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The minimum number of detours in graphs

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Abstract

A longest path in a graph is called a detour. It is not difficult to see that a connected graph of minimum degree at least 2 and order at least 4 has at least four detours. We prove that if the number of detours in such a graph of order at least 9 is odd, then it is at least nine, and this lower bound can be attained for every order. Thus the possibilities three, five and seven are never attained. The reason for this interesting phenomenon does not seem obvious, in view of the fact that the numbers four, six, eight and nine can be attained. Two related problems are posed.

1 Introduction

We consider finite simple graphs and use terminology and notation from [7]. Following Kapoor, Kronk, and Lick [6], we call a longest path in a graph G a *detour* of G. This concise term has now been widely used (e.g. [1] and [3]). In 1966, Gallai [4] asked whether all detours in a connected graph share a common vertex. The answer is no in general. However, for some classes of special graphs, the answer is yes. See [5] and the references therein. It is natural to consider the number of detours in a graph, but the author cannot find any results on this problem in the literature. Even results on the number of paths with a given length are few [2].

The *order* of a graph is its number of vertices. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively. For vertices x and y, an (x, y)-*path* is a path with endpoints x and y. We denote by $\delta(G)$ the minimum degree of a graph G, and by $N(x)$ the neighborhood of a vertex x. If u, v are two vertices on a path P, then $P[u, v]$ denotes the subpath of P with i endpoints u and v. A basic fact about detours [6] is that a detour of a connected graph G of order n has order at least $\min\{2\delta(G) + 1, n\}$.

We will consider the minimum number of detours in a given class of graphs. If the graph is disconnected, it suffices to consider its components, while if the minimum degree of a graph is 1, then the graph may contain only one detour. Thus we will consider only connected graphs of minimum degree at least 2. In Section 2 we prove

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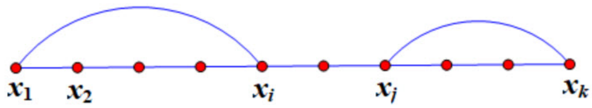

Figure 1: The case $i \le j$

that a connected graph of minimum degree at least 2 and order at least 4 has at least four detours and if the number of detours in such a graph of order at least 9 is odd, then it is at least nine, and this lower bound can be attained for every order. Thus the possibilities three, five and seven are excluded. In Section 3 we pose two related unsolved problems.

2 Main results

Notation. $f(G)$ denotes the number of detours in a graph G.

Theorem 1. *The minimum number of detours in a connected graph of minimum degree at least* 2 *and order at least* 4 *is four.*

Proof. Let G be a connected graph of order at least 4 with $\delta(G) \ge 2$ and let $P:\ x_1, x_2, \ldots, x_k$ be a detour of G. Then $N(x_1) \subseteq V(P)$ and $N(x_k) \subseteq V(P)$. Since $\delta(G) \ge 2$, x_1 has a neighbor x_i with $i \ge 3$ and x_k has a neighbor x_j with $j \le k-2$. If $i = k$ or $j = 1$, then G contains a k-cycle and we clearly have $f(G) \ge 4$. Next suppose $3 \le i \le k-1$ and $2 \le j \le k-2$.

Case 1. $i \le j$.

G has at least the following four detours:

$$P, \quad P[x_1, x_j] \cup x_j x_k \cup P[x_k, x_{j+1}],$$

$$P[x_{i-1}, x_1] \cup x_1 x_i \cup P[x_i, x_k], \ \ P[x_{i-1}, x_1] \cup x_1 x_i \cup P[x_i, x_j] \cup x_j x_k \cup P[x_k, x_{j+1}].$$

See Figure 1.

Case 2. $i > j$.

G has at least the following six detours:

$$P, \ \ P[x_1, x_j] \cup x_j x_k \cup P[x_k, x_{j+1}], \ \ P[x_{i-1}, x_1] \cup x_1 x_i \cup P[x_i, x_k],$$

$$P[x_{j-1}, x_1] \cup x_1 x_i \cup P[x_i, x_k] \cup x_k x_j \cup P[x_j, x_{i-1}],$$
$$P[x_{i+1}, x_k] \cup x_k x_j \cup P[x_j, x_1] \cup x_1 x_i \cup P[x_i, x_{j+1}],$$
$$P[x_{j-1}, x_1] \cup x_1 x_i \cup P[x_i, x_j] \cup x_j x_k \cup P[x_k, x_{i+1}].$$

See Figure 2.

