上海 HPM 通讯

SHANGHAI HPM NEWSLETTER

2012 年第 1 卷第 8 期

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(Felix Christian Klein, 1849–1925)

《上海 HPM 通讯》第 1 卷第 8 期编委会名单

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刊首语

江南十二月，景秀照山河。

在各位作者和编委会的共同努力下,《上海HPM通讯》第8期与大家见面了。

本期的封面人物是菲利克斯•克莱因，德国著名数学家、数学教育家、数学史家。

作为19世纪末20世纪初伟大的数学家，世界最有影响力的数学学派——哥廷根学派的创始人，F·克莱因在数学界享有崇高的声誉，在非欧几何、连续群论、代数方程论、自守函数论等方面都取得了杰出的成就，"爱尔兰根纲领"更是载入史册。F·克莱因著作颇丰，主要有:《论所谓非欧几何学》(1871)、《新近几何学研究的比较考察》(1872)、《二十面体及五次方程解讲义》(1884)、《椭圆模函数论讲义》(1897、1912)、《高观点下的初等数学》(1908、1909)等。此外，F·克莱因所著的《19世纪数学发展史讲义》(1926~1927)一书，标志着断代体近现代数学史研究的开始。他的许多观点至今仍然对数学家、数学史家有所启迪。

F·克莱因对数学教育十分关注。作为"培利——克莱因运动" 发起人之一，他提倡改革中等数学教学的内容和方法，强调要用近代数学的观点改造传统的中学数学内容，这些都深深地影响了近代数学教学。

F·克莱因还是HPM思想的倡导与实践者。他认为，按照历史顺序教授数学能使学生"看清一切数学观念的产生是如何迟缓；所有观念最初出现时，几乎是草创的形式，只有经过长期改进，才结晶为确定方法，成为大家熟悉的有系统的形式"。 F·克莱因以历史指导数学教学的HPM思想在其著作《高观点下的初等数学》中得到淋漓尽致的体现，其中很多观点对今天的教学仍有重要的借鉴价值。比如关于中学无理数如何教的问题，在回顾无理数的历史发展之后他指出："无理数的精确理论既未必适合大多数学生的兴趣，也超过他们的接受能力。一般来说，学生对有限精确性的结果已感到满足。……对于普通程度的学生，只要通过例子一般讲明白无理数就够了,平常也是这么做的。特别有天资个别学生肯定会要求更完整的解释，给予这些学生以补充解释而不牺牲多数人的兴趣，在教师方面来说，就是值得赞扬的教学技巧了"。

经典回顾，古为今用。我们相信，在大师的指引下，当前的HPM研究会得到更多的灵感与启示，为当前的数学教学实践提供更多的指导与帮助。

参考资料：蒲淑萍. F • 克莱因的HPM思想及其教学启示. 浙江教育学院学报,2010(3):16-21.

经典选读

HISTORICAL CONCEPTUAL DEVELOPMENTS AND THE TEACHING OF MATHEMATICS: FROM PHYLOGENESIS AND ONTOGENESIS THEORY TO CLASSROOM PRACTICE

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1. INTRODUCTION

More than a century ago, Hieronymus Georg Zeuthen wrote a book about the history of mathematics (Zeuthen, 1902). Of course, this was not the first book on the topic, but what made Zeuthen's book different was that it was intended for teachers. Zeuthen proposed that the history of mathematics should be part of teachers' general education. His humanistic orientation fitted well with the work of Cajori, 1894 who, more or less by the same time, saw in the history of mathematics an inspiring source of information for teachers. Since then, mathematics educators have increasingly made use of the history of mathematics in their lesson plans, and the spectrum of its uses has widened. For instance, the history of mathematics has been used as a powerful tool to counter teachers' and students' widespread perception that mathematical truths and methods have never been disputed. The biographies of several mathematicians have been a source of motivation for students. By stressing how certain mathematical theories flourished in various countries, the diverse contributions of various cultures to contemporary mathematics become evident. Specialized study groups have emerged in the past years as a result of the increasing interest in the history of mathematics in educational circles. Two of these are the Commission INTER-IREM ´Epistemologie et Histoire des Math ́ematiques in France and the *International Study Group on the Relations between History and Pedagogy of Mathematics*, which is related to International Commission on Mathematical Instruction (ICMI). In addition, regular conferences are organized, such as the European Summer Universities on the History and the Epistemology in Mathematics Education (see Lalande, Jaboeuf, & Nouaz ́e, 1995, and Lagarto, Vieira, & Veloso, 1996, for proceedings). Concomitantly, an important number of books are now available to help teachers use the history of mathematics (Calinger, 1996; Chabert, Barbin, Guillemot, Michel-Pajus, Borowczyk, Djebbar, & Martzloff,1994; Dhombres, Dahan-Dalmedico, Bkouche, Houzel, & Guillemot, 1987; Fauvel&van Maanen, 2000; Katz, 2000; Reimer&Reimer, 1995; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995).

Instead of offering an overview of the different domains in which the pedagogical use of the history of mathematics is now ramified, we want, in this chapter, to focus on something that Cajori started and in which mathematics educators interested in the history of mathematics are still involved. That is, in considering history not only as a window from where to draw a better knowledge of the nature of mathematics but as a means to transform the teaching itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance the students' development of mathematical thinking. Cajori's 1894 ideas have led us to developments that he could not have suspected. Indeed, Cajori adopted a positivistic view of the formation of knowledge. He saw knowledge as an objective entity that grows gradually and cumulatively. His reading of the history of mathematics was framed by viewing history as an unfolding process.

The direction or completion of the process guaranteed by the idea of progress—an idea underpinning the Enlightenment philosophy and attitudes toward life from which modern thought arose. Nonpositivistic views about the formation of knowledge were later elaborated by philosophers and epistemologists such as Bachelard, Foucault, and Piaget, among others, and by anthropologists such as Durkheim, Levy-Bruhl, and L évi-Strauss, to mention but a few.

Bachelard presented an interpretation of the formation of knowledge in terms of ruptures and discontinuities. Piaget was interested in explaining genetic developments in terms of stages and the intellectual mechanisms allowing the passage from one level to another. Foucault was opposed to the conception of history as a date-labeling practice and investigated the problem of the constitution of knowledge in terms of the conditions of its emergence, which he related to the different spheres of human activity. Bachelard, Foucault, and Piaget had different goals, and thus their projects differed. But what is important for our discussion here is that, contrary to what Cajori and many other positivist thinkers believed, knowledge in general and mathematical knowledge in particular cannot be taken as an unproblematic concept. Behind a concept of knowledge there is an epistemological stance, and this epistemological stance conditions our understanding of the formation of students' mathematical thinking as it conditions the interpretation of historical conceptual developments (Grugnetti & Rogers, 2000; Radford, Boero, & Vasco, 2000). Nevertheless, the study of the development of students' thinking and of the conceptual development of mathematics belong to two different domains—the psychological and the historical, respectively. Each has its specific problems as well as the tools with which to investigate them. Students' conceptualizations can be investigated through classroom observations, interviews, tests, and soforth. The same cannot be done in the historical domain, where historical records are the only available material for study. The difference in methodologies in both domains is, in fact, a token of more profound differences. These cannot be ignored in the context of a pedagogical use of the history of mathematics as a useful tool to enhance the development of students' mathematical thinking. Despite their differences, the psychological and historical domains need to be weighed and articulated in a specific way. One of today's more controversial themes concerns the terms in which such an articulation must be understood. More specifically, the question is how to relate the development of students' mathematical thinking to historical conceptual mathematical developments. Psychological recapitulation, which transposes the biological law of recapitulation, claims that in their intellectual development our students naturally traverse more or less the same stages as mankind once did; it has been taken as a guarantee (sometimes implicitly) to ensure the link between both domains. In its different variants, however, psychological recapitulation has been subject to a deep revision recently, in part because of the emergence of new conceptions about the role of culture in the way we come to know and think.

The purpose of this chapter is to discuss in some detail the basic problems referred to in this introduction. In the next section, we deal with psychological recapitulation and mention some of the current arguments against it. In section 3, we examine key ideas about ontogenesis and phylogenesis as found in the works of Piaget and in the works of Vygotsky. In section 4, we present some paradigmatic examples of mathematicians who commented on phylogenesis and its relation to ontogenesis. Section 5 focuses on a particular interpretation of the recapitulation law that led to the so-called "genetic approach", which had an obvious impact on early mathematics education. In section 6, we discuss some examples of teachers who take into consideration the history of mathematics to improve their teaching; determining how interpretations of the recapitulation law articulate the teachers' goals and actions guides our discussion.

Section 7 provides a brief account of a few current approaches in contemporary mathematics education that relate to the history of mathematics regarding either theoretical or practical links between the development of students mathematical thinking and historical conceptual developments. In the last section, we offer a critical assessment of the law of recapitulation and recommend ideas for conceptual and applied research in the 21st century regarding historical and ontogenetic developments in mathematics education.

