



A Conjecture Generalizing Thomassen's Chord Conjecture in Graph Theory

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Abstract

Thomassen's chord conjecture from 1976 states that every longest cycle in a 3-connected graph has a chord. This is one of the most important unsolved problems in graph theory. We pose a new conjecture which implies Thomassen's conjecture. It involves bound vertices in a longest path between two vertices in a k -connected graph. We also give supporting evidence and analyze a special case. The purpose of making this new conjecture is to explore the surroundings of Thomassen's conjecture.

Keywords Longest cycle · Longest path · Chord in a cycle · k -connected graph

Mathematics Subject Classification 05C38 · 05C40 · 05C35

We consider finite simple graphs and use standard terminology and notations as in the book [9]. Thomassen's famous chord conjecture from 1976 is as follows ([1, Conjecture 8.1] and [6, Conjecture 6]).

Conjecture 1 (Thomassen, 1976) *Every longest cycle in a 3-connected graph has a chord.*

Conjecture 1 has been proved to be true by Thomassen himself for cubic graphs [7, 8]. This beautiful and important conjecture was selected as No. 65 of the unsolved problems in the textbook [4], as Conjecture 5 in the survey article [2] collecting problems posed by Thomassen, and in Section 13 of the survey article [3] entitled "Beautiful conjectures in graph theory". It challenges our understanding of the structure of a graph. Since a 3-connected graph has minimum degree at least 3, Conjecture 1 is implied by the following conjecture [5, p.6].

Conjecture 2 (Harvey, 2017) *Every longest cycle in a 2-connected graph with minimum degree at least 3 has a chord.*

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We denote by $V(G)$ the vertex set of a graph G , by $E(G)$ the edge set of G , and by $N_G(v)$ the neighborhood in G of a vertex v .

Definition. Let H be a subgraph of a graph G . A vertex v of H is said to be H -bound if all the neighbors of v in G lie in H ; i.e., $N_G(v) \subseteq V(H)$.

A vertex of a path P is called an *internal vertex* if it is not an endpoint of P . For two distinct vertices x and y , an (x, y) -path is a path whose endpoints are x and y . We pose the following conjecture which implies Conjecture 2, and hence Conjecture 1.

Conjecture 3 *Let G be a k -connected graph with $k \geq 2$ and let x, y be two distinct vertices of G . If P is a longest (x, y) -path in G , then P contains $k - 1$ internal P -bound vertices.*

A computer search shows that Conjecture 3 holds for all graphs of order at most 10, for cubic graphs of order at most 18, for 4-regular graphs of order at most 14, for triangle-free graphs of order at most 12 and for C_4 -free graphs of order at most 13.

The case $k = 2$ of Conjecture 3 without the word "internal" (weaker version) has the following form:

Conjecture 4 *Let G be a 2-connected graph and let x, y be two distinct vertices of G . If P is a longest (x, y) -path in G , then P contains a P -bound vertex.*

Proof that Conjecture 4 implies Conjecture 2 Let C be a longest cycle in a 2-connected graph G with minimum degree at least 3. Choose two consecutive vertices x, y on C . Clearly x and y partition C into two paths, the longer of which we denote by $P = C[x, y]$. Then P is a longest (x, y) -path in G . By Conjecture 4, P contains a P -bound vertex v . Since v has degree at least 3, and all neighbors of v lie in C , it follows that C has a chord that is incident to v . \square

Note that in the above proof we do not require that the P -bound vertex is an internal vertex.

The following result shows that the conclusion in Conjecture 3 holds for longest paths in the whole graph. We can even relax the connectivity condition to minimum degree. This observation is due to Guantao Chen (Private communication in July 2023).

Theorem 5 *If Q is a longest path in a graph of minimum degree d with $d \geq 2$, then Q contains $d - 1$ internal Q -bound vertices.*

Proof Let $Q = u_1, u_2, \dots, u_k$. Since Q is a longest path, u_1 is Q -bound. Since the minimum degree of the graph is d , u_1 has d distinct neighbors u_2, w_1, \dots, w_{d-1} where $w_j = u_{i_j}$. Denote $f_j = u_{i_j-1}$, the predecessor of w_j on Q . Clearly the $d - 1$ vertices f_1, f_2, \dots, f_{d-1} are internal vertices of Q . For every j with $1 \leq j \leq d - 1$, consider the path $Q_j = Q[f_j, u_1] \cup u_1 w_j \cup Q[w_j, u_k]$. Then Q_j is an (f_j, u_k) -path with the same vertex set $V(Q)$ as Q . Thus every Q_j is a longest path in the graph. Consequently f_j is Q_j -bound. Since $V(Q_j) = V(Q)$, f_j is Q -bound. We have shown that f_1, f_2, \dots, f_{d-1} are $d - 1$ internal Q -bound vertices. \square

Remark. Since the endpoints of a longest path P in a graph are P -bound, Theorem 5 has the following corollary: If P is a longest path in a graph of minimum degree d with $d \geq 1$, then P contains $d + 1$ P -bound vertices.

