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THE DIAMETER AND RADIUS OF RADIALLY MAXIMAL GRAPHS

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Abstract

A graph is called radially maximal if it is not complete and the addition of any new edge decreases its radius. Harary and Thomassen [‘Anticritical graphs’, *Math. Proc. Cambridge Philos. Soc.* **79**(1) (1976), 11–18] proved that the radius r and diameter d of any radially maximal graph satisfy $r \le d \le 2r - 2$. Dutton *et al.* [‘Changing and unchanging of the radius of a graph’, *Linear Algebra Appl.* **217** (1995), 67–82] rediscovered this result with a different proof and conjectured that the converse is true, that is, if r and d are positive integers satisfying $r \le d \le 2r - 2$, then there exists a radially maximal graph with radius r and diameter d. We prove this conjecture and a little more.

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1. Introduction

We consider finite simple graphs. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively. The complement of G is denoted by $\overline{G}$. The radius and diameter of G are denoted by rad(G) and diam(G), respectively.

DEFINITION 1.1. A graph G is said to be *radially maximal* if it is not complete and

$$\text{rad}(G + e) < \text{rad}(G) \quad \text{for any } e \in E(\overline{G}).$$

Thus, a radially maximal graph is a noncomplete graph in which the addition of any new edge decreases its radius. Since adding edges in a graph cannot increase its radius, every graph is a spanning subgraph of some radially maximal graph with the same radius. It is well known that the radius r and diameter d of a general graph satisfy $r \le d \le 2r$ [4, page 78]. In 1976, Harary and Thomassen [3, page 15] proved that the radius r and diameter d of any radially maximal graph satisfy

$$r \le d \le 2r - 2. \tag{1.1}$$

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In 1995, Dutton *et al.* [1, page 75] rediscovered this result with a different proof and posed the conjecture that the converse is true, that is, if r and d are positive integers satisfying (1.1), then there exists a radially maximal graph with radius r and diameter d [1, page 76]. We prove this conjecture and a little more.

We denote by $d_G(u, v)$ the distance between two vertices u and v in a graph G. The *eccentricity*, denoted by $e_G(v)$, of a vertex v in G is the distance to a vertex farthest from v. The subscript G might be omitted if the graph is clear from the context. Thus, $e(v) = \max\{d(v, u) \mid u \in V(G)\}$. If $e(v) = d(v, x)$, then the vertex x is called an *eccentric vertex* of v. By definition the radius of a graph G is the minimum eccentricity of all the vertices in $V(G)$, whereas the diameter of G is the maximum eccentricity. A vertex v is a *central vertex* of G if $e(v) = \mathrm{rad}(G)$. A graph G is said to be *self-centred* if $\mathrm{rad}(G) = \mathrm{diam}(G)$. Thus, self-centred graphs are those graphs in which every vertex is a central vertex. We denote by $N_G(v)$ the neighbourhood of a vertex v in G. The *order* of a graph is the number of its vertices. The symbol C_k denotes a cycle of order k.

2. Main results

We will need the following operation on a graph. The *extension* of a graph G at a vertex v, denoted by $G\{v\}$, is the graph with $V(G\{v\}) = V(G) \cup \{v'\}$ and $E(G\{v\}) = E(G) \cup \{vv'\} \cup \{v'x \mid vx \in E(G)\}$, where $v' \notin V(G)$. Clearly, if G is a connected graph of order at least two, then $e_{G\{v\}}(u) = e_G(u)$ for every $u \in V(G)$ and $e_{G\{v\}}(v') = e_{G\{v\}}(v) = e_G(v)$. In particular, $\mathrm{rad}(G\{v\}) = \mathrm{rad}(G)$ and $\mathrm{diam}(G\{v\}) = \mathrm{diam}(G)$.

Gliviak *et al.* proved the following result.

LEMMA 2.1 ([2, Lemma 5]). *Let G be a radially maximal graph. If $v \in V(G)$ is not an eccentric vertex of any central vertex of G, then the extension of G at v is radially maximal.*

Now we are ready to state and prove the main result.

THEOREM 2.2. *Let r, d and n be positive integers. If $r \geq 2$ and $n \geq 2r$, then there exists a self-centred radially maximal graph of radius r and order n. If $r < d \leq 2r - 2$ and $n \geq 3r - 1$, then there exists a radially maximal graph of radius r, diameter d and order n.*

PROOF. We first treat the easier case of self-centred graphs. Suppose that $r \geq 2$ and $n \geq 2r$. The even cycle C_{2r} is a self-centred radially maximal graph of radius r and order $2r$. Let v be an arbitrary but fixed vertex of C_{2r}. For $n > 2r$, by successively performing extensions at the vertex v starting from C_{2r}, we obtain a graph $G(r, n)$ of order n. The graph $G(4, 11)$ is depicted in Figure 1.