2. FROM BIOLOGICAL TO PSYCHOLOGICAL RECAPITULATION

The way in which people perceived psychological recapitulation at the beginning of the 20th century was linked to the way they perceived themselves in the overall view of the world. As long as humans thought of themselves as essentially different from animals and plants, no relation in terms of ancestry could be advocated. Even in the early 18th century, a common scholarly view to explain the origin of species and to understand the formation of living things was that species came from those beings fortunate enough to survive the deluge, as indicated in the Genesis (see, e.g., Osborn, 1929), by finding refuge on Noah's ark. But with the appearance of the early 19th-century philosophy of nature, humans came to join the greater kingdom of species. In their broader sense, however, recapitulationist ideas data back, to the pre-Socratic thinkers. They did not state them in terms of a telescoping or condensed process of lower life that culminates with humans. Often their reference point was the cosmos. Thus, Empedocles believed that the growth of the embryo echos in a foreshortened way the cosmogonic process: The embryo is submerged into amniotic fluid that evokes the originally fluid earth (de Santillana, 1961, p. 114). During the 18th and early 19th centuries, a vigorous debate separated two opposing schools with regard to the concept of recapitulation. One of them, which became known as preformation theory, stated that ontogenesis was the unfolding or growing of preformed structures, whereas the other school adopted a more dynamic stance, arguing that the embryo was neither the exact miniature of the developed species nor the unfolding of preformed structures, but a being in a state of development. The "causes" originating embryo's the unfolding or the changes were variously interpreted. Charles Bonnet (1720–1793), usually recognized as one of the leaders of the preformationists, saw change as coming from an affectionate God who had ordered the world according to increasing perfection and progress. Whereas in the early-19th century Naturphilosophen attributed development to a "natural" final cause, Lamarck and Darwin envisioned a new theory that replaced the philosophical idea of final cause with an efficient cause—individual development. (For a detailed discussion of preformationist and Naturphilosophen ideas, see Gould, 1977.) Indeed, from the mid-19th century onward, the "causes" were seen in the context of the theory of evolution."Heredity and adaptation are, in fact, the two constructive physiological functions of living things," wrote Haeckel (1912, p. 6), who, in one of the most famous statements ever made in the realm of anthropogenesis (which he modestly called the fundamental law of biogeny), declared that

> The series of forms through which the individual organism passes during its development from the ovum to the complete bodily structure is a brief, condensed repetition of the long series of forms which the animal ancestors of the said organism, or the ancestral forms of the species, have passed through from the earliest period of organic life down to the present day. (pp. 2–3)

Haeckel's law was more than the simple statement of a condensed repetition of steps. What he was suggesting was that embryos of man and dog, at a certain stage of their development, are almost indistinguishable. Indeed, to take one of Haeckel's favorite examples, "the human gill slits *are* (literally) the adult features of an ancestor" (Gould, 1977, p. 7).

How, then, was the discussion about the biological growth of humans transferred to the psychological domain? It was Haeckel who, after discussing the nervous system, said "we are enabled, by this story of the evolution of the nervous system, to understand at length *the natural development of the human mind* and its gradual unfolding" (1912, p. 8, italics as in the original). A sharper formulation was the following: "the psychic development of the child is but a brief repetition of the phylogenetic evolution" (Haeckel quoted by Mengal, 1993, p. 94). The adoption of the psychological version of biological recapitulation served as a general framework to conceive the functioning of child psyche as something traveling the same path as his or her ancestors. For instance, the child was seen as behaving as humans in previous stages of the chain of evolution (e.g., such as having, in an early stage of his or her development, an "animist" view of nature, that is, that immaterial forces animate the universe). Psychological recapitulation endorses a peculiar view of history and development. Concerning development, for Bonnet and the preformists, there was no development, strictly speaking, but only growing or unfolding. Environment cannot alter the preformed structures and their growth. For evolutionary-based recapitulation theories, in contrast the environment is supposed to play in the development of species a role. The individual is seen as an organism adapting to his or her environment; in the interplay between individual and environment, some of the biological and psychological functions may develop, whereas others may be lost according to the natural selection.

As for history, in contrast to views that conceived a world that underwent different creations, Bonnet saw the world as created at one time, with its whole history encapsulated within it. History was therefore the unfolding of a predetermined plan. The concept of history is much more problematic for recapitulationists. Indeed, from a theoretical point of view, history and recapitulation become difficult to reconcil because, on one hand, Haeckel's psychological recapitulation supposes that present intellectual developments are to some extent a condensed version of those of the past.On the other hand, natural selection is presented as a function of the environment against which individuals act. For recapitulation to be possible, therefore, such an environment must remain essentially the same, which obviously is not the case. Given that the environment changes, it becomes difficult to maintain that the children's intellectual development will undergo the same process as the one children experienced in the past. The variability that natural selection imposes on the course of events in history conflicts with the idea of recapitulation as condensed repetition of some intellectual aspects registered in past history. Indeed, this point was recognized as a weakness.

Werner (1957), for instance, advocated contextual factors and argued that it is impossible o equate a certain intellectual stage of a child in a modern society to the stage an adult could have reached in a ancient society because the respective environments, as well as the genetic processes involved in them, are completely different (see Radford,1997a). Elias also mentioned the differences that necessarily result as a consequence of variations in cultural settings. Whereas in traditional societies children participate directly in the life of the adults earlier and their learning is done *in situ* (as apprentices), "modern" children are instructed indirectly in mediating institutions, or schools (Elias,1991, pp. 66–67). Consider memory, an example that is addressed neither by Werner nor Elias but which conveniently clarifies the previous ideas. As many anthropological accounts clearly show (see e.g., Lévy-Bruhl, 1928), memory plays a central role in illiterate societies. In contrast, sign systems related to writing in literate societies dispense with memory to a certain and fundamental extent. They create a different way to handle and distribute knowledge and information between the members of the society and shapes attitudes about how to scrutinize the future (see Lotman, 1990).

The theoretical difficulties encompassing the crude version of psychological recapitulation encouraged new reflections to find more suitable explanations concerning the relations between phylogenesis and ontogenesis. In the next section, we will discuss two different views that have been influential in the use of history in mathematics education.

3. PIAGET AND VYGOTSKY ON ONTOGENESIS AND PHYLOGENESIS

Piaget was interested in understanding the process of the formation of knowledge. To do so, he considered knowledge as something that can be described in terms of levels and strived to describe those levels, as well as the passage from one level to a more complex one. He said, "The study of such transformations of knowledge, the progressive adjustment of knowledge, is what I call genetic epistemology" (Piaget cited in Bringuier, 1980, p. 7). As a reaction to the simplistic psychological version of recapitulationandthe positivist view of knowledge thatwementioned in the introduction, Piaget and Garcia elaborated the concept of *genetic development*. They envisioned the problem of knowledge in terms of the intellectual instruments and mechanisms allowing its acquisition. According to Piaget and Garcia, the first of those mechanisms is a general process that accounts for the individual's assimilation and integration of what is new on the basis of his or her previous knowledge. In addition to the assimilation mechanism, they identified a second mechanism, a process that leads from the *intraobject*, or analysis of objects, to the *interobject*, or analysis of the transformations and relations of objects, to the *transobject*, or construction of structures. This epistemological viewpoint led them to revisit the parallelism that recapitulationists had emphasized. Therefore, Piaget concluded,"We mustn't exaggerate the parallel between history and the individual development, but in broad outline there certainly are stages that are the same" (Bringuier, p. 48). The two mechanisms were hence considered as invariables, not only in time but also in space. That is, we do not have to specify what they are in a certain geographical space at a particular time because they do not change from place to place or from time to time. They are exactly the same, regardless of the period of history and the place of the individuals. In modern mathematics, at the level of algebraic geometry, of quantum mechanics, although it's a much higher level of abstraction, you find the same mechanisms in action—the processes of the development of knowledge or the cognitive system are constructed according to the same kinds of evolutionary laws.(Garcia in Bringuier, 1980, pp. 101–102) Thus, when Piaget and Garcia investigated the relations between ontogenesis and phylogenesis, they did not seek a parallelism of contents between historical and psychogenetical developments but of the mechanisms of passage from one historical period to the next. They tried to show that those mechanisms are analogous to those of the passage from one psychogenetic stage to the next.

The two mechanisms of passage discussed by Piaget and Garcia have a different theoretical background. The second, that of the intra-, inter- and trans-objectual relations, obeys a structural conception of knowledge and reflects the role that mathematical and scientific thinking played in Piaget's work. AsWalkerdine noted, "In the work of Piaget, an evolutionary model was used in which scientific and mathematical reasoning were understood as the pinnacle of an evolutionary process of adaptation" (Walkerdine, 1997, p. 59). The first one, the assimilation mechanism, has its roots in the conception of knowledge as the prolongation of the biological nature of the individuals: "Thehumanmind is a product of biological organization, a refined and superior product, but still a product like another" (Piaget in Bringuier, 1980, p. 108). Both intellectual mechanisms of knowledge development embody a general conception of rationality that has been contested by some critics who find missing, among other things, a more vivid role of the culture and the social practices in the formation of knowledge. For instance, the epistemologist Wartofsky, who has stressed an intimate link between knowledge and the activities from which knowledge arises and is used, said:

We are, in effect, the products of our own activity, in this way; we transform our own perceptual and cognitive modes, our ways of seeing and of understanding, by means of the representations we make. . . . Theoretical artifacts, in the sciences, and pictorial or literary artifacts, in the arts constitute the a priori forms of our perception and cognition. But contrary to the ahistorical and essentialist traditional forms of Kantianism, I propose instead that it is we who create and transform these a priori structures. Thus, they are neither the unchanging transcendental structures of the understanding, nor only the biologically evolved a priori structures which emerge in species evolution (as, for example, Piaget and the evolutionary epistemologists suggest). Piaget's dynamic, or genetic structuralism is important here, of course. His dictum, "no genesis without structure, no structure without genesis," suggests the dialectical interplay of the practical emergence and transformation *of* structures with the shaping of our experience and thought *by* structures. But the domain of this genesis I take to be the context of our social, cultural and scientific practice, and not that of biological species-evolution alone. . . . In a sense, then, our ways of knowing are themselves artifacts which we ourselves have created and changed, using the raw materials of our biological inheritance. (Wartofsky, 1979, p. xxiii)

Vygotsky, in many writings, dealt with the problem of recapitulation and, like Piaget, believed that the understanding of ontogenesis and phylogenesis had to be based on a deep understanding of our biological nature. (This is clear, for instance, in his book *Speech and Thinking*, as well as in the influence he had on his student Luria and the huge amount of physiological research that the latter conducted.) Instead of posing the problem of the formation of knowledge in terms of universal and atemporal mechanisms functioning beyond culture, however, he saw the cognitive functions allowing the production of knowledge as inevitably overlapping with the context in which individuals act and live. His basic distinction between lower and higher mental functions is reinforced by the idea that the former belong to the sphere of the biological structure, whereas the latter are intrinsically social. Thus, in a passage from *Tool and Symbol in Child Development*, when discussing the problem of the history of the higher psychological functions, Vygotsky and Luria commented:

Within this general process of development two qualitatively original main lines can already be distinguished: the line of biological formation of elementary processes and the line of the socio-cultural formation of the higher psychological functions; the real history of child behaviour is born from the interweaving of these two lines. (Vygotsky&Luria, 1994, p. 148)

The merging of the natural and the sociocultural lines of development in the intellectual development of the child definitely precludes any recapitulation:

