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Sparse graphs with an independent or foresty minimum vertex cut

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ABSTRACT

A connected graph is called fragile if it contains an independent vertex cut. In 2002 Chen and Yu proved that every connected graph of order n and size at most $2n-4$ is fragile, and in 2013 Le and Pfender characterized the non-fragile graphs of order n and size $2n-3$. It is natural to consider minimum vertex cuts. We prove two results. (1) Every connected graph of order n with $n \geq 7$ and size at most $\lfloor 3n/2 \rfloor$ has an independent minimum vertex cut; (2) every connected graph of order n with $n \geq 7$ and size at most $2n$ has a foresty minimum vertex cut. Both results are best possible.

1. Introduction and main results

We consider finite simple graphs and use standard terminology and notation from [1] and [9]. The *order* of a graph is its number of vertices, and the *size* is its number of edges. We denote by $V(G)$ the vertex set of a graph G, and for $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S. A vertex cut of a connected graph G is a set $S \subset V(G)$ such that $G - S$ is disconnected. A vertex cut S of a connected graph G is called an *independent vertex cut* if S is an independent set, and S is called a *foresty vertex cut* if $G[S]$ is a forest. There is a recent work involving independent vertex cuts [6].

Definition 1. A connected graph is called *fragile* if it contains an independent vertex cut.

Fragile graphs have applications in some decomposition algorithms [2]. The following result was conjectured by Caro (see [4]) and proved by Chen and Yu [4] in 2002.

Theorem 1. *[4] Every connected graph of order n and size at most $2n-4$ is fragile.*

The size bound $2n-4$ is sharp, and in 2013 Le and Pfender [7] characterized the non-fragile graphs of order n and size $2n-3$ (see [8] for a related work). Also in 2002 Chen, Faudree and Jacobson [3] proved the following result.

Theorem 2. *[3] Every connected graph of order n and size at most $(12n/7)-3$ contains an independent vertex cut S with $|S| \leq 3$.*

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Recently Chernyshev, Rauch and Rautenbach [5] have initiated the study of foresty vertex cuts of graphs. A vertex cut S of a graph of connectivity k is called *minimum* if $|S| = k$. It is natural to consider minimum vertex cuts.

In this paper we prove the following two results.

Theorem 3. *Every connected graph of order n with $n \geq 7$ and size at most $\lfloor 3n/2 \rfloor$ has an independent minimum vertex cut, and the size bound $\lfloor 3n/2 \rfloor$ is best possible.*

Theorem 4. *Every connected graph of order n with $n \geq 7$ and size at most $2n$ has a foresty minimum vertex cut, and the size bound $2n$ is best possible.*

We give proofs of Theorems 3 and 4 in Section 2.

We denote by $|G|$, $e(G)$ and $\kappa(G)$ the order, size and connectivity of a graph G, respectively. The neighborhood of a vertex x is denoted by $N(x)$ or $N_G(x)$, and the closed neighborhood of x is $N[x] \triangleq N(x) \cup \{x\}$. The degree of x is denoted by $\deg(x)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and maximum degree of G, respectively. For a vertex subset $S \subseteq V(G)$, we use $N(S)$ to denote the neighborhood of S; i.e., $N(S) = \{y \in V(G) \setminus S \mid y \text{ has a neighbor in } S\}$. For $x \in V(G)$ and $S \subseteq V(G)$, $N_S(x) \triangleq N(x) \cap S$ and the degree of x in S is $\deg_S(x) \triangleq |N_S(x)|$. Given two disjoint vertex subsets S and T of G, we denote by $[S, T]$ the set of edges having one endpoint in S and the other in T. The degree of S is $\deg(S) \triangleq |[S, \overline{S}]|$, where $\overline{S} = V(G) \setminus S$. We denote by C_n, P_n and K_n the cycle of order n, the path of order n and the complete graph of order n, respectively. $\overline{G}$ denotes the complement of a graph G. For two graphs G and H, $G \vee H$ denotes the *join* of G and H, which is obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H.

For graphs we will use equality up to isomorphism, so $G = H$ means that G and H are isomorphic.

2. Proofs

We will repeatedly use the following fact.

