A sufficient condition for pancyclic graphs

Xingzhi Zhan*

Department of Mathematics, East China Normal University, Shanghai 200241, China

Abstract

A graph G is called an $[s,t]$-graph if any induced subgraph of G of order s has size at least t. We prove that every 2-connected $[4,2]$-graph of order at least 7 is pancyclic. This strengthens existing results. There are 2-connected $[4,2]$-graphs which do not satisfy the Chvátal-Erdős condition. We also determine the triangle-free graphs among $[p+2,p]$-graphs for a general p.

Key words. Hamiltonian graph; pancyclic graph; $[s,t]$-graph; triangle-free

Mathematics Subject Classification. 05C38, 05C42, 05C45, 05C75

1 Introduction

We consider finite simple graphs and use standard terminology and notation from [3] and [8]. The *order* of a graph is its number of vertices, and the *size* its number of edges. A k-cycle is a cycle of length k. In 1971 Bondy [1] introduced the concept of a pancyclic graph. A graph G of order n is called *pancyclic* if for every integer k with $3 \leq k \leq n$, G contains a k-cycle.

Definition 1. Let s and t be given integers. A graph G is called an $[s,t]$-*graph* if any induced subgraph of G of order s has size at least t.

Denote by $\alpha(G)$ the independence number of a graph G. We have two facts. (1) Every $[s,t]$-graph is an $[s+1,t+1]$-graph; (2) $\alpha(G) \leq k$ if and only if G is a $[k+1,1]$-graph. Thus the concept of an $[s,t]$-graph is an extension of the independence number.

*E-mail address: zhan@math.ecnu.edu.cn

In 2005 Liu and Wang [6] proved the following result.

Theorem 1. *Every* 2*-connected* $[4,2]$*-graph of order at least* 6 *is hamiltonian.*

In 2007 Liu, Wang and Gao [7] improved Theorem 1 as follows.

Theorem 2. *Let* G *be a* 2*-connected* $[4,2]$*-graph of order* n *with* $n \geq 7$*. If* G *contains a* k*-cycle with* $k < n$*, then* G *contains a* $(k+1)$*-cycle.*

In this paper we further strengthen Theorem 2 by proving that every 2-connected $[4,2]$-graph of order at least 7 is pancyclic (Theorem 6). To do so, we will determine the triangle-free graphs among $[p+2,p]$-graphs. This preliminary result (Lemma 5) is of independent interest.

2 Main results

We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively, and denote by $|G|$ and $e(G)$ the order and size of G, respectively. Thus $|G| = |V(G)|$ and $e(G) = |E(G)|$. For a vertex subset $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S. The neighborhood of a vertex x is denoted by $N(x)$ and the closed neighborhood of x is $N[x] \triangleq N(x) \cup \{x\}$. The degree of x is denoted by $\deg(x)$. For $S \subseteq V(G)$, $N_S(x) \triangleq N(x) \cap S$ and the degree of x in S is $\deg_S(x) \triangleq |N_S(x)|$. Given two vertex subsets S and T of G, we denote by $[S,T]$ the set of edges having one endpoint in S and the other in T. The degree of S is $\deg(S) \triangleq |[S,\overline{S}]|$, where $\overline{S} = V(G) \setminus S$. We denote by C_n and K_n the cycle of order n and the complete graph of order n, respectively. $\overline{G}$ denotes the complement of a graph G.

We will need the following two lemmas on integral quadratic forms.

Lemma 3. *Given positive integers* $n \geq k \geq 2$*, let* $x_1, x_2, \ldots, x_k$ *be positive integers such that* $\sum_{i=1}^{k} x_i = n$*. Then*

$$n - 1 \leq \sum_{i=1}^{k-1} x_i x_{i+1} \leq \begin{cases} \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil & \text{if } k = 2,\, 3 \\ ab + k - 5 & \text{if } k \geq 4 \end{cases} \tag{1}$$

where $a = \lfloor (n-k+4)/2 \rfloor$ *and* $b = \lceil (n-k+4)/2 \rceil$*. For any* n *and* k*, the lower and upper bounds in (1) can be attained.*

Proof. Define a quadratic polynomial $f(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k-1} x_i x_{i+1}$. We first prove the left-hand side inequality in (1). Let $x_j = \min\{x_i \,|\, 1 \le i \le k\}$. We have

$$\begin{aligned} f(x_1, x_2, \ldots, x_k) &\ge x_1x_j + \cdots + x_{j-1}x_j + x_jx_{j+1} + x_jx_{j+2} + \cdots + x_jx_k \\ &= x_j(x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k) \\ &= x_j(n - x_j) \\ &\ge n - 1. \end{aligned}$$

This proves the first inequality in (1). The lower bound $n-1$ is attained at $x_1 = n-k+1$, $x_2 = \cdots = x_k = 1$.