In the development of the child, two types of mental development are represented (not repeated) which we find in an isolated form in phylogenesis: biological and historical, or natural and cultural development of behavior. In ontogenesis both processes have their analogs (not parallels). . . . By this, we do not mean to say that ontogenesis in any form or degree repeats or produces phylogenesis or is its parallel. We have in mind something completely different which only by lazy thinking could be taken to be a return to the reasoning of biogenetic law. (Vygotsky, 1997, p. 19)

For Vygotsky even the elementary intellectual functions of the individual are intrinsically human, acquired through the activities and actions on which are based the intercourse between individuals and between people and objects. One of the central reasons for this is that human activities are mediated by diverse kinds of tools, artifacts, languages, and other systems of signs which, Vygotsky argued, are a constitutive part of our cognitive functions. Most important, these systems of signs, as well as tools and artifacts, are much more than technical aids: They modify our cognitive functioning. The knowledge produced by the individuals hence becomes intimately related to the activities out of which knowledge arises and the conceptual and material "cultural tool kit" (to borrow Bruner's expression, see Bruner, 1990) with which the individuals are equipped. Of course, it does not mean that with every new generation, all knowledge must be constructed anew. As Tulviste (1991) noted, whereas rats are still doing what they did centuries ago, humans have, from one generation to the next, assimilated, produced, and passed on their knowledge. During this process, humans have changed their activities and the way in which they think about the world. In Vygotsky's view, knowledge appears as an individual and social creative reappropriation and coconstruction carried out using conceptual and material tools that culture makes available to its individuals. In turn, in the course of this process, the previous tools and signs may become modified, and new ones may be created. It is in this sense that tools and concepts have embodied the social characteristics from which they arose, and their insertion into other activities allows their transformation and eventually their growth. Because activities, sign use, and attitudes toward the meaning of scientific inquiry do not necessarily remain the same throughout time, changes are effected in phylogenetic lines (and the plural of lines needs to be emphasized here) serving as the historicocultural starting point to new genetic developments. Epistamologica reflexions have then to evidence the relation between cognitive context and action. As Wartofsky pointed out:

If, in fact, our modes of cognitive practice change with changes in our modes of production, of social organization, of technology and technique, then the connection between cognition and action, between theoretical and applied practice, between consciousness and conduct, has to be shown. (Wartofsky, 1979, p. xxii)

One implication of the previous remarks for the use of the history of mathematics in education is that the study of recapitulation can be advantageously replaced by the contextual study of the social elements in which the historical geneses of concepts are subsumed. This can be accomplished through a careful investigation of the cultural symbolic webs shaping the form and content of scientific inquiry and the ways in which mathematical concepts are semiotically represented (Radford, 1997a, 1998, 1999a, 2000a).We return to this point in section 7.

4. INTERPRETATION OF RECAPITULATION LAW BY MATHEMATICIANS

In the period when the treatises of Zeuthen and Cajori appeared, the history of mathematics was growing as a scientific discipline. The first journals dealing exclusively with the history of mathematics were appearing in that period. We have extensive evidence that mathematicians and mathematics educators were both looking at the history of mathematics with great interest. Mathematics educators were creating new areas of work in their field linked to changes in societies. As discussed in Furinghetti (2000) and in Furinghetti and Somaglia (1998), the history of mathematics was considered a suitable means to find efficient ways of teaching in different situations.Among mathematicians, the axiomatization and the foundational works were undertaken. These themes were addressing mathematicians' attention to reflections on the nature of mathematics and on the activity of doing mathematics. The history of mathematics was considered a field that offered inspiration to discuss these kinds of problems. In this context, we consider some interpretations of recapitulation law made by important mathematicians.

In thefirst issue (1899) of *L'enseignement math ́ematique*, an important journal devoted to the teaching of mathematics, the eminent mathematician Henri Poincar ́e clearly stated his position on the relations between conceptual and historical developments:

Without a doubt, it is difficult for a teacher to teach a reasoning that does not satisfy him completely. . . . But the teacher's satisfaction is not the sole purpose of teaching. . . above all one should be concerned with the student's mind and of what we want him to become.

Zoologists claim that the embryonal development of animals summarizes in a very short time all the history of its ancestors of geologic epochs. It seems that the same happens to the mind's development. The educators' task is to make children follow the path that was followed by their fathers, passing quickly through certain stages without eliminating any of them. In this way, the history of sciences has to be our guide. (Poincar ́e, 1899, p. 159; our translation)

Poincar ́e gave examples of concepts to be taught at an intuitive stage before presenting them rigorously. Among these examples were fractions, continuity, and area. As far as we know, Poincar ́e never used his ideas on the efficacy of recapitulation law directly with teachers. This makes Poincar ́e's position different fromthat of Felix Klein, another supporter of the use of history in mathematics in teaching. In contrast, Klein applied his ideas in courses for prospective teachers and in related texts that he wrote.

Klein supported the German translation of the famous book *A study of Mathematical Education* by Benchara Branford (1921) in which, according to Fauvel (1991, p. 3), the theory of recapitulation "reached its apogee." This can be considered evidence of Klein's agreement to the recapitulation law (Fauvel, 1991, p. 3). Nevertheless, from what Klein wrote in his articles and books (see Klein, 1924), we understand that the application of the law was not advocated in a literal sense. As in the case of Poincar ́e, his opinion on the use of history was born of his wish to abolish the use of mathematical logic and the excesses of rigor advocated by some of his colleagues. Klein was interested in the dichotomy of "intuition versus rigor" and, as far as school is concerned, was in favor of intuition.Hesingled out the history of mathematics as being the suitable context for bringing intuition back into the teaching and learning process:

I maintain that mathematical intuition . . . is always far in advance of logical reasoning and covers a wider field. . . . I might now introduce a historical excursus, showing that in the development of most of the branches of our science [mathematics], intuition was the starting point, while logical treatment followed. This holds in fact, not only of the origin of the infinitesimal calculus as a whole [this issue was discussed at the beginning of Klein's paper] but also of many subjects that have come into existence only in the present [19th] century. (Klein, 1896, p. 246)

Klein claimed that in school, as well as in research, the phase of formalization must be preceded by a phase of exploration based on intuition.

We find an analogous statement in a secondary school geometry book written by a famous Italian mathematician, Francesco Severi, which clearly refers to school practice:

We need to take inspiration from the principle that in learning new notions, the mind tends to follow a process analogous to that according to which science has developed.One who is aware of the value of foundation theories [in Italian *critica dei principi*] does not make the dangerous mistake of giving to the elementary teaching a critical and excessively abstract style. It is necessary to know foundation theories for personal intellectual maturity; but in the elementary teaching they are not to be considered as a pedagogical means. (Severi, 1930, p. IX; our translation)

Both Klein and Severi do not clearly state what "intuition" means for them, but both state to what intuition is opposed: rigor, excessive abstraction, and formal logic used at the beginning of the presentation of a mathematical notion. (It may be interesting to note that Severi, famous during the first half of the 20th century, is one of the scholars of the Italian school of algebraic geometry who based his results on intuition to such a degree that these were published without being careful verified by a mathematical proof, as reported by Hanna, 1996).

5. THE GENETIC APPROACH

Using the history of mathematics in teaching does not necessarily entail a direct assumption of the recapitulation law; it also may be used in the so-called *genetic approach* to teaching. The term "genetic" is an ambiguous one because it is used with different meanings. In particular, in the foundation literature, the term *genetic method* is used in contrast to *axiomatic method*. David Hilbert probably introduced this term, which was popularized by Edward V. Huntington. Before Hilbert, we find other uses of the word "genetic." Immanuel Kant stated that all mathematical definitions are genetic; after Kant, the term "genetic definition" is present in all major logic treatises.

In addition to its use among mathematicians and philosophers, we find the word "genetic" in other fields of research. Piaget and Garcia used it in their epistemological studies. As to mathematics education, Ed Dubinsky, who dealt with genetic decomposition, used the word.

Here we are concerned with the word "genetic" as it is used in connection with history. In the 1920s the idea of a genetic principle was taking shape, as evidenced by the work of N. A. Izvolsky.

Gusev and Safuanov (2000) report that, according to Izvolsky, nor teachers nor textbooks try to explain the origin of geometrical theorems. He suggested that, when attempts to do this are done, students see geometry in a different way. Moreover sometimes students themselves guess that a given theorem was not originated by a mere wish of the teacher or textbooks' authors, but by questions arisen in previous works. It happens that students try to imagine the origin of a theorem. According to Izvolsky, even if their hypotheses are not correct from the historical point of view, this approach to the teaching of geometry is valuable.

The idea of a genetic approach later took a definite form in a work by Otto Toeplitz that he wrote to describe a method of presenting analysis to university students. The following passage illustrates the ideas underlying the genetic method:

Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, e.g., mean-value theorem, Taylor series, the concept of convergence, the definite integral, and the differential quotient itself, the question is never raised "Why so?"

or "How does one arrive at them?" Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut and dried facts and instead take on fresh and vibrant life again.

Burn explains in this way Toeplitz's ideas:

> The question which Toeplitz was addressing was the question of how to remain rigorous in one's mathematical exposition and the teaching structure while at the same time unravelling a deductive presentation far enough to let a learner meet the ideas in a developmental sequence and not just in a logical sequence. While the genetic method depends on careful historical scholarship it is not itself the study of history. For it is selective in its choice of history, and it uses modern symbolism and terminology (which of course have their own genesis) without restraint. (Burn, 1999, p. 8)

It is not by chance that this alternative approach developed in the domain of teaching calculus. It is in this domain where the notion that learning mathematics takes place in a sequence predetermined by mathematical logic has shown its pedagogical limitations. Indeed, when organized around their logical basis, the definitions of the main concepts of calculus (integrals, limits, derivatives) are abstract, and therein lies the burden of formal rules and theorems. Students have difficulty grasping the meaning of that with which they are asked to work. At present there are projects (not based on history) that take into account these difficulties and organize the teaching of calculus according to different patterns. (See, for example, the Harvard project based on giving an informal, operative approach to concepts in Hughes-Hallet et al., 1994).