Denote $G(r, 2r) = C_{2r}$. Since $G(r, n)$ has the same diameter and radius as C_{2r}, it is self-centred with radius r. Let xy be an edge of the complement of $G(r, n)$. Denote by S the set consisting of v and the vertices outside C_{2r}. Then S is a clique. If one end of xy, say x, lies in S, then $y \notin N[v]$, the closed neighbourhood of v in $G(r, n)$, and so $e(x) < r$. Otherwise $x, y \in V(C_{2r}) \setminus S$ and we have $e(x) < r$ and $e(y) < r$. In both cases, $\mathrm{rad}(G(r, n) + xy) < \mathrm{rad}(G(r, n))$. Hence, $G(r, n)$ is radially maximal.

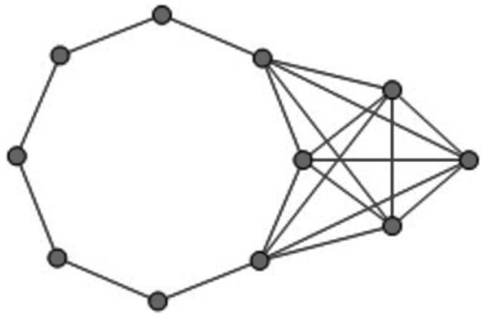

FIGURE 1. The graph $G(4, 11)$.

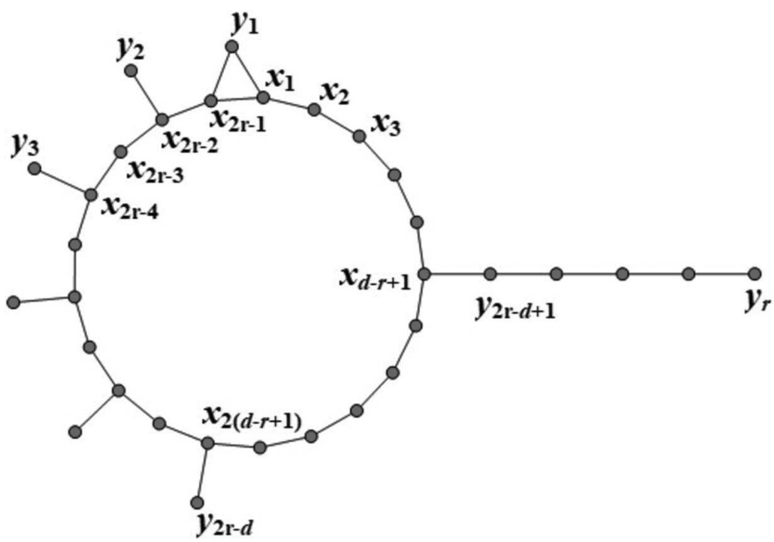

FIGURE 2. The graph $H(r, d, 3r - 1)$.

Next suppose that $r < d \leq 2r - 2$ and $n \geq 3r - 1$. We define a graph $H = H(r, d, 3r - 1)$ of order $3r - 1$ as follows: $V(H) = \{x_1, x_2, \ldots, x_{2r-1}\} \cup \{y_1, y_2, \ldots, y_r\}$ and

$$E(H) = \{x_i x_{i+1} \mid i = 1, 2, \ldots, 2r - 1\} \cup \{x_{2r-1} y_1\} \cup \{x_{2r-2j+2} y_j \mid j = 1, 2, \ldots, 2r - d\}$$
$$\cup \{x_{d-r+1} y_{2r-d+1}\} \cup \{y_t y_{t+1} \mid t = 2r - d + 1, \ldots, r - 1 \text{ if } d \geq r + 2\},$$

where $x_{2r} = x_1$. That is, H is obtained from the odd cycle C_{2r-1} by attaching edges and one path. The graph H is depicted in Figure 2 and the graphs $H(6, d, 17)$ with $d = 7, 8, 9, 10$ are depicted in Figure 3.

Clearly, H has radius r, diameter d and order $3r - 1$. To see this, note that x_{d-r+1} is a central vertex and $e_H(y_r) = d$.