This shows $f(G) \ge 4$. Conversely, for every order $n \ge 4$ we construct a graph G_n of order n with $\delta(G) \ge 2$ satisfying $f(G_n) = 4$. $G_4 = C_4$, the 4-cycle. G_5 is the

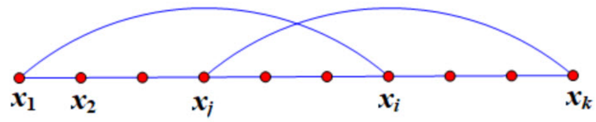

Figure 2: The case $i > j$

bowtie, the graph consisting of two triangles sharing one vertex. G_6 consists of a triangle and a 4-cycle sharing one vertex. G_7, G_8 and G_9 are depicted in (a), (b) and (c) of Figure 3, respectively.

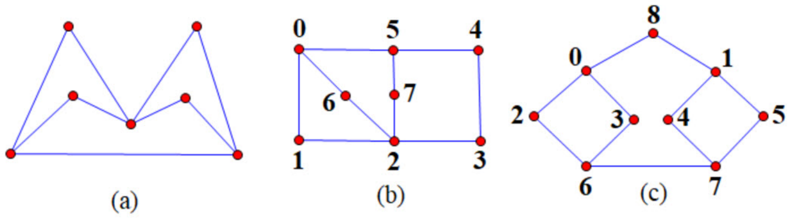

Figure 3: The graphs G_7, G_8 and G_9

G_8 has the four detours:

(1) $1, 0, 6, 2, 7, 5, 4, 3$; (2) $1, 0, 6, 2, 3, 4, 5, 7$;

(3) $3, 4, 5, 7, 2, 1, 0, 6$; (4) $6, 0, 1, 2, 3, 4, 5, 7$

and G_9 has the four detours:

(1) $2, 6, 3, 0, 8, 1, 5, 7, 4$; (2) $2, 6, 3, 0, 8, 1, 4, 7, 5$;

(3) $3, 6, 2, 0, 8, 1, 5, 7, 4$; (4) $3, 6, 2, 0, 8, 1, 4, 7, 5$.

Observe that each of the four detours in G_9 contains the path $0, 8, 1$. For $n \geq 10$, G_n is obtained from G_9 in (c) of Figure 3 by replacing the path $0, 8, 1$ by a $(0, 1)$-path of order $n - 6$. □

Remark 1. Note that for $n \geq 7$, the graphs G_n in the above proof of Theorem 1 are 2-connected. Thus, if we replace "minimum degree at least 2" by "2-connected" in Theorem 1, we obtain the same conclusion for graphs of order at least 7.

We make the following conventions: (1) For a positive integer r, "r detours" means "r pair-wise distinct detours"; (2) for an edge e of G and a detour D, we say that e appears on D if $e \in E(D)$.

Lemma 2. *Let $P : x_1, x_2, \ldots, x_k$ be a detour in a graph of order at least 4 and suppose that x_1 has a neighbor x_i with $i \geq 3$ and x_k has a neighbor x_j with $j \leq k-2$. Then an edge e of P appears on at least four detours unless (1) $i \leq j$ and $e = x_{i-1}x_i$ or $e = x_jx_{j+1}$ or (2) $i = j+1$ and $e = x_ix_j$. Each of the three edges in the exceptional cases (1) and (2) appears on at least two detours.*

Proof. This can be verified by checking the proof of Theorem 1. □

Recall that $f(G)$ denotes the number of detours in a graph G.

Theorem 3. *Let G be a connected graph of minimum degree at least 2 and order at least 9. If $f(G)$ is an odd number, then $f(G) \geq 9$. Furthermore, the lower bound 9 can be attained for every order by both graphs of connectivity 1 and graphs of connectivity 2.*

Proof. We first prove that if $f(G)$ is an odd number, then $f(G) \geq 9$. By Theorem 1 it suffices to show that either $f(G) \geq 8$ or $f(G) = 4$ or $f(G) = 6$.