What Toeplitz proposed is realistic and may be considered a compromise between the two ways of thinking about teaching mathematics, the logical versus developmental sequences. Toeplitz's historically based approach aims to provide a slow process of understanding that the student performs through a sequence of steps. Because Toeplitz's aim is to provide teaching materials that facilitate the learning of calculus, the main concern of the author is not to teach history, but to find learning sequences. Burn (1999) elaborated on these ideas. If we analyze Toeplitz's proposals or the more recent ideas of Burn, we may find an example of the history of mathematics used as a key element in the construction of a teaching sequence (on calculus) from intuition to logical deduction. The role of history is therefore that of providing materials on which to develop intuition. The presentation of the historical materials is not shaped according to recapitulationist principles because it uses modern symbols, verbal expressions, and cultural tools that are different from those of past authors.

An older example of the use of the genetic method (intertwined with a na¨ıve heuristic approach) is in the treatise on geometry by Alexis-Claude Clairaut (1771). The preface of his book is an early example of predidactic literature. Its importance lies in the traces of Clairaut's thought that can be found in works on mathematics education through the 20th century. Clairaut wrote:

> Even if geometry is abstract in itself, we nonetheless must agree that the difficulties suffered by beginners come mostly from the way it is taught in usual treatises. They always start with a great deal of definitions, questions, axioms, and preliminary principles, which

only seem to promise dry issues for readers. . . . To avoid this dry quality that is naturally linked to the study of geometry, some authors put examples after each proposition to show it is possible to do them; but in this way, they only prove the usefulness of geometry without making it any easier to learn. Because each proposition is presented before its use, the mind reaches concrete ideas after having toiled with abstract ideas. Having realized this fact, I intended to find out what may have given birth to geometry and tried to explain principles with the most natural methods, which I suppose were adopted by the first inventors, while trying to avoid the wrong attempts they had necessarily made. (Clairaut, 1771, pp. 2–4; our translation).

According to Glaeser (1983), Clairaut contributed greatly to the introduction of the genetic method. Glaeser commented on Clairaut's work with the following observations: "Giving up the dogmatic exposition, and to follow the true historical development of discovery, this method consists on imagining a road that learned peoples "could have followed"! Thus this is pretense education". (Glaeser, 1983, p. 341, our translation).

In spite of Glaeser's criticism, Clairaut's attempts present interesting features, even more so if we consider that in the period when this author conceived his project, the paradigm of geometrical teaching was based on the hypothetical-deductive Euclidean method. If we compare the passage from Toeplitz's book and Clairaut's passage, we see an extraordinary coincidence of intentions and didactic observations (i.e., the idea of "dryness" that is present in the work of both authors).

Freudenthal (1973) provided an interpretation of the genetic method:

Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now. . . . It is not the historical footprints of the inventor we should follow but an improved and better *guided* course of history. (Freudenthal, 1973, pp. 101, 103; our italics).

Freundenthal termed this way of using history "guided reinvention." It implies an active and aware participation of the teacher in designing and carrying out teaching with history.

6. THE HISTORY OF MATHEMATICS IN THE CLASSROOM FROM THE TEACHER'S POINT OF VIEW

We have argued elsewhere (Furinghetti, 1997) that to study the applications of the history of mathematics in the classroom; we need a systemic net of experiments to analyze. For this reason, one of the authors (F. F.) has constituted a permanent monitor to keep track of the use of history in mathematics teaching in Italy. This means that teachers experimenting with the use of the history of mathematics, or only wishing to do so, are invited to contact the monitor and to discuss their ideas. In this way, it has been possible to create a file containing a range of different situations. The examples that we shall present in what follows come from these data.

First, we report on a workshop of teachers held by Jan van Maanen in Italy to present and discuss the ICMI Study document, "The role of the history of mathematics in the teaching and learning of mathematics," together with Italian researchers in mathematics education and high school teachers. Teachers participating in the workshop were asked if they use history in their classrooms. The answer, in general, was negative because of the constraints of the school system. Nonetheless, all the teachers expressed the strong interest in using it if they were given the opportunity. When asked to explain why they consider the use of history fruitful, the answer was something echoing—usually unintentionally—the recapitulation law. Some of the paradigmatic statements (quoted literally) include the following: "The students' development of concepts follows the historical sequence," "The historical genesis of the concept may help teachers understand the genesis of the concept in students' minds," and "If I present the students with how algebra developed in history, they feel differently about their difficulties in learning it."

Although not necessarily in a conscious or explicit way, the answers exhibit an understanding of the relation between ontogenesis and phylogenesis that is close to Haeckel's psychological version of the law of recapitulation. The following three examples illustrate, in a more detailed way, some teachers' positions about recapitulation.

We will see that in these cases the initial stimulus to consider the history of mathematics in their teaching is the vague idea that some parallelism between child development and mathematical development exists. Nonetheless, the kind and amount of adaptations that result from changes due to differences in historical periods and their cultural contexts are so significant that it is not possible to talk about some form of genuine recapitulation.

6.1. First Example

The first teacher is a mathematics instructor in a middle school (students aged 11 to 13), who studied biological sciences in college (and hence does not have a substantially deep understanding of mathematics) but is fond of mathematics and of teaching.

She confesses her difficulties in teaching because of students' lack of motivation and her personal incapacity to interpret their difficulties. She has never carried out experiments in the classroom encompassing the use of history in mathematics teaching; nonetheless, she wrote (see also Gallo, 1999):

I feel that my mathematical preparation lacks a historical perspective. I think I could find in history some answer to my teaching problems.

In my opinion, to follow the evolution of the mathematical thinking could help the teacher understand how learning mathematics develops in children and preadolescents.

As an example, I mention the use of fractions by the Egyptians: It is closer to the intuitive concept held by a primary pupil. I gave my 10-year-old daughter an Egyptian problem of dividing loaves among men taken from a seventh-grade mathematics textbook. She solved the problem in the way that the Papyrus Rhind solves it.

I think it could be interesting to show students other issues taken from history: the geometrical representations of numbers, the geometrical representations of algebraic situations offered by Euclid. I think that the latter are more illuminating than the usual modern presentations.

The division problem the teacher used is the following problem in the Rhind Papyrus (ca. 1650 BCE): "*Example of reckoning out 100 loaves for 10 men, a sailor, a foreman and a watchman with double*" (see Peet, 1923, p. 109). Here we have an example of a teacher who does not have historical preparation; she only has some scattered ideas taken from notes in books and articles. She never carried out experiments using history in the classroom. Her experience is based on anecdotal facts.We interpret what she writes about history as being representative of the ideas that teachers in similar situations have about the use of history in teaching: There is a parallel between history and the way students learn.

6.2. Second Example

Other examples of the relationship of teachers with history that are more precise focus on experiments performed in the classroom. In these cases, the ideas expressed by the teacher are not mere intuition but are based on fact. The first case concerns a class of twenty-one 15-year-old high school students. We only briefly report on this experiment. (For a wider account, see Paola, 1998.) The teacher has a extensive experience in instruction and research in mathematics education. In the experiment, he acted as a teacher and as an observer. His purpose was to work with students on the concept of probability, which they had already encountered in previous school years. He chose to work with history to return to the concept of probability using a different (historical) approach. The work in the classroom was centered on a problem that is treated in many books of arithmetic from the Middle Ages "How can the stake be divided in a game where the two players are of the same value (in modern terms, have the same probability of winning) if the game is interrupted before one of the two players has realized the winning score?" This problem is known as "the problem of partition." Luca Pacioli gave his solution (based on proportionality) to this problem in his famous treatise *Summa de arithmetica geometria proportioni et proportionalit`a* (Printed in 1494). The classroom activity was developed through discussion of the problem between students divided into groups. The teacher not only orchestrated the discussion but also acted as an observer and reported all that happened in the classroom. Initially all students agreed that the best way to solve the problem would be to divide the stake in parts that were proportional to the scores earned by each player. The teacher easily refuted this solution by proposing that one of the two players had a score of zero when the game was interrupted. After a discussion on this particular case, another group of students proposed other ways of solving it that did not satisfy their classmates. At this point, the teacher read Pacioli's solution, which is similar to that of the students, allowing them to see that an important historical personage followed the same process they did. The students seemed ready to approach the concept of fair division of the stake. Additional classes were dedicated to discussing this concept, but the students did not arrive at effective results on their own (i.e., they were not able to grasp the concept of probability). The teacher expounded Pascal's the solution to the problem, as reported in (Pascal, 1954), and thus introduced students to the concept of probability.

As we said previously, the teacher acted as an observer, and he accurately reported the activity in the classroom (Paola, 1998). Even if some elements of probability had been taught to these students in the previous school years, it is clear from the chronicle of the classroomactivity that their strategies were based on proportionality, as Pacioli's were. The teacher believes this experiment shows that students follow the path of history: The voice of history is again evoked by the teacher to give dignity to the students' solutions which actually follow the path hinted by mathematicians before Pascal and Fermat. (Paola 1998, p. 34) There are many passages suggesting that the teacher is concerned with the mistakes in the ancient attempts of solving

Pacioli's problem. For example: "The incursion into history had the goal of giving dignity to the mistake made by students: it was not a trivial mistake if a mathematician made it" (p. 33). The teacher showed interest in the parallels between the strategies his pupils and Pacioli used, but he did not draw general theoretical conclusions concerning the recapitulation law. Fromhis conclusions, we see only that he has a certain confidence in the validity of following the stages of the historical development for didactic purposes:

With another session I could have read and commented on the Pascal–Fermat letters in the classroom and thus I would have stressed the role of history [in helping students to bypass some obstacles in constructing concepts of probability theory] (Paola, p. 35).

6.3. Third Example

The last case we present concerns a high school mathematics and physics teacher who works with students ranging in age from 16 to 19 years. The teacher has researched the history of mathematics. She is interested in proof and tries to develop students' abilities on this subject using historical examples. To this end, she uses the method of analysis and synthesis, found in the Pappus's *Collectiones Mathematicae*. We describe this method with the following passage taken from Hintikka and Remes (1974):

Now analysis is the way from what is sought—as if it were admitted—through its concomitants [the usual translation reads consequences] in order to something admitted in synthesis. For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order.

