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ORIGINAL PAPER

Graphs with Many Independent Vertex Cuts

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Abstract

Cycles are the only 2-connected graphs in which any two nonadjacent vertices form a vertex cut. We generalize this fact by proving that for every integer $k \geq 3$ there exists a unique graph G satisfying the following three conditions: (1) G is k-connected; (2) the independence number of G is greater than k; (3) any independent set of cardinality k is a vertex cut of G. However, the edge version of this result does not hold. We also consider the problem when replacing independent sets by the periphery.

Keywords Vertex cut · Connectivity · Independent set

Mathematics Subject Classification 05C40 · 05C69

We consider finite simple graphs. For terminology and notations we follow the books [2, 5]. It is known [4, p. 46] that cycles are the only 2-connected graphs in which any two nonadjacent vertices form a vertex cut. We will generalize this fact and consider two related problems.

We denote by $V(G)$ the vertex set of a graph G. The order of G, denoted by $|G|$, is the number of vertices of G. For $S \subseteq V(G)$, the notation $G[S]$ denotes the subgraph of G induced by S. Let $K_{s,t}$ denote the complete bipartite graph whose partite sets have cardinality s and t, respectively.

Notation. The notation $K_{s,s} - PM$ denotes the graph obtained from the balanced complete bipartite graph $K_{s,s}$ by deleting all the edges in a perfect matching of $K_{s,s}$.

Note that $K_{s,s} - PM$ is an $(s-1)$-connected $(s-1)$-regular graph, $K_{3,3} - PM$ is the 6-cycle C_6 and $K_{4,4} - PM$ is the cube Q_3.

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Theorem 1 *Let $k \geq 3$ be an integer. Then $K_{k+1,k+1} - PM$ is the unique graph G satisfying the following three conditions: (1) G is k-connected; (2) the independence number of G is greater than k; (3) any independent set of cardinality k is a vertex cut of G.*

Proof It is easy to verify that the graph $K_{k+1,k+1} - PM$ indeed satisfies the three conditions in Theorem 1.

Conversely, let G be a graph satisfying the three conditions in Theorem 1. We first assert that G has order at least $2k+2$. Let S be an independent set of G with cardinality $k+1$. Since G is k-connected, every vertex has degree at least k. Let T be the neighborhood of one vertex in S. Then $|T| \geq k$. Thus $|G| \geq |S| + |T| \geq 2k+1$. If $|G| = 2k+1$, then T would be the common neighborhood of all the vertices in S. But now any k vertices in S do not form a vertex cut, contradicting condition (3). This shows that $|G| \geq 2k+2$.

Choose an arbitrary but fixed independent set $A = \{x_1, x_2, \ldots, x_{k+1}\}$ of cardinality $k+1$ in G. By condition (3), for every i with $1 \leq i \leq k+1$, the graph $H_i \triangleq G - (A \setminus \{x_i\})$ is disconnected. Let G_i denote the union of all the components of H_i except the component containing x_i. Note that each G_i is disjoint from the set A.

Let Q and W be subgraphs of G or subsets of $V(G)$. We say that Q and W are *adjacent* if there exists an edge with one endpoint in Q and the other endpoint in W; otherwise Q and W are *nonadjacent*. Next we prove three claims.

Claim 1. $V(G_i) \cap V(G_j) = \phi$, G_i and G_j are nonadjacent for $1 \leq i < j \leq k+1$.

In the sequel, for notational simplicity, a vertex v may also mean the set $\{v\}$. We will use the fact that if T is a minimum vertex cut of G, then every vertex in T has a neighbor in every component of $G - T$. Clearly, G has connectivity k. Since $A \setminus x_j$ is a minimum vertex cut of G, the subgraph $G[x_i \cup V(G_j)]$ is connected and it is contained in the component of H_i containing x_i. By the definition of G_i, we deduce that $(x_i \cup V(G_j)) \cap V(G_i) = \phi$, implying $V(G_i) \cap V(G_j) = \phi$.