Now we prove the second inequality in (1). The case $k = 2$ is an elementary fact: $f(x_1, x_2) = x_1x_2 \le \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ where equality holds when $x_1 = \lfloor n/2 \rfloor$ and $x_2 = \lceil n/2 \rceil$. The case $k = 3$ reduces to the case $k = 2$ as follows:

$$f(x_1, x_2, x_3) = x_2(x_1 + x_3) \le \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$$

where equality holds when $x_2 = \lfloor n/2 \rfloor$ and $x_1 + x_3 = \lceil n/2 \rceil$. Next suppose $k \ge 4$. Denote by $f_{\max}$ the maximum value of f. If $x_1 > 1$, with $x_1' = 1$, $x_2' = x_2$, $x_3' = x_3 + x_1 - 1$ and $x_i' = x_i$ for $i \ge 4$ we have

$$f(x_1', x_2', \ldots, x_k') - f(x_1, x_2, \ldots, x_k) = (x_1 - 1)x_4 > 0.$$

Similarly analyzing the variable x_k, we deduce that $f_{\max}$ can only be attained at some $x_1, \ldots, x_k$ with $x_1 = x_k = 1$, which we assume now. With $x_2' = 1$, $x_3' = x_3$, $x_4' = x_4 + x_2 - 1$, and $x_i' = x_i$ for $i = 5, \ldots, k-1$ we have

$$f(1, 1, x_3', x_4', \ldots, x_{k-1}', 1) - f(1, x_2, x_3, \ldots, x_{k-1}, 1) = (x_2 - 1)(x_5 - 1) \ge 0.$$

Hence $f_{\max}$ can be attained at a certain $(1, 1, x_3, \ldots, x_{k-1}, 1)$. Successively applying this argument we deduce that $f_{\max}$ can be attained at $(1, 1, \ldots, 1, x_{k-2}, x_{k-1}, 1)$. Now $(x_{k-2} + 1) + (x_{k-1} + 1) = n - k + 4$. We have

$$\begin{aligned} f(1, 1, \ldots, 1, x_{k-2}, x_{k-1}, 1) &= (x_{k-2} + 1)(x_{k-1} + 1) + k - 5 \\ &\le \lfloor (n - k + 4)/2 \rfloor \cdot \lceil (n - k + 4)/2 \rceil + k - 5. \end{aligned}$$

This proves the second inequality in (1). The upper bound is attained at $x_1 = x_2 = \cdots = x_{k-3} = x_k = 1$, $x_{k-2} = \lfloor (n - k + 2)/2 \rfloor$ and $x_{k-1} = \lceil (n - k + 2)/2 \rceil$. □

Lemma 4. [9, Theorem 1] *Given positive integers $n \ge k \ge 2$, let $x_1, x_2, \ldots, x_k$ be positive integers such that $\sum_{i=1}^{k} x_i = n$. Then*

$$2n - k \le \sum_{i=1}^{k} x_ix_{i+1} \tag{2}$$

where $x_{k+1} \triangleq x_1$. For any n and k, the lower bound in (2) can be attained.

The sharp upper bound on the quadratic form in (2) is also determined in [9], but we do not need it here.

Definition 2. Given a graph H and a positive integer k, the *k-blow-up of H*, denoted by $H^{(k)}$, is the graph obtained by replacing every vertex of H with k different vertices where a copy of u is adjacent to a copy of v in the blow-up graph if and only if u is adjacent to v in H.

For example, $C_5^{(2)}$ is depicted in Figure 1.

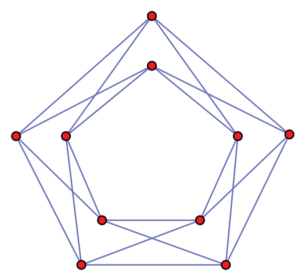

Figure 1: The 2-blow-up of C_5

Now we are ready to determine the triangle-free graphs among $[p+2,p]$-graphs. $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of a graph G, respectively. We regard isomorphic graphs as the same graph. Thus for two graphs G and H, the notation $G = H$ means that G and H are isomorphic.

