected if for every edge e of G, the graph $G - e$ is not hamiltonian-connected.

In 2016, Modalleliyan and Omoomi posed the

Problem. What are the possible maximum degrees of a minimally hamiltonian-connected graph of order n?

Theorem 4. Let $n \geq 4$ be an integer. There exists a minimally hamiltonian-connected graph of order n with maximum degree Δ if and only if $3 \leq \Delta \leq n-1$ and $\Delta \neq n-2$, where $\Delta = 3$ occurs only if n is even.

Our construction of $G(n, \Delta)$

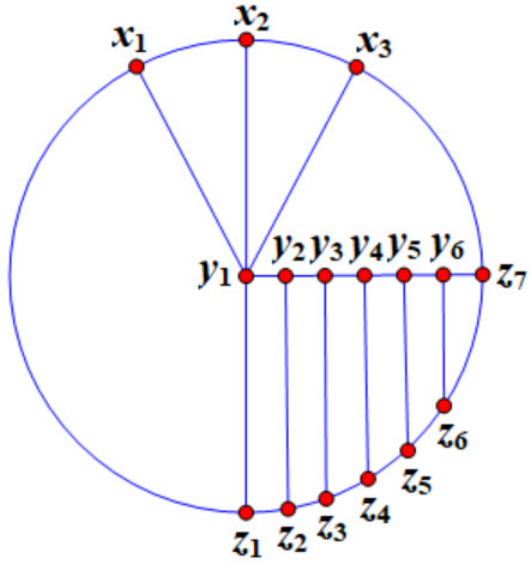

Fig. 1. The graph G(16, 5)

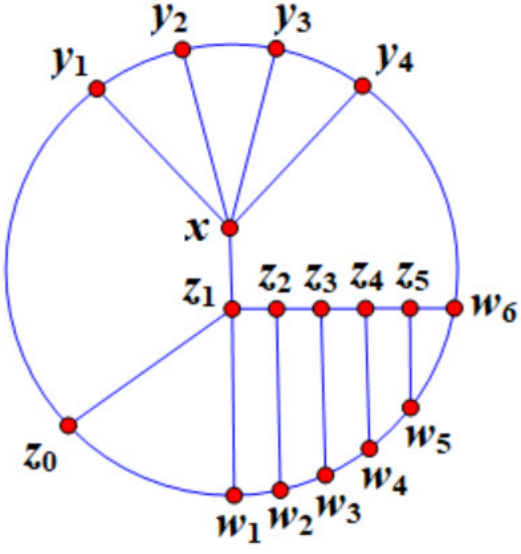

Fig. 2. The graph H(17, 5)

Two unsolved problems

Problem 1. What are the possible values of the minimum degree of a minimally hamiltonian-connected graph of order n?

A computer search shows that
every minimally hamiltonian-connected graph of order n with $n \leq 10$ has minimum degree 3.

Problem 2. Does there exist a minimally hamiltonian-connected graph with minimum degree at least 4?

References

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THANK YOU