Lemma 5. *If S is a minimum vertex cut of a connected graph G, then every vertex in S has a neighbor in every component of $G - S$.*

A 3-regular graph is called a *cubic graph*.

Lemma 6. *Every connected cubic graph of order at least eight has an independent minimum vertex cut.*

Proof. Let G be a connected cubic graph of order at least 8. Then $\kappa(G) \in \{1, 2, 3\}$. Lemma 6 holds trivially in the case $\kappa(G) = 1$. Next we consider the remaining two cases.

Case 1. $\kappa(G) = 2$.

Let $S = \{x, y\}$ be a minimum vertex cut of G. If x and y are nonadjacent, then S is what we want. Now suppose that x and y are adjacent. Let H be a component of $G - S$. We assert that for any $v \in V(H)$, $\deg_S(v) \leq 1$. Otherwise v would be a cut-vertex of G, contradicting our assumption $\kappa(G) = 2$. Since $\deg(x) = 3$ and x and y are adjacent, x has exactly one neighbor p in H. By the above assertion, $N_S(p) = \{x\}$, and consequently p has two neighbors in H. Then $\{p, y\}$ is an independent minimum vertex cut of G.

Case 2. $\kappa(G) = 3$.

Choose a vertex $v \in V(G)$ and denote $S = N(v) = \{x, y, z\}$. If S is an independent set, then it is an independent minimum vertex cut of G. Next suppose that S is not an independent set. Without loss of generality, suppose that x and y are adjacent. Since G is cubic and S is a minimum vertex cut, $\Delta(G[S]) = 1$. It follows that $G[S] = K_2 + K_1$.

Denote $T = V(G) \setminus N[v]$. We assert that for any $w \in T$, w is adjacent to at most one of x and y. Otherwise $\{w, z\}$ would be a vertex cut of G, contradicting our assumption $\kappa(G) = 3$. Let $\{p\} = N_T(x)$ and $\{q\} = N_T(y)$.

We assert that at least one of p and q is nonadjacent to z. To the contrary, suppose that both p and q are adjacent to z. Since G has order at least 8, $T \setminus \{p, q\} \neq \emptyset$. Then $\{p, q\}$ is a vertex cut, contradicting our assumption $\kappa(G) = 3$. If p is nonadjacent to z, then $\{p, y, z\}$ is an independent minimum vertex cut of G; if q is nonadjacent to z, then $\{q, x, z\}$ is an independent minimum vertex cut of G. This completes the proof. □

The graph in Fig. 1 shows that the lower bound 8 for the order in Lemma 6 is sharp.

Proof of Theorem 3. We first use induction on the order n to prove the statement that every connected graph of order n with $n \geq 7$ and size at most $\lfloor 3n/2 \rfloor$ has an independent minimum vertex cut.

The basis step. $n = 7$.

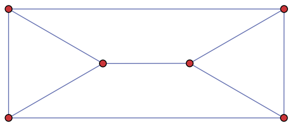

Fig. 1. A cubic graph of order 6 without independent minimum vertex cut.

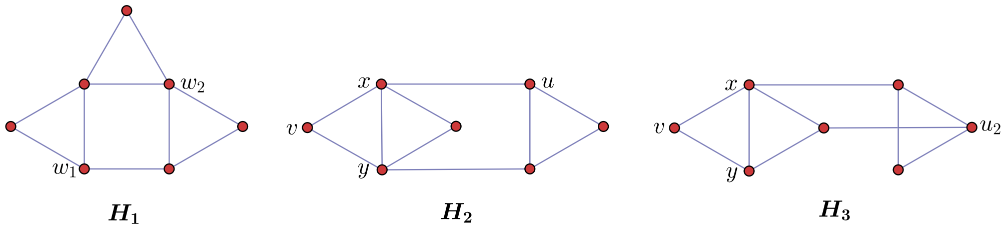

Fig. 2. H_1, H_2 and H_3.