Let $P : x_1, x_2, \ldots, x_k$ be a detour of G. If there is another detour Q with $V(Q) \neq V(P)$, by the proof of Theorem 1, there are at least four detours with the same vertex set $V(P)$ and there are at least four detours with the same vertex set $V(Q)$. These detours are clearly distinct. Hence we have $f(G) \geq 8$. Next suppose that all detours of G have $V(P)$ as their vertex set.

Recall that an edge e of G is called a *chord* of a path R if the two endpoints of e lie in R but $e \notin E(R)$. A chord e of R is called an *inner chord* if both endpoints of e are internal vertices of R. Otherwise e is called a *boundary chord*. A detour D is called a *basic detour* if no inner chord of P is an edge of D; otherwise D is called a *non-basic detour*.

Let the order of G be n. If G is hamiltonian, then $f(G) \geq n \geq 9$. Next assume that G is non-hamiltonian.

Since P is a detour, $N(x_1) \subseteq V(P)$ and $N(x_k) \subseteq V(P)$. The condition $\delta(G) \geq 2$ implies that x_1 has a neighbor x_i with $i \geq 3$ and x_k has a neighbor x_j with $j \leq k-2$. If $i = k$ or $j = 1$, then G has a k-cycle C which contains P. Since P is a detour, C must be a Hamilton cycle, contradicting our assumption that G is non-hamiltonian. Hence $3 \leq i \leq k-1$ and $2 \leq j \leq k-2$. We distinguish two cases.

Case 1. Every detour of G is a basic detour.

We need consider only the boundary chords of P.

Subcase 1.1. P contains exactly two boundary chords.

As analyzed in the proof of Theorem 1, in this case $f(G) = 4$ or $f(G) = 6$, where we have used the assumptions that all detours of G have $V(P)$ as their vertex set and every detour of G is a basic detour.

Subcase 1.2. P contains exactly three boundary chords.

Without loss of generality, let x_1x_q be the third chord of P with $q \neq i$. Note that the two boundary chords x_1x_i and x_1x_q are in symmetric positions. If $q > i$ we may interchange the roles of x_1x_i and x_1x_q. Thus we may and do assume that $q < i$.

Suppose $i \leq j$. We have four basic detours not containing the edge x_1x_q. If $3 \leq q \leq i-2$, we have exactly the following two detours containing the edge x_1x_q:

$$P[x_{q-1}, x_1] \cup x_1x_q \cup P[x_q, x_k], \quad P[x_{q-1}, x_1] \cup x_1x_q \cup P[x_q, x_j] \cup x_jx_k \cup P[x_k, x_{j+1}].$$

Hence $f(G) = 6$. If $q = i-1$, we have exactly the following four detours containing the edge x_1x_q:

$$P[x_2, x_q] \cup x_qx_1 \cup x_1x_i \cup P[x_i, x_k], \quad P[x_2, x_q] \cup x_qx_1 \cup x_1x_i \cup P[x_i, x_j] \cup x_jx_k \cup P[x_k, x_{j+1}],$$

$$P[x_{q-1}, x_1] \cup x_1x_q \cup P[x_q, x_k], \;\; P[x_{q-1}, x_1] \cup x_1x_q \cup P[x_q, x_j] \cup x_jx_k \cup P[x_k, x_{j+1}].$$

Hence $f(G) = 8$.

Suppose $i > j$. We have six basic detours not containing the edge x_1x_q. In this case it is easy to check that there are at least two detours containing the edge x_1x_q by considering the subgraph $P \cup x_1x_q \cup x_kx_j$. Thus $f(G) \geq 8$.

Subcase 1.3. P contains at least four boundary chords.

Based on Subcase 1.2, we deduce that $f(G) \geq 8$ in this case.

Case 2. G contains a non-basic detour.

Claim 1. Every edge in a detour appears on at least two detours.

This claim follows from Lemma 2.

Since G contains a non-basic detour, some inner chord e of P is an edge of a detour. By Claim 1, there are at least two detours containing e as an edge. Thus G has at least two non-basic detours. If $i > j$, we have six basic detours, and consequently $f(G) \geq 8$. By Subcases 1.2 and 1.3, if P contains at least three boundary chords, then G contains at least six basic detours. Again we have $f(G) \geq 8$. If an inner chord of P appears on at least four detours, then we have at least four non-basic detours. It follows that $f(G) \geq 8$.