To show the second conclusion, just note that any vertex in G_i and any vertex in G_j lie in different components of the graph $G - (A \setminus x_i)$.

Claim 2. $A \cup (\bigcup_{i=1}^{k+1} V(G_i)) = V(G)$.

To the contrary, suppose that $F \triangleq V(G) \setminus \{A \cup (\bigcup_{i=1}^{k+1} V(G_i))\}$ is not empty. Let $F_1, F_2, \ldots, F_s$ be the components of $G[F]$.

Recall that by definition, for $1 \leq i \leq k+1$, G_i denotes the union of all the components of $G - (A \setminus x_i)$ except the component R_i that contains x_i. Hence, for every p with $1 \leq p \leq s$, F_p is a subgraph of R_i, implying that G_i is nonadjacent to F_p. Note that

$$R_i = G\left[x_i \cup F \cup \left(\bigcup_{j \neq i} V(G_j)\right)\right].$$

Since R_i is connected, x_i is adjacent to every component of G_j with $j \neq i$ and x_i is adjacent to each F_p for $1 \leq p \leq s$. Thus, every F_p is adjacent to every vertex in A.

We choose one vertex y_i from G_i for each $1 \le i \le k$. Then $B \triangleq \{y_1, y_2, \dots, y_k\}$ is an independent set of G. We assert that every component of $(\bigcup_{i=1}^{k+1} G_i) - B$ is adjacent to A, since otherwise G would have a cut-vertex. It follows that $G - B$ is connected, contradicting condition (3). This shows that F is empty and claim 2 is proved.

Claim 3. $|G_i| = 1$ for every $1 \le i \le k+1$.

To the contrary, we suppose that some G_i has order at least 2. Without loss of generality, suppose $|G_k| \ge 2$. Let z_j be a neighbor of x_{k+1} in G_j for $j = 1, \dots, k-1$. Since x_{k+1} is adjacent to G_k, x_{k+1} has a neighbor $w \in G_k$. The condition $|G_k| \ge 2$ ensures that G_k has a vertex z_k distinct from w. Denote $C = \{z_1, z_2, \dots, z_k\}$. Then C is an independent set. We assert that every component of $(G_1 \cup G_2 \cup \cdots \cup G_k) - C$ is adjacent to $A \backslash x_{k+1}$, since otherwise some z_j and x_{k+1} would form a vertex cut of G, contradicting the condition that G is k-connected and $k \ge 3$. Also, every component of G_{k+1} is adjacent to every vertex in $A \setminus x_{k+1}$. It follows that the graph $G - (C \cup x_{k+1})$ is connected. But x_{k+1} is adjacent to w, a vertex in $G_k - z_k$. Hence $G - C$ is connected, contradicting condition (3). This shows that each G_i consists of one vertex.

Combining the information in the above three claims, we deduce that $|G| = 2k+2$ and the neighborhood of x_i is $\{G_1, G_2, \dots, G_{k+1}\} \setminus \{G_i\}$ for $1 \le i \le k+1$. It follows that $G = K_{k+1,k+1} - PM$. This completes the proof. □

Feng Liu [3] asked whether the edge version of Theorem 1 holds. The following result shows that the answer is negative.

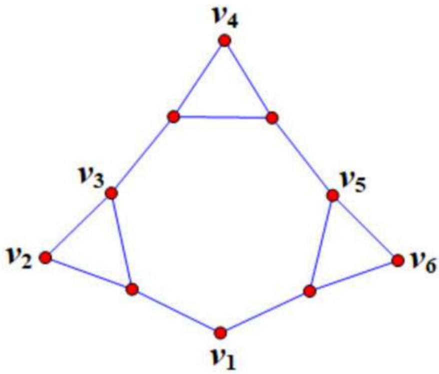
Corollary 2 *Let $k \ge 3$ be an integer. If a graph G is k-edge-connected with matching number greater than k, then G contains a matching M of cardinality k such that $G - M$ is connected.*