Let F be a connected graph of order 7 and size at most $10 = \lfloor 3 \times 7/2 \rfloor$. We have $\delta(F) \le 2$, since otherwise we would have $e(F) \ge 11 > 10$, a contradiction. It follows that $\kappa(F) \le \delta(F) \le 2$. The result holds trivially if $\kappa(F) = 1$. Thus it suffices to consider the case when $\kappa(F) = \delta(F) = 2$.

Let v be a vertex of degree 2 and let $N(v) = \{x, y\}$. If x and y are nonadjacent, then $\{x, y\}$ is an independent minimum vertex cut of F. Next suppose that x and y are adjacent. Applying Lemma 5 and using the size restriction of F we deduce that $F - \{x, y\}$ has at most four components; i.e., $F - N[v]$ has at most three components. Then

$$F - N[v] \in \{2K_1 + K_2,\ 2K_2,\ K_1 + P_3,\ K_1 + C_3,\ K_1 \vee \overline{K_3},\ P_4,\ C_4,\ K_1 \vee (K_1 + K_2),\ K_4^-\}$$

where K_4^- is the graph obtained from K_4 by deleting one edge.

Let $R = V(F) \setminus N[v]$ and let H_1, H_2, H_3 be the graphs illustrated in Fig. 2.

- $F - N[v] \in \{2K_1 + K_2, 2K_2, K_1 + P_3, K_1 \vee \overline{K_3}, C_4\}$. Since $e(F) \le 10$ and $\kappa(F) = \delta(F) = 2$, there exists a vertex in R with degree 2 whose neighborhood is an independent set of F, as required.
- $F - N[v] = P_4$. If $F = H_1$ (see Fig. 2), then $\{w_1, w_2\}$ is an independent minimum vertex cut of F. Next assume that $F \ne H_1$. Then there exists a vertex in R with degree 2 whose neighborhood is an independent set of F, as required.
- $F - N[v] = K_1 + C_3$. Since $e(F) \le 10$, by Lemma 5, we have $F = H_2$. Thus $\{y, u\}$ is an independent set of F, as required.
- $F - N[v] = K_1 \vee (K_1 + K_2)$. If $F = H_1$, then $\{w_1, w_2\}$ is an independent minimum vertex cut of F. If $F = H_3$, then $\{x, u_2\}$ is an independent minimum vertex cut of F. Now assume that $F \notin \{H_1, H_3\}$. Then there exists a vertex in R with degree 2 whose neighborhood is an independent set of F, as desired.
- $F - N[v] = K_4^-$. Since $e(F) \le 10$, by Lemma 5, $|[N(v), R]| = 2$. Since $\kappa(F) = 2$, we have $N_R(x) \cap N_R(y) = \emptyset$. Then $\{x\} \cup N_R(y)$ is an independent minimum vertex cut of F, as desired.

The induction step. $n \ge 8$.

Let G be a connected graph of order $n \ge 8$ and size at most $\lfloor 3n/2 \rfloor$ and suppose that the above statement holds for all graphs of order $n-1$. It suffices to consider the case $\kappa(G) \ge 2$. Since $e(G) \le 3n/2$, we have $2 \le \kappa(G) \le \delta(G) \le 3$.

Case 1. $\delta(G) = 3$.

Since $\delta(G) = 3$ and $e(G) \le 3n/2$, we have $\Delta(G) = 3$ and hence G is cubic. The statement holds by Lemma 6.

Case 2. $\delta(G) = 2$.

In this case $\kappa(G) = 2$. Choose a vertex v of degree 2 and let $N(v) = \{x, y\}$. If x and y are nonadjacent, then $\{x, y\}$ is an independent minimum vertex cut. Next we assume that x and y are adjacent. Denote $H = G - v$. Then H is a connected graph of order $n-1$ and

$$e(H) = e(G) - 2 \le \frac{3n}{2} - 2 = \frac{3n-4}{2} < \frac{3(n-1)}{2},$$

which implies that $\delta(H) \le 2$ and hence $\kappa(H) \le 2$. On the other hand, since x and y are adjacent, the condition $\kappa(G) = 2$ implies that $\kappa(H) \ge 2$. Thus $\kappa(H) = 2$. By the induction hypothesis, H has an independent vertex cut M with $|M| = 2$. Clearly M is an independent minimum vertex cut of G.