It remains to treat the case when (1) $i \leq j$, (2) P contains exactly two boundary chords, and (3) every inner chord of P appears on at most three detours. Next we make these three assumptions.

Let $D : y_1, y_2, \ldots, y_k$ be a detour of G. Suppose y_c is a neighbor of y_1 and y_d is a neighbor of y_k with $3 \leq c \leq k-1$ and $2 \leq d \leq k-2$. As in the proof of Theorem 1, there are four detours (if $c \leq d$) or six detours (if $c > d$) whose edges belong to $E(D) \cup \{y_1y_c,\, y_ky_d\}$. We denote by $\Psi(D)$ the set of these four or six detours according as $c \leq d$ or $c > d$. When we write $\Psi(D)$ we assume that the two boundary chords y_1y_c and y_ky_d have been prescribed.

Claim 2. If an inner chord h of P appears on a detour D such that $\Psi(D) \cap \Psi(P) \neq \emptyset$, then one of the two endpoints of h belongs to the set $\{x_{i-1}, x_{j+1}\}$.

Let $D = y_1, y_2, \ldots, y_k$ with $h \in E(D)$. Suppose y_c is a neighbor of y_1 and y_d is a neighbor of y_k with $3 \leq c \leq k-1$ i and $2 \leq d \leq k-2$. Since h appears on at most three detours, by Lemma 2, if $c > d$ we must have $c = d + 1$ and then $V(D)$ is contained in a cycle which must be a Hamilton cycle since D is a detour, contradicting our assumption that G is non-hamiltonian. Thus $c \leq d$ and then by Lemma 2, either $h = y_{c-1}y_c$ or $h = y_dy_{d+1}$. Note that since h is an inner chord of P, the two endpoints of h cannot be x_1 or x_k. Let $R \in \Psi(D) \cap \Psi(P)$. Then R does not contain h. Each of the two detours in $\Psi(D)$ not containing h has one endpoint which is an endpoint of h. Thus one endpoint v of R is an endpoint of h. Since the endpoints of the four detours in $\Psi(P)$ are $x_1, x_k, x_{i-1}, x_{j+1}$, we deduce that $v \in \{x_1, x_k, x_{i-1}, x_{j+1}\}$ but $v \notin \{x_1, x_k\}$. Hence $v \in \{x_{i-1}, x_{j+1}\}$.

Subcase 2.1. G has a detour which contains at least two inner chords of P.

Let h and e be two inner chords of P that appear on one common detour. Consider the subgraph $G' = P \cup x_1x_i \cup x_kx_j \cup h \cup e$. The path $T = P[x_{i-1}, x_1] \cup x_1x_i \cup P[x_i, x_j] \cup x_jx_k \cup P[x_k, x_{j+1}]$ is a detour of G' with endpoints x_{i-1} and x_{j+1}, which is also a detour of G. Since we have assumed that G is non-hamiltonian, x_{i-1} and x_{j+1} are non-adjacent. By Claim 2, each of h and e has exactly one endpoint in the set $\{x_{i-1}, x_{j+1}\}$. Now in G', the detour T has four boundary chords $x_{i-1}x_i$, $x_{j+1}x_j$, h and e. By Subcase 1.3 above (replacing P there by T), we obtain $f(G) \geq f(G') \geq 8$.

Subcase 2.2. Every non-basic detour contains exactly one inner chord of P.

Denote by Ω the set of the inner chords of P that appear on at least one detour. By the above Claim 1, if one inner chord of P appears on a detour, then there are at least two detours containing that chord. Thus, if $|\Omega| \geq 2$ then we have at least four non-basic detours, and consequently we have $f(G) \geq 8$. Next suppose $|\Omega| = 1$ and let $\Omega = \{x_sx_t\}$ with $2 \leq s \leq t-2$. Recall that we have assumed $i \leq j$. Using Claim 2, we deduce that $f(G) = 6$ if (1) $t = i-1$; (2) $s = j+1$; (3) $s = i-1$ and $i+2 \leq t \leq j$; (4) $i \leq s \leq j-2$ and $t = j+1$. In all other cases $f(G) \geq 8$. This completes the proof that if $f(G)$ is an odd number, then $f(G) \geq 9$.

Next for every integer $n \geq 9$ we construct a graph H_n of order n and connectivity 1 which contains exactly nine detours. Every H_n is traceable. We depict H_9, H_{10} and H_{11} in Figure 4.