Lemma 5. *Let G be a $[p+2,p]$-graph of order n with $\delta(G) \geq p \geq 2$ and $n \geq 2p+3$. Then G is triangle-free if and only if p is even, $p \geq 6$ and $G = C_5^{(p/2)}$.*

Proof. We will repeatedly use the condition that G is a $[p+2,p]$-graph without mentioning it possibly. Denote $\Delta = \Delta(G)$ and choose a vertex $x \in V(G)$ such that $\deg(x) = \Delta$. Let $S = N(x)$ and $T = V(G) \setminus S$. Then $|S| = \Delta$.

Next suppose that G is triangle-free. Then S is an independent set. Since G is a $[p+2,p]$-graph, $\Delta \leq p+1$. We assert that $\Delta = p$ and hence G is p-regular, since $\delta(G) \geq p$ by the assumption. Otherwise $\Delta = p+1$. Since $n \geq 2p+3$, $|T| \geq p+2$. Thus $G[T]$ contains an edge uv. $|\{u\} \cup S| = p+2$ implies that $\deg_S(u) \geq p$. Similarly $\deg_S(v) \geq p$. Since $p + p = 2p > p+1 = |S|$, we have $N_S(u) \cap N_S(v) \neq \emptyset$. Let $w \in N_S(u) \cap N_S(v)$. Then $wuvw$ is a triangle, a contradiction. This shows that G is p-regular.

Let $y \in S$ and denote $C = N(y)$. Then C is an independent set and $|C| = p$. Denote

$D = T \setminus C$. See the illustration in Figure 2.

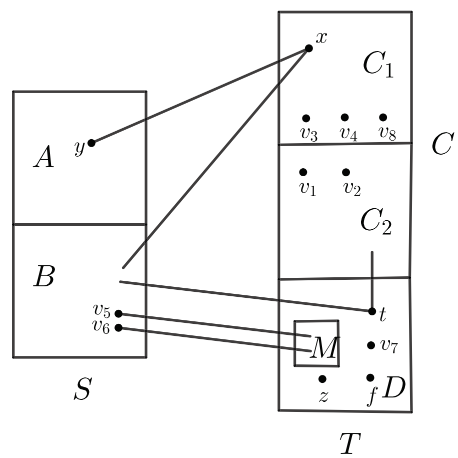

Figure 2: The structure of G

Since $n \geq 2p+3$, we have $|D| = n-2p \geq 3$. Thus D is not a clique, since G is triangle-free. Let z and f be any two distinct nonadjacent vertices in D. Since $|\{z, f\} \cup S| = p+2$, $|\{z, f\} \cup C| = p+2$, and S and C are independent sets, we have

$$\deg_S(z) + \deg_S(f) = |[\{z, f\}, S]| \geq p \quad \text{and} \quad \deg_C(z) + \deg_C(f) = |[\{z, f\}, C]| \geq p.$$

Note that $S \cap C = \emptyset$, $\deg(z) = \deg(f) = p$. We must have

$$|[\{z, f\}, S]| = p \quad \text{and} \quad |[\{z, f\}, C]| = p. \tag{3}$$

We assert that D is an independent set. Otherwise D contains two adjacent vertices u_1 and u_2. Let $u_3 \in D \setminus \{u_1, u_2\}$. Since G is triangle-free, u_3 is nonadjacent to at least one vertex in $\{u_1, u_2\}$, say, u_1. Setting $z = u_1$ and $f = u_3$ in (3) we deduce that $|[\{u_1, u_3\}, S \cup C]| = 2p$. On the other hand, since both u_1 and u_3 have degree p, and u_1 already has a neighbor $u_2 \notin S \cup C$, we have $|[\{u_1, u_3\}, S \cup C]| \leq 2p-1$, a contradiction.

Observe that now (3) holds for any two distinct vertices z and f in D. (3) has the equivalent form

$$\deg_S(z) + \deg_S(f) = p \quad \text{and} \quad \deg_C(z) + \deg_C(f) = p. \tag{4}$$