Now for every integer $n \ge 7$ we construct a graph G_n of order n and size $\lfloor 3n/2 \rfloor + 1$ such that G_n has no independent minimum vertex cut. Hence the size bound $\lfloor 3n/2 \rfloor$ in Theorem 3 is best possible.

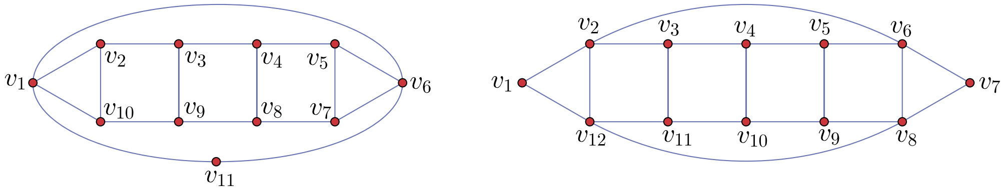

Fig. 3. G_{11} and G_{12}.

If n is odd, let $C: v_1v_2\dots v_{n-1}v_1$ be an $(n-1)$-cycle. Add a vertex v_n to C and then add edges $v_1v_{(n+1)/2}$, v_1v_n, $v_{(n+1)/2}v_n$, v_iv_{n+1-i} for $i=2,3,\dots,(n-1)/2$ to obtain G_n. If n is even, let $D: v_1v_2\dots v_nv_1$ be an n-cycle. Then in D add edges $v_2v_{n/2}$, $v_{(n+4)/2}v_n$, v_iv_{n+2-i} for $i=2,3,\dots,n/2$ to obtain G_n. We depict G_{11} and G_{12} in Fig. 3.

G_n has order n and size $\lfloor 3n/2\rfloor+1$. If n is odd, $\{v_1, v_{(n+1)/2}\}$ is the unique minimum vertex cut of G_n, which induces an edge. If n is even, G_n has exactly two minimum vertex cuts: $\{v_2, v_n\}$ and $\{v_{n/2}, v_{(n+4)/2}\}$, each of which induces an edge. Thus G_n has no independent minimum vertex cut. This completes the proof. $\square$

Now we prepare to prove Theorem 4.

Let S and T be two disjoint vertex subsets of a graph G. An (S,T)-*path* is a path P with one endpoint in S and the other in T such that $S\cup T$ contains no internal vertex of P. The following fact is well-known [9, p. 174] and it follows from Menger's theorem ([1, p. 208] or [9, p. 167]).

Lemma 7. *Let G be a k-connected graph. If S and T are two disjoint subsets of $V(G)$ with cardinality at least k, then G has k pairwise vertex disjoint (S,T)-paths.*

A k-*matching* is a matching of cardinality k.

Lemma 8. *Let S be a vertex cut of a k-connected graph G and let H be a component of $G-S$. If $|H|\ge k$, then the set $[S,\ V(H)]$ contains a k-matching.*

Proof. Since G is k-connected, $|S|\ge k$. By Lemma 7, G contains k pairwise vertex disjoint $(S, V(H))$-paths P_i, $i=1,\dots,k$. Clearly each P_i must be an edge, and hence $\{P_1, P_2,\dots,P_k\}$ is a k-matching in $[S,\ V(H)]$. $\square$

Lemma 9. *Every connected 4-regular graph of order at least seven has a foresty minimum vertex cut.*

Proof. Let G be a 4-regular graph of order n with $n\ge 7$. We will show that G has a foresty minimum vertex cut. We have $\kappa(G)\le 4$. If $\kappa(G)\le 2$, the result holds trivially. Next suppose $\kappa(G)\ge 3$ and we distinguish two cases.

Case 1. $\kappa(G)=3$.