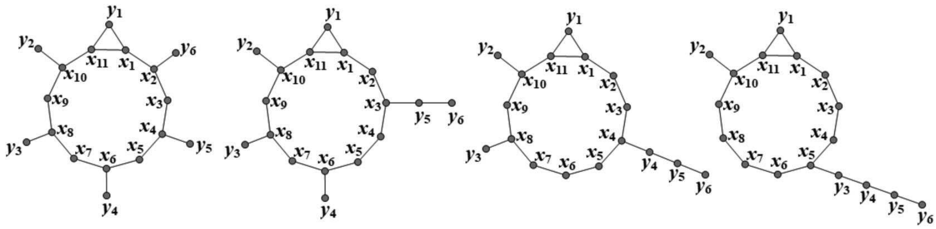

FIGURE 3. The graphs $H(6, d, 17)$ with $d = 7, 8, 9, 10$.

Now we show that H is radially maximal. Let C be the cycle of length $2r-1$, that is, $C = x_1x_2\cdots x_{2r-1}x_1$. We specify two orientations of C. Call the orientation $x_1, x_2, \ldots, x_{2r-1}, x_1$ *clockwise* and call the orientation $x_{2r-1}, x_{2r-2}, \ldots, x_1, x_{2r-1}$ *counterclockwise*. For two vertices $a, b \in V(C)$, we denote by $\overrightarrow{C}(a, b)$ the clockwise (a, b)-path on C and by $\overleftarrow{C}(a, b)$ the counterclockwise (a, b)-path on C.

For $uv \in E(\overline{H})$, denote $T = H + uv$. To show that $\mathrm{rad}(T) < r$, it suffices to find a vertex z such that $e_T(z) < r$. Denote

$$A = V(C) = \{x_1, x_2, \ldots, x_{2r-1}\} \quad \text{and} \quad B = V(H) \setminus V(C) = \{y_1, y_2, \ldots, y_r\}.$$

We distinguish three cases.

Case 1. $u, v \in A$. Let $u = x_i$ and $v = x_j$ with $i > j$.

Since $d - r + 1 \le 2r - 3$, the vertex y_2 is a leaf whose only neighbour is x_{2r-2}. Note that in H, the three vertices x_r, x_{r-1} and x_{r-2} are central vertices, y_1 is the unique eccentric vertex of x_r and y_2 is the unique eccentric vertex of x_{r-1} and x_{r-2}. If $j \ge r$ or $i \le r$, then $e_T(x_r) < r$. For, in the former case $\overrightarrow{C}(x_r, v) \cup vu \cup \overrightarrow{C}(u, x_{2r-1}) \cup x_{2r-1}y_1$ is an (x_r, y_1)-path of length less than r and, in the latter case, $\overleftarrow{C}(x_r, u) \cup uv \cup \overleftarrow{C}(v, x_1) \cup x_1y_1$ is an (x_r, y_1)-path of length less than r.

Next suppose that $i > r > j$. If $|(i - r) - (r - j)| \ge 2$, then in T there is an (x_r, y_1)-path of length less than r, which implies that $e_T(x_r) < r$. It remains to consider the case $|(i - r) - (r - j)| \le 1$. If $(i - r) - (r - j) = 0$ or 1, then in T there is an (x_{r-1}, y_2)-path of length less than r and hence $e_T(x_{r-1}) < r$. If $(r - j) - (i - r) = 1$, then in H there is an (x_{r-2}, y_2)-path of length $r - 1$ and hence $e_T(x_{r-2}) < r$.

Case 2. $u, v \in B$. Let $u = y_i$ and $v = y_j$ with $1 \le i < j \le r$.

Subcase 2.1. $i = 1$ and $j \le 2r - d$. In what follows, the subscript arithmetic for x_k is taken modulo $2r - 1$. Vertex x_{r-2j+2} is a central vertex of H whose unique eccentric vertex is y_j. To see this, note that if $r - 2j + 2 \le d - r + 1$, then

$$d_H(x_{r-2j+2}, y_r) \le d - r + 1 - (r - 2j + 2) + r - (2r - d) = 2d - 3r + 2j - 1 \le r - 1$$

since $j \le 2r - d$ and, if $r - 2j + 2 > d - r + 1$, then

$$d_H(x_{r-2j+2}, y_r) \le r - 2j + 2 - (d - r + 1) + r - (2r - d) = r - 2j + 1 \le r - 3$$

since $j \ge 2$.