And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. And this we call synthesis. (p. 8)

The method of analysis is described in a manual for teachers (Smith, 1911) as follows:

I can prove this proposition if I can prove this thing; I can prove this thing if I can prove that; I can prove that if I can prove a third thing," and so the reasoning runs until the pupil comes to the point where he is able to add, "but I can prove that." This does not prove the proposition, but it enables him to reverse the process, beginning with the thing he can prove and going back, step by step, to the thing that he is to prove. Analysis is, therefore, his method of discovery of the way in which he may arrange his synthetic proof. (Smith, 1911, pp. 161–162)

Historically this method originated in the field of geometry, but it hassince been used in other branches of mathematics. For example, the method of analysis is at the heart of algebra: The introduction of symbols made byVi`ete in the 16th century did not arise spontaneously but was a consequence of having adopted the method of analysis for solving algebraic problems (Charbonneau, 1996). The method of analysis also is not specific to mathematics; for example, in Marchi (1980), it is applied to chemistry. The method of analysis represents a link between history and education. In their chapter on proof, Alibert and Thomas (1991) proposed a method of proving that is similar to the method of analysis, probably without considering the history of mathematics. The teaching experiment with this method that the teacher in this example carried out lasted for many years. We report on only briefly this experiment; for a lengthier account, see Somaglia (1998). At the beginning of the lesson, the teacher presents herstudents with the method of analysis in the field of Euclidean geometry. Students experience the application of this method in different problems until the method is mastered and recognized as a tool for attacking geometrical problems. Afterward, the teacher has the students apply the method to other parts of mathematics (algebra and calculus) so that they become aware of the transversality of the method (i.e., that the method is not linked to a particular domain of knowledge but can be generalized).

Students are then ready to attack problems in physics and in chemistry using this method (see Clavarino & Somaglia, 2001). In the description of her work, the teacher never mentioned any parallel between the strategies of her students and those of past mathematicians, nor the persistence of errors. In our experience, this fact is unusual among teachers dealing with mathematics history. There are two developments in the work of this high school teacher, the historical and the educational, that interact, and her way of looking at these processes is very positive. The teacher looks for what can give students the means to realize the condensation of concepts (see Sfard, 1991). This teacher has an excellent knowledge of mathematics history, and moreover it is quite natural for her to work with original sources.

Thus, history is an integral part in her mathematics teaching. Her contact with the past is not that of someone who looks at the past with the eyes of the present but one who sees the concepts of the past as real and important content—as foundations in an architectonic sense—upon which our modern concepts and methods are based. She puts in action Gadamer’s way of looking at the past, that is, as “a dialogical process in which two horizons (the past and the present) are fused together” (Radford, 1997a, p. 27).

7. THE RECOURSE TO HISTORY IN CONTEMPORARY MATHEMATICS EDUCATION

In the previous sections, we discussed some interpretations of recapitulation law made by past mathematicians and teachers. Let us now examine a few examples of contemporary mathematics educators, confining our discussion to two specific cases.

The first emphasizes (mainly although not exclusively) a theoretical interest. The second appears closer to specific contexts arising from the needs to enhance teaching and learning processes in mathematics instruction. In the first case, the history of mathematics appears as a theoretical tool to understand developmental aspects of mathematical thinking. The purpose of the second case is to facilitate, through explicit pedagogical interventions, students’ learning of mathematics by attempting to relate the development of students’ mathematical thinking to historical conceptual developments.

7.1. The Interface Between History and Developmental Aspects of Mathematical Thinking

The work of Sfard (1995) provides a clear example of contemporary views on the relation between history and the developmental aspects of mathematical thinking. She analyzed the development of algebra by blending historical and psychological perspectives. At the beginning of her article, she claimed that

> there are good reasons to expect that,

when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities. At least, this is what follows from the constructivist view according to which learning consists in the reconstruction of knowledge. (p. 15)

The similarities between the phylogenetic and ontogenetic domains result in this account from "inherent properties of knowledge." For Sfard, who follows a Piagetian epistemological perspective, knowledge can be theoretically described in terms of genetic structural levels, and it is precisely the nature of the relationship between the different levels that accounts for the similarity of phenomena appearing in the historical and in the individual's construction of knowledge. As she noted, "difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians" (Sfard, 1995, pp. 15–16). A large part of the text is devoted to the discussion of the development of algebraic language. Indeed, using Nesselmann's (1842) distinction between rhetorical, syncopated, and symbolic algebra, Sfard endeavored to locate those "constants" (more precisely, those "developmental invariants") that ensure the passage from rhetorical and syncopated algebra to symbolic algebra. Rhetorical algebra refers to the reliance on nonsymbolic, verbal expressions to state and solve a problem, as it appears, for instance, in Arabic, Hindu, and Italian Medieval texts. Syncopated algebra is seen as a more elaborate algebra in that, although still relying heavily on verbal expressions, it introduces some symbols, the work of Diophantus being the canonical example. Vi`ete's systematic introduction of letters epitomizes symbolic algebra.

After confronting experimental classroom results with the traditional view of the historical development of algebra, Sfard concluded that one of the development invariants underpinning the passage from rhetorical and syncopated algebra to symbolic (Vietan) algebra is the precedence of operational over structural thinking. Operational thinking, in this context, means a way of thinking about algebraic objects in terms of computational operations. Structural thinking is related to more abstract objects conceived structurally on a higher level.

As we can see, the use of history in Sfard's approach is an attempt to corroborate parallelisms between ontogenetic and phylogenetic developments. As she said, "history will be used here only to the extent which is necessary to substantiate the claims about historical and psychological parallels" (Sfard, 1995, p. 17). Although she stressed the importance for teachers to be aware of the historical development of mathematics, the intention is not that of creating a historically inspired classroom activity. This is the goal of another perspective in contemporary mathematics education, discussed in section 7.2. For the time being, we want to mention a sociocultural approach that shares Sfard's use of history for epistemological reasons but, in contrast, emphasizes the crucial link between cognition and the practical human activity in which cognition is embedded. This approach (see Radford, 1997a; Radford et al., 2000), inspired by key ideas of the Vygotskian and cultural perspectives alluded to in section 3 of this chapter, is driven by a conception of knowledge that differs from Piagetian genetic structuralism, particularly in that knowledge and the individuals' intellectual means to produce it are seen as intimately and contextually related to their cultural setting. Knowledge, in fact, is conceived as the product of a *mediated cognitive reflexive praxis* (see Radford, 2000b). The mediated character of knowledge refers to the role played by artifacts, tools, sign systems, and other means to achieve and objectify the cognitive praxis. The reflexive nature of knowledge is to be understood in Ilyenkov's sense, that is, as the distinctive component that makes cognition an intellectual reflection of the external world in the forms of the individual's activity (Ilyenkov, 1977, p. 252). Knowledge as the result of a cognitive praxis (*praxis cogitans*) emphasizes the fact that what we know and the way we come to know it is framed by ontological stances and by cultural meaning-making processes that shape a certain kind of rationality out of which specific kinds of mathematical questions and problems are posed.

Theoretically, however, this does not mean that the study of knowledge is determined by social, economical, and political factors because these are also historically produced. Certainly, the link between culture and cognition is more subtle than the distinction between the "internal" and "external" realms employed in many historiographic approaches that see the external as mere stimulus for conceptual changes and development. Methodologically, this means that the study of the historical development of mathematics cannot be reduced to the sociology of knowledge. This also means that such a study cannot be done through the analysis of texts only. The "archive" (to borrow Foucault's expression), as a historical repository of previous experiences and conceptualizations, bears the sediments of social, economic, and symbolic human activities. Therefore, understanding the rationality within which a mathematical text was produced requires relocating the text with in its own context.

The goal of this kind of epistemological reflection is not to find a parallel between phylogenetic and ontogenetic developments. In the sociocultural approach that we advocate, mathematical texts from other cultures are investigated while taking into account the cultures in which they were embedded. This allows the researcher to scrutinize the way mathematical concepts, notations, and meanings were produced. Through an oblique contrast with the notations and concepts taught in contemporary curricula, we seek to gain insights about the intellectual requirements that learning mathematics demands of our students.We also seek to broaden the scope of our interpretations of classroom activities. In designing classroom activities, we aim at eventually adapting conceptualizations embedded in history to facilitate students' understanding of mathematics. Our work on Babylonian algebra and the teaching of second-degree equations (Radford & Gu ́erette, 2000) is an example of the latter. Our classroom research on the strategies students use to deal with the algebraic generalization of patterns and the way they conceive relations between the concrete and the abstract (see Radford, 1999b, 2000c)—research based on our investigation of pre- and Euclidean forms to convey generality (Radford, 1995a)—is an example of oblique contrast between past developments and contemporary students' conceptualizing processes.

Our classroom research on the introduction of algebraic symbolism also benefited from our epistemological inquiries based on editions of original texts from Medieval and Renaissance Italian mathematics (Radford, 1995b, 1997b). Space constraints do not allow us to go further, but this anthropological approach to the epistemology of mathematics offers a new view of the rise of symbolic algebra in the 16th century. The difference from traditional views stressing the passage from syncopated to abstract algebra in terms of abstractive processes is that, in our account, changes in development are related to changes in societal practices and the way in which mathematical conceptualizations are subsumed in them. Briefly, what we find in our analysis is that there were two main mathematical practices in the early Renaissance, that used by merchants and abacus mathematicians and that used by humanists and court mathematicians. While the latter were busy with the restoration of Greek texts, the former were applying Arabic algebraic techniques to practical as well as nonpractical problems (e.g., problems about numbers). Symbolic algebra was a timeconsuming effort made by Italian humanist and engineer mathematicians, such as the priest Francesco Maurolico, who eradicated all commercial content in his *Demonstratio Algebrae*, which was completed October 7, 1569 and edited by Napoli in the 19th Century (Napoli, 1876). Another example is the engineer Rafael Bombelli, who, after having learned that the first books of Diophantus' *Arithmetic* were on the shelves of a Roman library, studied them and ended up eliminating the commercial problems in his *Algebra*. Bombelli provided a final version of it that conformed much more to the humanist understanding of Greek mathematics. In France, a similar effort was made by the humanists Jacques Peletier and Guillaume Gosselin (although in this case, the promotion of French as a scientific language was an important drive; Cifoletti, 1992).