Let S be a vertex cut of G with $|S|=3$. If $G[S]\ne C_3$, then S is a foresty minimum vertex cut of G. Suppose $G[S]=C_3$. Since G is 4-regular, by Lemma 5 we deduce that $G-S$ has exactly two components, which we denote by G_1 and G_2. Without loss of generality, suppose $|G_1|\ge|G_2|$. Then $|G_1|\ge(n-|S|)/2\ge(7-3)/2=2$. Let $S=\{x,y,z\}$. We assert that $\deg_S(v)\le 1$ for any $v\in V(G_1)$. To the contrary, suppose that there is $v\in V(G_1)$ such that $\deg_S(v)\ge 2$. Without loss of generality, suppose $\{x,y\}\subseteq N_S(v)$. Then $\{v,z\}$ is a vertex cut of G, contradicting the assumption that $\kappa(G)=3$.

Let u be the neighbor of x in G_1. Then $\{u,y,z\}$ is a vertex cut of G which induces K_1+K_2. Hence it is a foresty minimum vertex cut.

Case 2. $\kappa(G)=4$.

Choose a vertex $v\in V(G)$ and denote $T=N(v)=\{x,y,z,u\}$. Then T is a minimum vertex cut of G. Denote $H=G[T]$. If H is a forest, then T is what we want. Next suppose H contains a cycle. By Lemma 5 and the condition that G is 4-regular, we have $\Delta(H)\le 2$. Thus $H\in\{C_4, C_3+K_1\}$.

Subcase 2.1. $H=C_4$.

Let $W=V(G)\setminus N[v]$. We assert that for any $w\in W$, $\deg_T(w)\le 1$. Otherwise there exists a $w\in W$ with $\deg_T(w)\ge 2$. Since the order $n\ge 7$ and G is 4-regular, $N(w)\ne T$. Now $\{w\}\cup T\setminus N(w)$ is a vertex cut of cardinality at most 3, contradicting $\kappa(G)=4$.

Since G is 4-regular, by Lemma 5 we deduce that every vertex in T has exactly one neighbor in W. Let f be the neighbor of x in W. Then $R\triangleq\{f,y,z,u\}$ is a minimum vertex cut of G and $G[R]=K_1+P_3$ is a forest.

Subcase 2.2. $H=C_3+K_1$.

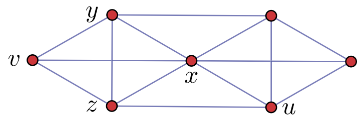

Fig. 4. The graph Z.

Without loss of generality, suppose that $G[A] = C_3$ where $A = \{x, y, z\}$. We assert that every vertex in W has at most one neighbor in A. Otherwise, there exists a vertex $w \in W$ which has at least two neighbors in A. Then $\{w, u\} \cup A \setminus N(w)$ is a vertex cut of G of cardinality at most 3, contradicting $\kappa(G) = 4$.

Let p be the neighbor of x in W. Then $\{p, y, z, u\}$ is a foresty minimum vertex cut of G. □

Remark. There is only one 4-regular graph of order 6, which has connectivity 4 and has no foresty minimum vertex cut. Thus the lower bound 7 for the order in Lemma 9 is sharp.

Proof of Theorem 4. The first part of Theorem 4 is the following

Statement. Every connected graph of order n with $n \geq 7$ and size at most $2n$ has a foresty minimum vertex cut.

We use induction on the order n to prove this statement.

The basis step. $n = 7$.

Let M be a graph of order 7 and size at most 14. The condition $e(M) \leq 14$ implies $\kappa(M) \leq \delta(M) \leq 4$. If $\kappa(M) \leq 2$ then the statement holds trivially. Next suppose $3 \leq \kappa(M) \leq \delta(M) \leq 4$.

If $\delta(M) = 4$, then M is 4-regular and by Lemma 9, M has a foresty minimum vertex cut. Now suppose $\delta(M) = 3$. Then $\kappa(M) = 3$. Choose a vertex $v \in V(M)$ with $\deg(v) = 3$, let $S = N(v)$ and let $R = V(M) \setminus N[v]$. If $M[S]$ is a forest, then S is a foresty minimum vertex cut. Now suppose that $M[S] = C_3$. If R is an independent set, then the condition $\delta(M) = 3$ implies that $e(M) = 15$, contradicting $e(M) \leq 14$. Hence $M[R] \in \{K_2 + K_1, P_3, C_3\}$.

Let $S = \{x, y, z\}$ and let Z be the graph illustrated in Fig. 4.