If $r-2j+2 \geq 1$, in T there is the (x_{r-2j+2}, y_j)-path $\overleftarrow{C}(x_{r-2j+2}, x_1) \cup x_1y_1 \cup y_1y_j$. Hence, $d_T(x_{r-2j+2}, y_j) \leq r-2j+2-1+2 = r-2j+3 \leq r-1$ since $j \geq 2$, implying that $e_T(x_{r-2j+2}) < r$. If $r-2j+2 \leq 0$, there is the path $\overrightarrow{C}(x_{r-2j+2}, x_{2r-1}) \cup x_{2r-1}y_1 \cup y_1y_2$ in T. Hence, $d_T(x_{r-2j+2}, y_j) \leq 0-(r-2j+2)+2 = 2j-r \leq r-2$ since $j \leq 2r-d$ and $d \geq r+1$, implying that $e_T(x_{r-2j+2}) < r$.

Subcase 2.2. $i = 1$ and $2r-d+1 \leq j \leq r$. First suppose that $j = r$. Observe that $x_{2d-3r+1}$ is a central vertex of H whose unique eccentric vertex is y_r. Also, the condition $d \leq 2r-2$ implies that $2d-3r+1 < d-r+1$. On the other hand, if $2d-3r+1 \geq 1$, then $d_T(x_{2d-3r+1}, y_r) \leq 2d-3r+1-1+2 \leq r-2$. If $2d-3r+1 \leq 0$, then $d_T(x_{2d-3r+1}, y_r) \leq 0-(2d-3r+1)+2 \leq r-1$, where we have used the fact that $d \geq r+1$. Hence, $e_T(x_{2d-3r+1}) < r$.

Next suppose that $2r-d+1 \leq j \leq r-1$. Observe that x_r is a central vertex of H whose unique eccentric vertex is y_1. Note also that $r > d-r+1$. Now in T there is the (x_r, y_1)-path $\overleftarrow{C}(x_r, x_{d-r+1}) \cup x_{d-r+1}y_{2r-d+1} \ldots y_j \cup y_jy_1$. Hence,

$$d_T(x_r, y_1) \leq r-(d-r+1)+j-(2r-d)+1 = j \leq r-1,$$

implying that $e_T(x_r) < r$.

Subcase 2.3. $i \geq 2$ and $j \leq 2r-d$. First suppose $2(j-i) \leq r-1$. Then $2r-2j+2 \geq r-2i+3$. Clearly, $x_{2r-2j+2}$ is the unique neighbour of y_j in H. By considering the two possible cases $r-2i+3 \leq d-r+1$ and $r-2i+3 > d-r+1$, it is easy to verify that x_{r-2i+3} is a central vertex of H whose unique eccentric vertex is y_i. In T there is the (x_{r-2i+3}, y_i)-path $\overrightarrow{C}(x_{r-2i+3}, x_{2r-2j+2}) \cup x_{2r-2j+2}y_j \cup y_jy_i$. It follows that $d_T(x_{r-2i+3}, y_i) \leq 2r-2j+2-(r-2i+3)+1+1 = r-2(j-i)+1 \leq r-1$, implying that $e_T(x_{r-2i+3}) < r$.

Next suppose that $2(j-i) \geq r$. Then $r-2i+2 \geq 2r-2j+2$. Observe that x_{r-2i+2} is a central vertex of H whose unique eccentric vertex is y_i. Also, $j-i \leq 2r-d-2$. As before,

$$\begin{aligned} d_T(x_{r-2i+2}, y_i) &\leq r-2i+2-(2r-2j+2)+1+1 \\ &= 2-r+2(j-i) \leq 2-r+2(2r-d-2) \leq r-2, \end{aligned}$$

implying that $e_T(x_{r-2i+2}) < r$.

Subcase 2.4. $2 \leq i \leq 2r-d$ and $2r-d+1 \leq j \leq r$. First suppose that $2r+2 \leq 2i+d$. Then $d-r+1 \geq r-2i+3$. Note that x_{r-2i+3} is a central vertex of H whose unique eccentric vertex is y_i. Thus, $\overrightarrow{C}(x_{r-2i+3}, x_{d-r+1}) \cup x_{d-r+1}y_{2r-d+1} \ldots y_j \cup y_jy_i$. is an (x_{r-2i+3}, y_i)-path in T and

$$\begin{aligned} d_T(x_{r-2i+3}, y_i) &\leq d-r+1-(r-2i+3)+j-(2r-d)+1 \\ &\leq d-r+1-(r-2i+3)+r-(2r-d)+1 \\ &= 2d-3r+2i-1 \leq r-1, \end{aligned}$$

implying that $e_T(x_{r-2i+3}) < r$.