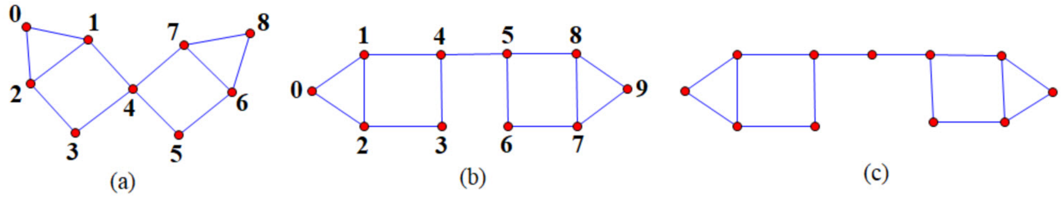

Figure 4: The graphs H_9, H_{10} and H_{11}

For $n \geq 11$, H_n is obtained from H_{10} by subdividing the edge $(4, 5)$ precisely $n - 10$ times, i.e., replacing the edge $(4, 5)$ by a $(4, 5)$-path of order $n - 8$. Note that the vertex 4 is a cut-vertex of H_9. The nine detours in H_9 are

$(0, 1, 2, 3, 4, 5, 6, 7, 8)$, $(0, 1, 2, 3, 4, 5, 6, 8, 7)$, $(0, 1, 2, 3, 4, 7, 8, 6, 5)$,
$(1, 0, 2, 3, 4, 5, 6, 7, 8)$, $(1, 0, 2, 3, 4, 5, 6, 8, 7)$, $(1, 0, 2, 3, 4, 7, 8, 6, 5)$,
$(3, 2, 0, 1, 4, 5, 6, 7, 8)$, $(3, 2, 0, 1, 4, 5, 6, 8, 7)$, $(3, 2, 0, 1, 4, 7, 8, 6, 5)$.

Finally, for every integer $n \geq 9$ we construct a graph M_n of order n and connectivity 2 which contains exactly nine detours. Every M_n is traceable. We depict M_9 and M_{10} in Figure 5.

For $n \geq 10$, M_n is obtained from M_9 by subdividing the edge $(7, 8)$ $n - 9$ times. Observe that M_9 is obtained from H_9 in Figure 4(a) by adding the edge $(2, 6)$, and any detour of M_9 cannot contain the edge $(2, 6)$. Hence M_9 and H_9 have the same set of detours, in particular, the same number of detours, i.e., nine. Note that each detour of M_9 contains the edge $(7, 8)$. Thus for every $n \geq 10$, M_n has the same number of detours as M_9. □

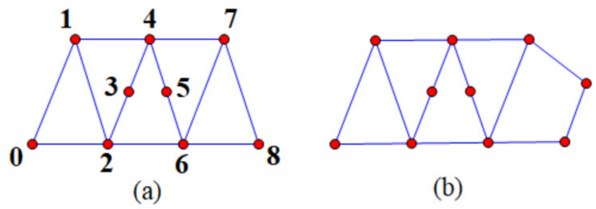

Figure 5: The graphs M_9 and M_{10}

Remark 2. The condition "of order at least 9" in Theorem 3 cannot be dropped. A computer search shows that the minimum odd number of detours in a connected graph of order 8 and minimum degree at least 2 is 11, and the minimum odd number of detours in a connected graph of order 7 and minimum degree at least 2 is 7.

Next we show that the numbers six and eight can be attained.

Theorem 4. *For every integer $n \geq 4$, there exists a 2-connected graph of order n with exactly six detours, and for every integer $n \geq 6$, there exists a 2-connected graph of order n with exactly eight detours.*

Proof. For every integer $n \geq 4$, we construct a 2-connected graph D_n of order n with exactly six detours. We depict D_4 and D_5 in Figures 6(a) and (b), respectively.