The underlying reason for the effort to introduce a specific symbolism in algebra was not due to the limitations of vernacular language. Mathematicians working within the possibilities offered by rhetorical algebra produced many difficult problems involving several unknowns, as can be seen in Fibonacci's *Il Flos* (Picutti, 1983). These problems could not be simplified by the introduction of letters because what was symbolized in the emergence of symbolic algebra did not include all of the unknowns mentioned in a problem but only one of them. (See, for instance Bombelli's symbolism or the neogeometrical example in Piero della Francesca's *Trattato d'abaco*, edited by Arrighi, 1970.) It was only later that some in Germany began using letters for several unknowns (see Radford, 1997b). In our approach, the emergence of algebraic symbolism appears to be related to the effort made by humanists and court-related mathematicians to render the merchant's algebra noble and Court worthy (details in Radford, 2000b). This was accomplished by the lawyer and mathematician François Viète, at the French court, who followed the prestigious Greek traditions typified by Diophantus' *Arithmetic* rather than the multitude 15th- and 16th-century of abacus treatises.

We now discuss a second reference to the use of history in contemporary mathematics education, that which aims at enhancing, through explicit pedagogical interventions the students' learning of mathematics.

7.2. Enhancing Students' Mathematical Thinking Through Historically Based Pedagogical Actions

Boero and collaborators (see Boero, Pedemonte, & Robotti, 1997; Boero, Pedemonte, Robotti,&Chiappini, 1998) made use of the mathematics history to investigate the nature of theoretical knowledge and the conditions by which it emerges. Their historicoepistemological analysis aims at looking for elements considered typical of mathematical thinking, such as organization, coherence, and systematic character. They have investigated the role played by definitions and proofs, as well as by the type of theoretical discourse. The framework draws from Bakhtin's theory of discourse, mainly from the theoretical construct of "voice"

(Bachtin, 1968, Wertsch, 1991) and from Vygostky's distinction between scientific and everyday concepts (Vygotsky, 1962).

The historico-epistemological inquiry is subsequently invested in the design and implementation of teaching settings based on a careful selection of primary sources of which the main objective is to allow the students to echo the voice of past mathematicians. In the students' echoing process, the students bring their individual subjective and cultural backgrounds to build from it a "voices and echoes game," which proves to be fruitful for the acquisition of theoretical knowledge. The voices from the past are not listened to passively but actively appropriated through an effort of interpretation. Usually the students' echoes may take various forms. Boero and his team have provided a categorization of some of the ways in which the students enter the dialogical game. For instance, a "mechanical echo" consists in precise paraphrasing of a verbal voice, whereas an "assimilation echo" refers to the transfer of the content and method conveyed by a voice to other problem situations. A "resonance" is a student's appropriation of a voice as a way of reconsidering and representing his or her experience.

Among the concrete instance of theoretical knowledge examined by the authors are the theories of the falling bodies of Galileo and Newton, Mendel's probabilistic model of the transmission of hereditary traits, and theories of mathematical proof and algebraic language, all of which feature aspects of a counterintuitive character.

Another example of the contemporary use of history in the classroom is the research of Sierpinska and collaborators. One of the goals of this research is to provide an alternative, based on the use of the Cabri-G ́eom`etre software, to the traditional axiomatic approach to the teaching linear algebra in undergraduate courses. A problem examined in this research, which underlies important aspects of the learning of basic linear algebra, is that of understanding key differences in the representations of mathematical objects. In this line of thought, Sierpinska has emphasized the distinction between a "numerical" and "geometrical" space. The objects of the arithmetic spaces are sets of n-tuples of real numbers defined by conditions (in the form of equations, inequalities, etc.) on the terms of the n-tuples belonging to the sets. It stresses the fact that these objects can be represented by geometric figures (e.g., lines, surfaces). Geometric objects, in contrast, are defined as a locus of points verifying some conditions (e.g., the "geometric circle" means the locus of points equidistant from a given point). The geometric objects can be represented by sets of n-tuples defined by conditions on their terms (e.g., by equations). Thus, in the case of arithmetical spaces, the geometrical aspect is derived from the numerical one; in the case of geometrical spaces, the numerical aspect results from the geometrical one. A suitable understanding of elementary linear algebra requires the students to establish a convenient relation between the geometrical and the numerical views of the objects of linear algebra and to grasp that the roles of objects and representations are reversed.

The difference between geometrical and numerical space is clear in the history of linear algebra. Sierpinska, Defence, Khatcherian, and Saldanha (1997) identified three modes of reasoning, which they labeled "synthetic-geometric," "analytic-arithmetic," and "analytic-structural." As they noted (a more detailed report is in Bartolini Bussi& Sierpinska, 2000), the concepts of linear algebra do not all have the same meaning and, in the classroom, they are not equally accessible to beginning students. The design of the teaching activities as well as the understanding of students' answers took into account the modes of reasoning as determined in the historico-epistemological analysis. (An extended account of the teaching activities can be found in Sierpinska, Trgalová, Hillel, & Dreyfus, 1999a and Sierpinska, Dreyfus, & Hillel, 1999b.)

8. SYNTHESIS AND CONCLUSION

In this chapter, we dealt with one of the many uses of the history of mathematics in mathematics education, namely, a use that can be characterized as an attempt to investigate historical conceptual developments to deepen our understanding of mathematical thinking and to enhance the students' conceptual achievement. In the first part of the article, we saw how psychological recapitulation was imported from biological recapitulation and gave rise to a discourse that framed much of the discussions about child development since the beginning of the 20th century. Psychological recapitulation was adopted by some eminent mathematicians who, in one form or another, supported the idea that in developing their mathematical thinking, children would traverse similar steps as those followed by humans. Within this conception, children will supposedly find during their development some similar problems, difficulties, or obstacles as those encountered by past mathematicians. Recapitulationism, we argued, served the cause of some mathematicians as a means to counter the teaching orientation based on commitments to rigor and logical structures arising in the flow of the research on the foundations of mathematics at the turn of the 20th century.

Nonetheless, one of the problems with the recapitulationist approach is that conceptual developments are seen as chronologically self-explanatory, and psychological evolution is taken for granted. Furthermore, knowledge is conceived as having little (if any) bond to its context, and the idea of history is reduced to a linear sequence of events judged from the vantage point of the modern observer. In all likelihood, the extremely low number of studies that attempt to check the validity of recapitulation law is evidence of the impossibility of reproducing the conditions in which ideas developed in the past. As Dorier and Rogers noted, "'naive recapitulationism' has persisted in many forms and now we accept that the relation between ontogenesis and phylogenesis is universally recognized to be much more complex than was originally believed" (Dorier & Rogers, 2000, p. 168).

This statement corresponds well with recent nonpositivist epistemological and anthropological trends.Indeed, in emphasizing the relation between knowledge and social practices, these trends have raised some criticisms to the acultural stance conveyed by the general and universal character of the recapitulation law, thereby opening new ways to reconceptualize the relations between historical conceptual developments and the teaching of mathematics. In the course of our discussion, we mentioned two different and critical stances toward the relation between ontogenesis and phylogenesis as elaborated by Piaget and Garcia on one hand and by Vygotsky and his collaborators on the other. The way Piagetian and Vygotskian epistemologies have inspired current work on contemporary mathematics education was made clear in the brief presentations of specific traits in the works of Sfard, Radford, Boero, and Sierpinska, works that attempt to contrast (with different purposes and in different senses) ontogenetic and phylogenetic developments to shed light on the nature of mathematical knowing as well as on the teaching and learning of mathematics.

Regarding recommendations for future research, it can be suggested, in light of the previous discussion, that a pedagogical use of the history of mathematics committed to enhance students' conceptual achievements requires a critical reflexion about the conceptions of ontogenesis and phylogenesis and, of course, of knowledge itself. But to be fruitful in practical terms, such a critical reflexion must be clear about its classroom implications. In particular, efforts to include teachers in the reflexive enterprise must be made. The work of Furinghetti suggests that to reach effectiveness in using history, teachers' willingness is not enough. To use history productively, teachers need to gain an appropriate understanding of differences between ontogenetic and phylogenetic developments and to bear a critical stance toward recapitulation views. As the sophisticated methodology of Boero's approach suggests, this requires teachers to be amply comfortable in handling cognitive and historical aspects. Let us make three suggestions concerning actions for research.

1. Ona theoretical level, discussions about recapitulation and its different meanings should be promoted among historians, epistemologists,psychologists,anthropologists and mathematics educators.

2. On a practical level, models of contrasts and conceptualizations between ontogenetic and phylogenetic developments also should be considered further. Models of contrast may help us to better grasp specific traits of mathematical thinking, its relation to the cultural settings, and the mathematical concepts thus produced. This can lead to a better understanding of the kind of practical pedagogical interventions that can be envisioned.

3. Theoretical reconceptualizations of recapitulation and contrasts and comparisons between ontogenetic and phylogenetic domains should be explicit as to how they can frame the engineering of material and teaching sequences. We consider these related research topics as being interactively fed by theoretical enquiries, historical studies, and also classroom observations.

The course of the three aforementioned actions for future research will ultimately depend on the very conception of mathematical knowledge to be adopted. At this point, two main contrasting trends seem to be emerging. In the first trend, what makes the specificity of mathematical knowledge is its systemic, objective, and logical nature (see Fujimura, 1998). In the second trend, which is much more anthropologically driven, knowledge is conceived as a kind of culturally framed activity enabling individuals to enquire about their world and themselves. Here "systematicity" and "logicality" are seen as circumscribed characteristics of knowledge that can be different from culture to culture (see Radford, 1999c). Between them, of course, many possibilities can be envisaged. To theoretically elaborate on some of those possibilities, to build practical and conceptual reflexions about historical and contemporary "developments," and to deepen our understanding of mathematics and facilitate the way students learn is a challenge for the years to come.