Then (4) and $|D| \geq 3$ imply that for any vertex $z \in D$,

$$\deg_S(z) = \deg_C(z) = p/2. \tag{5}$$

To see this, to the contrary, first suppose $\deg_S(z) > p/2$. Then by the first equality in (4), for any two other vertices $f, r \in D$ we have $\deg_S(f) < p/2$ and $\deg_S(r) < p/2$, yielding $\deg_S(f) + \deg_S(r) < p$, which contradicts (4). If $\deg_S(z) < p/2$, the same argument gives a contradiction. A similar analysis with C in place of S shows $\deg_C(z) = p/2$. Thus we have proved (5). In particular, $q \triangleq p/2$ is a positive integer; i.e., p is even. Now choose an arbitrary but fixed vertex $t \in D$ and denote $B = N_S(t)$, $C_2 = N_C(t)$, $A = S \setminus B$ and $C_1 = C \setminus C_2$. See the illustration in Figure 2. We have

$$|A| = |B| = |C_1| = |C_2| = q.$$

Since G is p-regular of order $n \geq 2p + 3$, it is impossible that $p = 2$. Otherwise G would be a 2-regular graph of order ≥ 7, which is not a $[4, 2]$-graph. Thus $p \geq 4$ and $q \geq 2$.

Choose any two distinct vertices $v_1, v_2 \in C_2$. $|\{v_1, v_2\} \cup S| = p + 2$ implies that $|[\{v_1, v_2\}, S]| \geq p$. Since G is triangle-free, $N(v_i) \cap B = \emptyset$, $i = 1, 2$. Hence $N_S(v_i) = N_A(v_i)$, $i = 1, 2$. However, $|A| = q$. We have $N_S(v_i) = A$ and $\deg_A(v_i) = q$, $i = 1, 2$, implying that every vertex in C_2 is adjacent to every vertex in A.

Choose any two distinct vertices $v_3, v_4 \in C_1$. Then $|\{v_3, v_4\} \cup C_2 \cup B| = p + 2$. Since $\{v_3, v_4\} \cup C_2$ is an independent set and $[C_2, B] = \emptyset$, we have $|[\{v_3, v_4\}, B]| \geq p$. However, $|B| = q$. Hence $N_B(v_j) = B$, $j = 3, 4$. This shows that every vertex in C_1 is adjacent to every vertex in B. Consequently, every vertex in B has exactly q neighbors in D.

Choose any vertex $v_5 \in B$. Denote $M = N_D(v_5)$. We have $|M| = q$. Since G is triangle-free and every vertex in B is adjacent to every vertex in C_1, the neighborhood of any vertex in M is disjoint from C_1. Thus the q neighbors of any vertex of M in C are exactly the vertices of C_2, implying that every vertex in M is adjacent to every vertex in C_2. The neighborhood of any vertex in C_2 is $A \cup M$. For the same reason, for any vertex $v_6 \in B$ with $v_6 \neq v_5$, we must have $N_D(v_6) = M$. Hence the neighborhood of any vertex in M is $B \cup C_2$.

We assert that $M = D$. Otherwise let $v_7 \in D \setminus M$. Take a vertex $v_8 \in C_1$. Note that $B \cup C_2$ is an independent set of cardinality p and $[v_7, B \cup C_2] = \emptyset$. Denote $R = \{v_7, v_8\} \cup B \cup C_2$. Then $|R| = p+2$ and hence $G[R]$ has size at least p. However, the size of $G[R]$ is at most $|[v_8, B]| + 1 = q + 1 < p$, a contradiction. Finally, since G is p-regular, every vertex in A must be adjacent to every vertex in C_1. Denote $V_1 = A$, $V_2 = C_1$, $V_3 = B$, $V_4 = D$, $V_5 = C_2$ and set $V_6 = V_1$. Then each V_i is an independent set of cardinality $q = p/2$ and every vertex in V_i is adjacent to every vertex in V_{i+1} for $i = 1, 2, \ldots, 5$. This proves that $G = C_5^{(q)}$. Note that we have shown above that $q = |D| \geq 3$, implying that

$p = 2q \geq 6$.

Conversely let $H = C_5^{(q)}$ where $q = p/2$ and $p \geq 6$ is even. We will prove that H is a triangle-free $[p+2, p]$-graph. Write $H = H_1 \vee H_2 \vee H_3 \vee H_4 \vee H_5 \vee H_1$ where each $H_i = \overline{K_q}$ and $\vee$ is the join operation on two vertex-disjoint graphs. If H contains a triangle, it must lie in $H[V(H_i) \cup V(H_{i+1})]$ for some i ($H_6 \triangleq H_1$). However, this is a bipartite graph, containing no triangle.

Let $U \subseteq V(H)$ with $|U| = p + 2$. We need to show $e(H[U]) \geq p$. Denote $I = \{i \,|\, U \cap V(H_i) \neq \emptyset,\ 1 \leq i \leq 5\}$. Since $|H_i| = q$, $1 \leq i \leq 5$ and $|U| = p + 2$, we have $|I| \geq 3$. Denote $x_i = |U \cap V(H_i)|$ for $1 \leq i \leq 5$. Then $0 \leq x_i \leq q$. We distinguish three cases.