- $M[R] = K_2 + K_1$. Let $R = \{w_1, w_2, w_3\}$ where $w_1w_2 \in E(M)$. Recall that $\kappa(M) = 3$. Then $\deg_S(w_3) = 3$ and $|N_S(w_1) \cup N_S(w_2)| = 3$, which implies that $N(w_1)$ is a foresty minimum vertex cut of M.
- $M[R] = P_3$. By Lemma 8, $[S, R]$ contains a 3-matching. Then there exists a vertex in R with degree 3 whose neighborhood induces a forest, as desired.
- $M[R] = C_3$. Since $e(M) \leq 14$, $|[S, R]| \leq 5$. By Lemma 8, $[S, R]$ contains a 3-matching. If $M = Z$ (see Fig. 4), then $\{x, y, u\}$ is a foresty minimum vertex cut. Next we assume that $M \neq Z$. Then there exists a vertex in R with degree 3 whose neighborhood induces a forest, as desired.

The induction step. $n \geq 8$.

Let G be a connected graph of order n with $n \geq 8$ and size at most $2n$, and suppose that the above statement holds for all graphs of order $n - 1$. The condition $e(G) \leq 2n$ implies $\kappa(G) \leq \delta(G) \leq 4$. If $\kappa(G) \leq 2$ then the statement holds trivially. Next suppose $3 \leq \kappa(G) \leq \delta(G) \leq 4$.

Case 1. $\delta(G) = 4$.

Since $e(G) \leq 2n$, G is 4-regular. The statement holds by Lemma 9.

Case 2. $\delta(G) = 3$.

We have $\kappa(G) = 3$. Choose a vertex $v \in V(G)$ with $\deg(v) = 3$ and denote $S = N(v)$. If $G[S]$ is a forest, then S is a foresty minimum vertex cut. Otherwise $G[S] = C_3$. Consider the graph $H = G - v$. H has order $n - 1$ and $e(H) = e(G) - 3 \leq 2n - 3 < 2(n - 1)$, which implies that $\delta(H) \leq 3$. Hence $\kappa(H) \leq 3$. Since any vertex cut of H is also a vertex cut of G and $\kappa(G) = 3$, we deduce that $\kappa(H) = 3$. By the induction hypothesis, H has a foresty minimum vertex cut T. Clearly T is also a foresty minimum vertex cut of G.

Now for every integer $n \geq 7$ we construct a graph F_n of order n and size $2n + 1$ such that F_n has no foresty minimum vertex cut. This shows that the size bound $2n$ in Theorem 4 is best possible. Recall that a chord xy of a cycle D is called a k-chord if the distance between x and y on D is k. Let $C : v_1v_2 \ldots v_{n-1}v_1$ be a cycle of order $n - 1$. Add all the 2-chords to C to obtain a 4-regular graph R. Finally adding a new vertex v_n to R and adding the edges v_nv_1, v_nv_2 and v_nv_3, we obtain F_n. We depict F_{11} and F_{12} in Fig. 5.

It is easy to see that $\kappa(F_n) = 3$ and $\{v_1, v_2, v_3\}$ is the unique minimum vertex cut, which induces a triangle. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

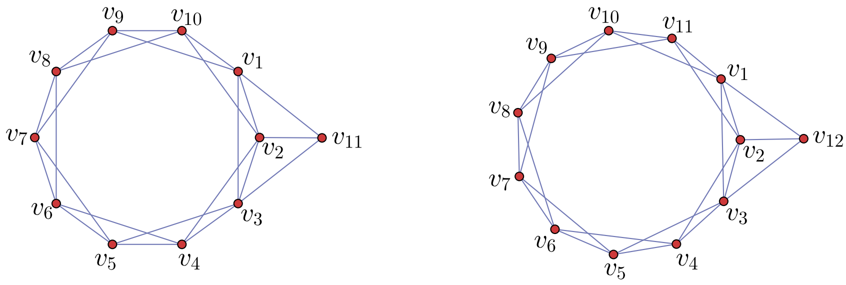

Fig. 5. F_{11} and F_{12}.

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Data availability

No data was used for the research described in the article.

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