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研究综述

数学史与数学教育研究综述

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早在 19 世纪，数学史与数学教育之间的关系已经受到欧美数学家和数学教育家的关注。1972 年，在英国 Exeter 召开的第二届国际数学教育大会上，成立了数学史与数学教学关系国际研究小组（*International Study Group on the Relations between History and Pedagogy of Mathematics*, 简称 HPM），1976 年开始隶属于国际数学教育委员会。自此，数学史与数学教育关系（通常我们也称之为 HPM）成了数学教育的重要学术研究领域之一。

HPM 领域的研究工作主要包括以下几个方面。

1 关于“为何”的探讨

数学教学中为什么要运用数学史？欧美学者早在 19 世纪就有讨论。法国数学家泰尔凯（O. Terquem, 1782～1862）、英国数学家德摩根（A. De Morgan, 1806～1872）、丹麦数学家和数学史家邹腾（H. G. Zeuthen, 1839～1920）等都强调数学史的教育价值。（汪晓勤，2001；汪晓勤，2002；汪晓勤等，2003；赵瑶瑶等，2007）HPM 先驱者、美国数学史家卡约黎（F. Cajori, 1859～1930）指出，一门学科的历史知识乃是“使面包和黄油更加可口的蜂蜜”，“有助于使该学科更具吸引力”（Cajori, 1899），能够激发学生学习兴趣、使他们树立正确的价值观。他在《数学史》前言里指出，通过数学史的介绍，“教师可以让学生明白：数学并不是一门枯燥呆板的学科，而是一门不断进步的生动有趣的学科。”（Cajori, 1911）另一位 HPM 先驱者、美国数学史家和数学教育家史密斯（D. E. Smith, 1860～1944）则认为，数学史展现了不同方法的成败得失，因而今人可从中汲取思想养料，少走弯路，获取最佳教学方法。他指出：

> “若要考虑该学科（数学）的任何改革问题，就不可不知道几何如何演变为现今的形式；若要理解今天所提倡的改进教学的多种方法，就必须知道早期解方程的几何方法；若要将微积分从现今的地位中挽救出来，知道微积分的早期历史同样很重要。这些只是数学史对教学的无数教训中的几个例子而已。”（汪晓勤，2010）

以后的许多美国学者对数学史的教育价值都有讨论（汪晓勤，2004）。

HPM 成立之后，西方学者对数学史的教育价值进行了更为广泛的探讨。Fauvel（1991）总结了数学教学中运用数学史的各种理由，共有 15 条：

（1）增加学生的学习动机；

（2）改变学生的数学观；

（3）因为知道并非只有他们自己有困难，因而得到安慰；

（4）使数学不那么可怕；

（5）有助于保持对数学的兴趣；

（6）给予数学以人文的一面；

（7）有助于解释数学在社会中的作用；

（8）有助于发展多元文化进路；

（9）历史发展有助于安排课程内容顺序；

（10）告诉学生概念如何发展，有助于他们对概念的理解；

（11）通过古今方法的对比，确立现代方法的价值；

（12）提供探究的机会；

（13）过去的发展障碍有助于解释今天学生的学习困难；

（14）培养优秀生的远见卓识；

（15）提供跨学科合作的机会。

Tzanakis 和 Arcavi（2000）从数学学习、关于数学本质和数学活动观点的发展、教师的教学背景与知识储备、数学情感、对数学作为文化活动的鉴赏等五个方面总结了数学史对支持、丰富和改进数学教学的作用。Jankvist（2009）则将数学史对于数学教学的作用分成“工具”和“目标”两类。

2 关于“如何”的讨论

Fauvel 和 van Mannen 曾指出：“对于数学史引入数学教学的研究，乃是数学教学研究的重要组成部分。”（Bagni, 2000）HPM 成立以来，特别是 20 世纪 80 年代以来，许多数学教育家、数学教师对于数学史在数学教学上的具体运用方法作了理论探讨。如何将数学史融入数学教学，如今已成了近年来国际上 HPM 研究者们关注的中心课题之一。

Fauvel（1991）总结了数学教学中运用数学史的方式，共有 10 种：

（1）介绍历史上数学家的故事；

（2）运用数学史引入新概念；

（3）鼓励学生理解以所学概念为答案的数学史问题；

（4）讲授“数学史”课；

（5）利用历史上的数学教材设计课堂练习和作业；

（6）举办数学历史主题的展览；

（7）运用历史上的主要例子来说明方法和技术；

（8）探索过去的错误、另类观点以帮助今天的学习者理解并解决困难；

（9）借鉴历史发展设计一个话题的教学方法；

（10）基于历史信息设计大纲范围内主题的顺序和结构。

Tzanakis 和 Arcavi (2000)总结了数学史在数学教学中的三种运用方式：一是提供直接的历史信息；二是借鉴历史进行教学，即发生教学法；三是开发对数学及其社会文化背景的深刻意识。而 Jankvist (2009)则提出另三种方式：启发法、模块法和基于历史法。

我们将各种分类方法进行整合与改进，得到附加式、复制式、顺应式和重构式四类，见表 1。

表 1　数学教学中运用数学史的方式

类 别	描　述	Fauvel	Tzanakis & Arcavi	Jankvist
附加式	展示有关的数学家图片，讲述逸闻趣事等，去掉后对教学内容没有什么影响	方法 1	直接运用法	启发法
复制式	直接采用历史上的数学问题、解法等	方法 2-8	直接运用法	启发法
顺应式	根据历史材料，编制数学问题	—	—	—
重构式	借鉴或重构知识的发生、发展历史	方法 9-10	间接运用法	基于历史法

如，在对数概念的教学中，讲述对数发明者纳皮尔（J. Napier, 1550 ~ 1617）和常用对数发明者布里格斯（H. Briggs, 1561 ~ 1630）的故事（汪晓勤，2002b），即属于附加式。在引入复数概念时，抛开方程 $x^2+1=0$，而直接采用德国数学家莱布尼茨（G. W. Leibniz, 1646～1716 ）的二元二次方程组问题：已知 $x^2+y^2=2$，$xy=2$，分别求 $x+y$，x，y。（张小明等，2005）此即属于复制式。在数列概念的教学中，将毕达哥拉斯学派的多边形数理论改编成供学生探究的问题，即属于顺应式。

重构式是数学史最高层次的用法，发生教学法即属于该方式。托普利茨（O. Toeplitz,

1881～1940）曾指出，发生法的本质是追溯一种思想的历史起源，以寻求激发学习动机的最佳方式。（Edwards, 1977）但追溯历史起源、重演历史发展并非原原本本地、精确地复制历史，而是借鉴历史、重构历史。原原本本的历史往往很复杂，而发生法所重构的历史却是线性的。发生法强调知识的自然发生过程，即教学必须建立在学生已有的认知基础之上；同时也强调知识的必要性，即教学必须激发学生的学习动机。例如，通过借鉴椭圆知识的历史，基于古希腊的原始定义（截线定义），用圆柱中的旦德林球来引出椭圆的焦半径性质，（汪晓勤等, 2011；陈锋等，2012）即属于这种方式。

3 教育取向的数学史研究

早年，卡约黎曾研究过如下问题：未知数为什么用 x 来表示？指数记号是如何演进的？谁最早使用了数学归纳法？“数学归纳法”之名是如何产生的？“对数”之名是怎么来的？为什么等差和等比级数又叫算术和几何级数？这些问题具有明显的教育取向。教育取向的历史研究主要通过对数学课程中的概念、公式、定理、问题的历史进行研究，不是为历史而历史，而是为教育而历史。这是 HPM 研究的基础性工作，如果我们不了解一个概念、公式或定理的历史，就无从谈论概念理解的历史相似性以及借鉴历史的概念教学。

美国数学教师协会早在 1969 年就组织数学史家和数学教育家编写了《用于数学课堂的历史话题》（Hallerberg *et al*, 1969），供数学教师使用。Gulikers 和 Blom（2001）在有关几何历史与教学的文献综述中，教育取向的历史研究占了相当大的比例。笔者亦曾致力于这方面的工作（如汪晓勤，2002）。

HPM 为历史研究提供了丰富的问题。例如，“三角公式的几何渊源”、“古代文献中的数列问题”、“用字母表示数的历史”、“平方差公式的历史”等等课题无不源于 HPM 教学实践的需要。

数学教育取向的数学史研究的另一目的是获取相关知识点（概念、公式、定理等）的教学启示。卡约黎的《初等数学史》就是早期的例子。如，卡约黎根据负数的历史得出结论：“在教代数的时候，给出负数的图形表示是十分重要的。如果我们不用线段、温度等来说明负数，那么现在的中学生就会与早期代数学家一样，认为它们是荒谬的东西。”（Cajori, 1917）M 克莱因（M. Kline, 1908～1992）基于数学史的研究，对美国的“新数运动”提出尖锐的批评（Kline, 1958），并强调直觉对于数学理解的重要性（Kline, 1970）。

类似地，Malik（1980）通过对函数概念的历史考察获得启示：中学阶段应该教简单易懂的函数概念；Ponte（1993）通过对函数历史的考察获得启示：在中学，将函数定义为数集之间的对应关系是合适的；在中学数学中必须强调具有解析式的函数例子。Filep（2001）通过对分数概念历史考察，获得教学启示：分数概念时的引入必须与度量联系起来，而不是

两数相除。

数学史是一个宝藏，不论时代如何变迁，数学教育工作者总是可以并且也有必要从中汲取有益的思想养料。

4 历史相似性研究

所谓历史发生原理，指的是个体数学理解的发展遵循数学思想的历史发展顺序，这就是我们通常所说的“历史相似性”。对发生原理的认识可以上溯到 19 世纪法国哲学家孔德（A. Comte, 1798～1857）的著述。孔德认为，“个体教育必然在其次第连续的重大阶段，仿效群体的教育，在感情上如此，在思想上也是如此。”（孔德，1999）英国教育家斯宾塞（H. Spenser, 1820～1903）据此指出：

> “对儿童的教育在方式和顺序上都必须符合历史上人类的教育，换言之，个体知识的发生必须遵循人类知识的发生过程。…因此，在确定正确的教育方法时，研究一下文明中的方法，有助于为我们提供指南。”（Spencer, 1862）

发生原理又受到德国生物学家海克尔（E. Haeckel 1834～1919）所提出的生物发基本定律——“个体发育重演种族发展”的支持。（Radford, 2000）

早期学者似乎坚信个体数学理解过程与数学思想的历史发展之间存在严格的相似性。如，马塞（J. Macé, 1815～1894）认为，儿童的学习一如人类的学习，“尽管起点离我们已经很远，但人类漫长的教育在每一个儿童身上又会重新开始。”（Mace, 1862）布兰福德（Benchara Branford）则指出：“为教育之目的，几何学最有效的讲授方式乃是遵循科学历史演进的顺序”。（Branford, 1921）他还将人类与个体数学经验的发展进行了对比。