Case 1. $|I| = 3$.

There are at least two consecutive integers in I (1 and 5 are regarded as consecutive here). Without loss of generality, suppose $1, 2 \in I$. Then $1 \leq x_1,\ x_2 \leq q$ and $x_1 + x_2 \geq p + 2 - q = q + 2$. Hence $e(H[U]) \geq x_1 x_2 \geq 2q = p$.

Case 2. $|I| = 4$.

Without loss of generality, suppose $I = \{1, 2, 3, 4\}$. Then $e(H[U]) = x_1x_2 + x_2x_3 + x_3x_4$ where each x_i is a positive integer and $x_1 + x_2 + x_3 + x_4 = p + 2$. Applying Lemma 3 we have $e(H[U]) \geq (p + 2) - 1 = p + 1 > p$.

Case 3. $|I| = 5$.

Now $e(H[U]) = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$ where each x_i is a positive integer and $x_1 + x_2 + x_3 + x_4 + x_5 = p + 2$. Applying Lemma 4 we have $e(H[U]) \geq 2(p+2) - 5 = 2p - 1 > p$.

In every case H is a $[p + 2, p]$-graph. This completes the proof. □

Theorem 6. *Every 2-connected $[4, 2]$-graph of order at least 7 is pancyclic.*

Proof. Let G be a 2-connected $[4, 2]$-graph of order at least 7. Since G is 2-connected, $\delta(G) \geq 2$. By the case $p = 2$ of Lemma 5, G contains a triangle C_3. Then successively applying Theorem 2 we deduce that G is pancyclic. □

Remark. The Chvátal-Erdős theorem on hamiltonian graphs ([4], [1] and [8]) states that for a graph G, if $\kappa(G) \geq \alpha(G)$ then G is hamiltonian, where κ and α denote the connectivity and independence number, respectively. Bondy [2] proved that if a graph satisfies Ore's condition, then it satisfies the Chvátal-Erdős condition. A computer search for graphs of lower orders shows that there are many graphs which satisfy the condition in Theorem 6, but do not satisfy the Chvátal-Erdős condition. There are exactly 398 such graphs of order 9. For every integer $n \geq 7$ we give an example. Let $G_1 = K_{n-3}^-$ be the graph obtained from K_{n-3} by deleting one edge xy and let $G_2 = uvw$ be a triangle that is vertex-disjoint from G_1. Construct a graph Z_n form G_1 and G_2 by adding two edges xu and yv. The graph Z_9 is depicted in Figure 3.

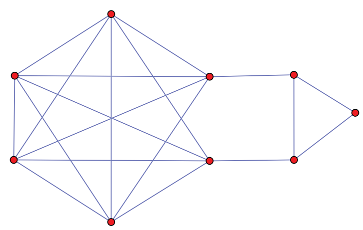

Figure 3: The graph Z_9

Clearly Z_n is a 2-connected $[4, 2]$-graph of order n, but $2 = \kappa(Z_n) < \alpha(Z_n) = 3$.

Acknowledgement. This research was supported by the NSFC grant 12271170 and Science and Technology Commission of Shanghai Municipality grant 22DZ2229014.

References

[1] J.A. Bondy, Pancyclic graphs. I, J. Combinatorial Theory Ser. B, 11(1971), 80-84.

[2] J.A. Bondy, A remark on two sufficient conditions for Hamilton cycles, Discrete Math., 22(1978), no.2, 191-193.

[3] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.

[4] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, Discrete Math., 2(1972), 111-113.

[5] J.C. George, A. Khodkar and W.D. Wallis, Pancyclic and Bipancyclic Graphs, Springer, 2016.

[6] C. Liu and J. Wang, $[s, t]$-graphs and their hamiltonicity, (Chinese), J. Shandong Normal Univ. Nat. Sci., 20(2005), no.1, 6-7.

[7] X. Liu, J. Wang and G. Gao, Cycles in 2-connected $[4, 2]$-graphs, (Chinese), J. Shandong Univ. Nat. Sci., 42(2007), no.4, 32-35.

[8] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., 1996.

[9] X. Zhan, Extremal numbers of positive entries of imprimitive nonnegative matrices, Linear Algebra Appl. 424(2007), no.1, 132–138.