Next suppose that $2r + 2 \geq 2i + d + 1$. Then $r - 2i + 2 \geq d - r + 1$. Observe that x_{r-2i+2} is a central vertex of H whose unique eccentric vertex is y_i. As before,

$$\begin{aligned} d_T(x_{r-2i+2}, y_i) &\leq r - 2i + 2 - (d - r + 1) + j - (2r - d) + 1 \\ &\leq r - 2i + 2 - (d - r + 1) + r - (2r - d) + 1 \\ &= r - 2i + 2 \leq r - 2, \end{aligned}$$

implying that $e_T(x_{r-2i+2}) < r$.

Subcase 2.5. $2r - d + 1 \leq i < j \leq r$. Observe that x_{r+1} is a central vertex of H whose unique eccentric vertex is y_r. Clearly, $e_T(x_{r+1}) < r$.

Case 3. $u \in A$ and $v \in B$. Let $u = x_i$ and $v = y_j$.

Observe that x_r is a central vertex of H whose unique eccentric vertex is y_1. If $j = 1$, then $e_T(x_r) < r$. Now suppose that $2 \leq j \leq 2r - d$. Then both x_{r-2j+2} and x_{r-2j+3} are central vertices of H whose unique eccentric vertex is y_j. If u lies on the path $\overrightarrow{C}(x_{2r-2j+2}, x_{r-2j+2})$, then $e_T(x_{r-2j+2}) < r$; if u lies on the path $\overleftarrow{C}(x_{2r-2j+2}, x_{r-2j+3})$, then $e_T(x_{r-2j+3}) < r$.

Finally, suppose that $2r - d + 1 \leq j \leq r$. We have $2d - 3r + 1 < d - r + 1 < r + 1$. Observe that both x_{r+1} and $x_{2d-3r+1}$ are central vertices of H whose unique eccentric vertex is y_r. If $2d - 3r + 1 \leq i \leq d - r + 1$, then $d_T(x_{2d-3r+1}, y_r) \leq r - 1$ and hence $e_T(x_{2d-3r+1}) < r$. Similarly, if $d - r + 2 \leq i \leq r + 1$, then $e_T(x_{r+1}) < r$.

It remains to consider the case when $u = x_i$ lies on the path $\overrightarrow{C}(x_{r+2}, x_{2d-3r})$. We assert that $e_T(u) < r$. First note that if $w \in \{y_{2r-d+1}, y_{2r-d+2}, \ldots, y_r\}$, then $d_T(x_i, w) \leq d - r \leq r - 2$. Also, if $w \in V(C)$, we have $d_T(x_i, w) \leq r - 1$ since $\mathrm{diam}(C) = r - 1$. Next suppose that $w = y_s$ with $1 \leq s \leq 2r - d$. Let x_k and x_{k+1} be the two vertices on C with $d_C(x_i, x_k) = d_C(x_i, x_{k+1}) = r - 1$. Since x_i lies on the path $\overrightarrow{C}(x_{r+2}, x_{2d-3r})$, we have $k \geq 2$ and $k + 1 \leq 2d - 2r < 2(d - r + 1)$. It follows that $d_H(x_i, w) \leq r - 1$ since $N_H(y_1) = \{x_{2r-1}, x_1\}$ and $N_H(y_{2r-d}) = \{x_{2(d-r+1)}\}$. This completes the proof that H is radially maximal.

Note that by the two inequalities in (1.1), any non-self-centred radially maximal graph has radius at least three. Obviously, the vertex x_{2r-2} is not an eccentric vertex of any vertex in H. Hence, by Lemma 2.1, the extension of H at x_{2r-2}, denoted H_{3r}, is radially maximal. Also, H_{3r} has the same diameter and radius as H and has order $3r$. Again, the vertex x_{2r-2} is not an eccentric vertex of any vertex in H_{3r}. For any $n > 3r - 1$, performing extensions at the vertex x_{2r-2} successively, starting from H, we can obtain a radially maximal graph of radius r, diameter d and order n. This completes the proof. □

Combining restriction (1.1) on the diameter and radius of a radially maximal graph and Theorem 2.2, we obtain the following corollary.

COROLLARY 2.3. *There exists a radially maximal graph of radius r and diameter d if and only if $r \leq d \leq 2r - 2$.*

3. Final remarks

Since any graph with radius r has order at least $2r$, Theorem 2.2 covers all the possible orders of self-centred radially maximal graphs.

Gliviak *et al.* [2, page 283] conjectured that the minimum order of a non-self-centred radially maximal graph of radius r is $3r - 1$. This conjecture is known to be true for the first three values of r; that is, $r = 3, 4, 5$ [2, page 283], but it is still open in general. If this conjecture is true, then Theorem 2.2 covers all the possible orders of radially maximal graphs with a given radius.

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