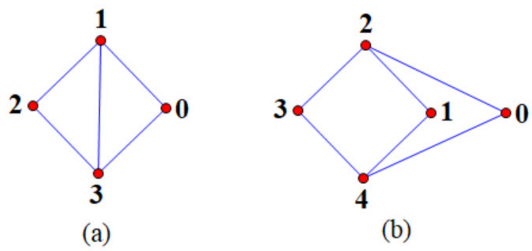

Figure 6: The graphs D_4 and D_5

D_4 has exactly the six detours:

$$(0,3,2,1),\ (0,1,3,2),\ (0,3,1,2),\ (0,1,2,3),\ (1,0,3,2),\ (2,1,0,3).$$

D_5 has exactly the six detours:

$$(0,2,3,4,1),\ (0,4,3,2,1),\ (0,2,1,4,3),\ (0,4,1,2,3),\ (1,2,0,4,3),\ (1,4,0,2,3).$$

For $n \geq 6$, D_n is obtained from D_5 by replacing the edge $(3,4)$ by a path $(3,4,\ldots,n-1)$; i.e., subdividing the edge $(3,4)$ precisely $n-5$ times. Clearly D_n has the same number of detours as D_5.

For every integer $n \geq 6$, we construct a 2-connected graph F_n of order n with exactly eight detours. We depict F_6, F_7 and F_8 in Figures 7(a), (b), and (c), respectively.

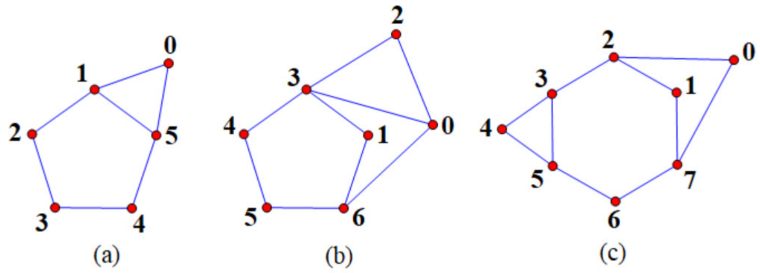

Figure 7: The graphs F_6, F_7 and F_8

Note that F_6, F_7 and F_8 are all traceable. Thus every detour is a Hamilton path. F_6 has exactly the eight detours:
$(0,5,4,3,2,1)$, $(0,1,5,4,3,2)$, $(0,5,1,2,3,4)$, $(0,1,2,3,4,5)$,
$(1,0,5,4,3,2)$, $(2,1,0,5,4,3)$, $(3,2,1,0,5,4)$, $(4,3,2,1,0,5)$.
F_7 has exactly the eight detours:
$(0,2,3,4,5,6,1)$, $(0,2,3,1,6,5,4)$, $(1,3,4,5,6,0,2)$, $(1,6,5,4,3,0,2)$,
$(1,3,2,0,6,5,4)$, $(1,6,0,2,3,4,5)$, $(2,0,3,1,6,5,4)$, $(2,0,6,1,3,4,5)$.
F_8 has exactly the eight detours:
$(0,2,3,4,5,6,7,1)$, $(0,7,6,5,4,3,2,1)$, $(0,2,1,7,6,5,4,3)$, $(0,2,1,7,6,5,3,4)$,
$(0,7,1,2,3,4,5,6)$, $(1,2,0,7,6,5,4,3)$, $(1,2,0,7,6,5,3,4)$, $(1,7,0,2,3,4,5,6)$.
For $n \geq 9$, F_n is obtained from F_8 by replacing the edge $(6,7)$ by a path $(6,7,\ldots,n-1)$; i.e., subdividing the edge $(6,7)$ exactly $n-8$ times. F_n has the same number of detours as F_8. This completes the proof. □

3 Unsolved problems

Finally we pose two problems. Recall that $f(G)$ denotes the number of detours in a graph G.

Problem 1. Let k and n be integers with $3 \leq k \leq n-2$. Denote by $\Gamma(k,n)$ the set of connected graphs with minimum degree k and order n. Define

$$a(k,n) = \min\{f(G) |\, G \in \Gamma(k,n)\}.$$

Determine $a(k,n)$.

Problem 2. Let k, n and $\Gamma(k,n)$ be as in Problem 1. Define

$$b(k,n) = \min\{f(G) |\, G \in \Gamma(k,n) \text{ and } f(G) \text{ is odd}\}.$$

Determine $b(k,n)$.

Perhaps for sufficiently large orders n, $a(k,n)$ and $b(k,n)$ are independent of n. We may also ask the two corresponding problems by replacing “with minimum degree k” in Problems 1 and 2 above by “with connectivity k”.

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