Proof To the contrary, suppose that for any matching M of cardinality k, $G - M$ is disconnected. Consider the line graph of G, denoted by $H \triangleq L(G)$. Since G is k-edge-connected, we deduce that [5, p. 283] H is k-connected. An independent set of vertices in H corresponds to a matching in G. Applying Theorem 1 to H we have $H = K_{k+1,k+1} - PM$, where we use the equality sign for graphs to mean isomorphism. It is known ([1] or [5, p. 282]) that any line graph of a simple graph cannot have the claw as an induced subgraph. However, for $k \ge 3$, $K_{k+1,k+1} - PM$ contains an induced claw (many in fact). This contradiction shows that G contains a matching M of cardinality k such that $G - M$ is connected. □

Remark As for the case $k = 2$ of Corollary 2, using the ideas in the above proof and using the fact mentioned at the beginning of this paper, we see that cycles are the only 2-edge-connected graphs in which any two nonadjacent edges form a separating set.

Finally, we consider replacing independent vertices in Theorem 1 by peripheral vertices. The *eccentricity* of a vertex v in a graph G, denoted by $e(v)$, is the distance to a vertex farthest from v. A vertex v is a *peripheral vertex* of G if $e(v)$ is equal to the diameter of G. The *periphery* of G is the set of all peripheral vertices. We pose the following conjecture.

Fig. 1 The graph F



Conjecture 3 *Let $k \geq 2$ be an integer. If G is a k-connected graph whose periphery has cardinality at least k, then G contains a set S of k peripheral vertices such that $G - S$ is connected.*

Observation 4 *The case $k = 2$ of Conjecture 3 is true.*

Proof To the contrary, suppose that any two peripheral vertices form a vertex cut of G. Denote by $d(u, v)$ the distance between two vertices $u,\ v$ and let the diameter of G be d. We have $d \geq 2$. Choose vertices $x,\ y$ such that $d(x, y) = d$. Let P be a shortest (x, y)-path, and let y' be the neighbor of y on P. Let H be a component of $G - \{x, y\}$ that does not contain the path $P - \{x, y\}$.

Since G is 2-connected, both x and y have a neighbor in H. Let x' be a neighbor of x in H. Then $d(x', y) \geq d - 1$. Since every (x', y')-path contains either x or y, we deduce that $d(x', y') = d$. Thus x' is also a peripheral vertex. By our assumption, $G - \{x, x'\}$ is disconnected. Let R be the component of $G - \{x, x'\}$ containing y. Clearly every component of $G - \{x, x'\}$ other than R is contained in H. Let Q be an arbitrary such component. We assert that every vertex in Q is adjacent to x'. Let $z \in V(Q)$. Any (z, y)-path must contain either x or x'. Since $d(x, y) = d$, a shortest (z, y)-path must contain x', which implies that z and x' are adjacent and z is a peripheral vertex, since $d(x', y) \geq d - 1$. Choose a vertex z_0 from any component of $G - \{x, x'\}$ other than R. Note that x' is adjacent to R, since $\{x, x'\}$ is a minimum vertex cut of G. Thus, the graph $G - \{x, z_0\}$ is connected, contradicting our assumption. □

The graph F in Fig. 1 shows that the connectivity condition in Conjecture 3 cannot be dropped. F has diameter 4 and periphery $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. With $k = 5$, any 5 peripheral vertices of F form a vertex cut.

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Data Availability No data set is used during the study.

Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose

References

1. Beineke, L.W.: Derived graphs and digraphs. In: Beiträge zur Graphentheorie, Teubner, 17-33 (1968)
2. Bondy, J.A., Murty, U.S.R.: Graph Theory, GTM 244. Springer (2008)
3. Liu, F.: Private communication (October 2022)
4. Lovász, L.: Combinatorial Problems and Exercises, 2nd edn. North-Holland Publishing Co., Amsterdam (1993)
5. West, D.B.: Introduction to Graph Theory. Prentice Hall, Inc. (1996)