An *independent set* in a graph is a set of vertices no two of which are adjacent. For a set S of vertices in a graph G , the *subgraph induced by S* , denoted $G[S]$, is the subgraph of G whose vertex set is S and whose edge set consists of all those edges of G which have both endpoints in S . The following conjecture is equivalent to Conjecture 4.

Conjecture 6 (The ST conjecture) *Suppose that the vertex set of a graph G consists of two disjoint sets S and T such that (1) $G[S]$ is an (x, y) -path P and T is an independent set; (2) every vertex in S has at least one neighbor in T ; (3) every vertex in T has at least two neighbors in S . Then P is not a longest (x, y) -path in G .*

Proof that Conjecture 6 is equivalent to Conjecture 4 Suppose that Conjecture 4 holds. Let a graph G and an (x, y) -path P satisfy the three conditions in Conjecture 6. Then G is connected.

Case 1. G is 2-connected. Since P has no P -bound vertex, by Conjecture 4 we deduce that P is not a longest (x, y) -path in G .

Case 2. G has connectivity 1. Clearly any cut-vertex of G is an internal vertex of P . On P from x to y , let z be the first cut-vertex of G . Denote $P_1 = P[x, z]$, the subpath of P with endpoints x and z , and let $S_1 = V(P_1)$. Let T_1 be the subset of T consisting of the vertices in T that have a neighbor in $S_1 \setminus \{z\}$. Observe the following facts: (1) T_1 is nonempty, since $N_G(x) \cap T \subseteq T_1$; (2) any vertex in T_1 has no neighbor in $S \setminus S_1$; (3) z has a neighbor in T_1 , since otherwise z would not be the first cut-vertex. Now the subgraph $G_1 = G[S_1 \cup T_1]$ is 2-connected and satisfies the three conditions in Conjecture 6 with S and T replaced by S_1 and T_1 , respectively. Since the (x, z) -path P_1 has no P_1 -bound vertex, it is not a longest (x, z) -path in G_1 by Conjecture 4. Hence, there exists an (x, z) -path Q in G_1 longer than P_1 . Thus the (x, y) -path $Q \cup P[z, y]$ is longer than P , implying that P is not a longest (x, y) -path in G . This proves Conjecture 6.

Conversely, suppose Conjecture 6 holds. To prove Conjecture 4, let G be a 2-connected graph, let x, y be two distinct vertices of G , and let P be a longest (x, y) -path in G . We assert that P contains a P -bound vertex. To the contrary, assume that P contains no P -bound vertex. Denote $S = V(P)$ and let T be the set of components of the graph $G - V(P)$. We define a new graph H for which $V(H) = S \cup T$ and $E(H) = E(P) \cup A$ where

$$A = \{uC \mid u \in S, C \in T \text{ and } u \text{ has a neighbor in } C\}.$$

Then $H[S] = P$ and T is an independent set of H . The assumption that P contains no P -bound vertex implies that every vertex in S has a neighbor in T . Since G is 2-connected, every vertex in T has at least two neighbors in S . By Conjecture 6, P is not a longest (x, y) -path in H . Let W be an (x, y) -path in H that is longer than P . Necessarily W contains at least one vertex in T . Let $V(W) \cap T = \{C_1, \dots, C_k\}$, and let the two neighbors of C_i on W be r_i and $f_i, i = 1, \dots, k$. Then $r_i, f_i \in V(G)$, since T is an independent set in H . Viewing C_i as a connected subgraph of G , we see that there is an (r_i, f_i) -path L_i in G whose internal vertices lie in C_i . On W replacing $r_i C_i f_i$ by L_i for every $i = 1, \dots, k$, we obtain an (x, y) -path W' in G that is not shorter than W . Hence W' is longer than P , contradicting the condition that P is

a longest (x, y) -path in G . This shows that P contains a P -bound vertex, and thus Conjecture 4 is proved. \square

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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