卡约黎和史密斯则从学生学习困难的角度来诠释历史发生原理。卡约黎指出：“学生所遭遇的困难往往是相关学科的创建者经过长期思索和探讨后所克服的实际困难。”（Cajori, 1899）而史密斯则认为：“困扰世界的东西也会困扰儿童，世界克服其困难的方式提示我们，儿童在其发展过程中会以类似的方式来克服类似的困难。”（Smith, 1900）美国数学家和数学史家 M 克莱因（M. Kline, 1908～1992）的观点与卡约黎和史密斯一脉相承：“历史上数学家所遇到的困难，正是学生也会遇到的学习障碍，因而数学史是教学的指南。”（汪晓勤，2004a）

F. 克莱因（F. Klein, 1849～1925）、庞加莱（Poincare, 1854～1912）、波利亚（G. Polya, 1887～1985）、弗赖登塔尔（H. Freudenthal, 1905～1990）等都是历史发生原理的支持者，当然，他们已经摆脱了严格相似性的狭隘观点。

克莱因（F. Klein, 1849～1925）指出：

> “生物发生学的一项基本定律指出，个体的成长要经历种族成长的所有阶段，顺序

相同，只是所经历的时间缩短；而教授数学和其他任何事情一样，至少在一般意义上要遵循这项定律。鉴于年轻人的本能，教学应慢慢将其引向更高级的事物，最终到达抽象形式。为此，教学应遵循人类从知识的原始状态到更高级形式的道路。……推广这种自然的真正科学的教学的主要障碍是缺乏历史知识。”（Klein, 1932）

这里，克莱因所说的借鉴生物发生定律、基于历史知识的“自然的”和“科学的”教学方法就是发生教学法。

庞加莱说：“教育工作者的任务就是让儿童的思维经历其祖先之所经历，迅速通过某些阶段而不跳过任何阶段；鉴于此，科学史应该是我们的指南。”（Poincaré, 1899）

波利亚指出：“只有理解人类如何获得某些事实或概念的知识，我们才能对人类的孩子应该如何获得这样的知识作出更好的判断。”（Pólya, 1965）

弗赖登塔尔（H. Freudenthal, 1905～1990）将人类看作一个“学习者”，其学习过程就是历史。他告诉我们：“数学史是一个图式化不断演进的系统化的学习过程，儿童无需重蹈人类的历史，但他们也不可能从前人止步的地方开始。从某种意义上说，儿童应该重蹈历史，尽管不是实际发生的历史，而是倘若我们的祖先已经知道我们今天有幸知道的东西，将会发生的历史。”（Freudenthal, 1980）

历史相似性的研究对数学教育具有重要意义。因为，如果某一概念的历史相似性得到检验，那么，我们可以参照历史来预测学生的认知障碍，从而有针对性地制订相关教学策略，最有效地让学生跨越学习障碍。

一些欧美学者就符号代数、角的概念、平面概念、数轴上序关系、函数极限等，对有关被试的理解进行了实证研究，印证了历史相似性的存在。Harper (1987) 通过对两所文法学校 1-6 年级 144 名学生的测试，发现学生对符号代数的理解过程经历修辞代数、缩略代数和符号代数三个阶段，具有历史相似性；Bagni (2000) 通过对 88 名 16-18 岁、尚未学过无穷级数概念的高中生的测试和访谈发现，就发散的无穷级数而言，历史发展与个体认知发展是相似的。

Keiser (2004)通过课堂观察和访谈调查发现，六年级学生对角的理解涉及“质”、“量”和“关系”三个方面，与古希腊数学家的理解具有相似性；Zormbala & Tzanakis (2004)通过对 51 名大学非数学专业毕业、从事各类社会职业的人员的调查，发现被试对平面概念的理解与历史上巴门尼德（Parmenides, 前 5 世纪）、海伦（Heron, 1 世纪）、莱布尼茨（G. W. Leibniz, 1646～1716）、辛松（R. Simson, 1687～1768）、高斯（C. F. Gauss, 1777～1855)、皮埃里（M. Pieri, 1860～1930）等数学家的理解具有相似性；Thomaidis & Tzanakis（2007）研究发现，学生对数轴上序关系的理解与历史上数学家的理解也存在一定的相似性。Juter (2006)的研究则表明，关于函数的极限，大一学生中只有成绩优秀的学生才表现出理解上的历史相似性。

类似地，针对学生对虚数、发散级数、函数、古典概率、切线、符号代数、数列极限等的理解，我们也进行了一些初步的实证研究（如汪晓勤等, 2005；汪晓勤等, 2006；任明俊等, 2007）。以切线为例。对鄂、苏、沪、皖四地 332 名高中生的调查表明：绝大多数高中生对切线的理解只达到古典几何阶段，他们只是根据公共点个数、直线与曲线相对位置或直线与圆半径位置关系来判别切线，与古希腊欧几里得、阿波罗尼斯、阿基米德等的理解具有相似性；绝大多数被试未能从特殊曲线的切线顺利过渡到一般曲线的切线，这也表现出高度的历史相似性，因为从古典几何阶段过渡到近代分析阶段，历史上的切线概念也经历了漫长而艰辛的过程。（殷克明，2011）

5 教学实践

关于数学史融入数学教学，迄今已积累了相当多各层次的实践案例，我们按改进后的分类法依次对其作一概述。

5.1 附加式

早在 1977 年，McBride 和 Rollins 就进行了一项为期 12 周的 HPM 教学实验。在每一次代数课上，教师用 5 分钟时间附加式地介绍数学史，结果发现，教学中使用数学史，有助于提高学生的学习数学积极性。（Marshall & Rich, 2000）

Brown (1991) 利用每节数学课的开头五分钟作“微型演讲”：向学生讲述数学故事、趣闻、数学家传记、数学名家名言、甚至数学小诗等。Gardner (1991)让小学生探究“风走得有多快”，借机讲述梅森（Merssene, 1588～1648）测声速的故事以及伽利略（G. Galilei, 1564～1642）测光速的实验。Ofir (1991) 在数学课堂中设计了以下活动：在七年级，介绍古代埃及、巴比伦、罗马的记数制度，并让学生将其与今天的记数制作比较；在八年级，介绍古埃及的乘法和分数的拆分、古代巴比伦人、古希腊阿基米德、中国刘徽以及犹太人求圆周率的方法和结果。Führer（1991）在他的数学课上多次使用三个故事：一是古希腊数学家埃拉托色尼（Eratosthenes）测量地球大圆周长，二是圆周率的历史，三是复数的历史。

5.2 复制式

Ransom(1991) 利用 1747 年拉丁文版《几何原本》第一卷命题 47 来讲授勾股定理，并从英国数学家波尼卡斯特（Bonnycastle）《测量与实用几何引论》中选取勾股定理应用问题供学生探究，如：

- 顶撑距建筑物直柱 12 英尺，支撑一离地 20 英尺高的门窗，求顶撑长度；
- 河岸悬崖高 103 英尺，自崖顶引线至对岸，线长 320 英尺，求河宽。

Perkins (1991) 在女子学校通过让学生解决历史上数学家觉得很难的概率问题来增加她们的学习自信心：

- 一次同时掷三颗骰子，比较 9 点和 10 点的概率大小。(伽利略与图斯坎尼大公)
- 一颗骰子掷 4 次，至少一次出现 6 点的概率，与两颗骰子掷 24 次，至少一次出现双 6 点的概率大小。（德 梅勒）
- 一次同时掷 6 颗骰子，至少出现一个 6 点的概率，与一次同时掷 12 颗骰子，至少出现两个 6 点的概率大小。（裴皮斯致牛顿）

同时，用 16 世纪初德国百科全书《知识明珠》（*Margarita Phylosophica*）中的两幅插图来引发学生对“女性与数学”这一话题的探讨。两幅插图中，各有一位优雅的女性，一个代表几何学，一个代表算术。Thomaidis (1991) 在平面几何课上向学生讲解几何定理在古代天文学上的应用，如托勒密定理是如何帮助天文学家亚里士塔丘（Aristachus）计算出月半圆时日地距离与月地距离的比值在 18 和 20 之间的。Van Maanen (1991)选择 17 世纪法国数学家洛必达（L’Hopital, 1661～1704）《无穷小分析》中的力学问题作为其所教文法学校毕业考试的题目。Chun Ip Fung 等人将刘徽《海岛算经》第一题、海伦的测量问题以及达 芬奇的“猫眼图”用于教学设计。（Fauvel & van Maanen, 2000）

Kool (2003) 选择 16 世纪荷兰数学教科书上的问题，对 20 名 11-12 岁的小学生进行了课堂教学实验：

- 甲从帕里希出发到安特卫普去，日行 8 英里；乙从安特卫普出发到帕里希去，日行 12 英里。两人走同一条路，且同时出发。帕里希和安特卫普之间的距离为 60 英里。问几天后甲乙二人相遇？（Peter van Halle, 1568）
- 如果忘记九九表中某两数的乘积，如何求出这个乘积呢？（Peter van Halle, 1568）
- 皮尔特对简说: “你 50 岁了！”“不”，简说，“我没到 50 岁。如果我取我年龄的一半，加上我年龄的 1/3，加上我年龄的 1/4，最后加上我的年龄，总数才是 50 岁。”问简的年龄多大？（Christianus van Varenbraken, 1532）

作者将原教科书上的解法称为“班级里的一名额外学生”。

Paola 将历史上著名的“点数问题”（或称“分配问题”）用于概率的教学，且在课堂上发现了学生解法的历史相似性（Furinghetti & Radford, 2002；Furinghetti & Paola, 2003）。Massa 等（2006）将古希腊天文学家亚里士塔丘《论日月的大小与距离》中的命题 7（即上文提到的日地、月地距离关系推导）、《几何原本》第一卷命题 47（勾股定理）以及雷格蒙塔努斯（Regiomontanus, 1436～1476）《论各种三角形》卷 1 命题 27（已知直角三角形三边，求各角）用于三角学的教学。

5.3 顺应式

Van Maanen (1992)利用荷兰数学家舒腾（F. van Shooten, 1615～1660）的圆锥曲线作图工具，编制高中毕业考试题目。

舒腾的其中一种椭圆作图工具如图 1 所示。短杆 *AB* 可绕 *A* 转动，*AB* = *a*；长杆 *BE* 通过 *B* 处铰链与 *AB* 连接，*BE* = *b*。*BE* 上的钉子 *D* 可沿 *KL* 移动，*AB* = *BD*。舒腾说，当 *D* 在 *KL* 上移动时，*E* 处的笔尖即画出了椭圆。

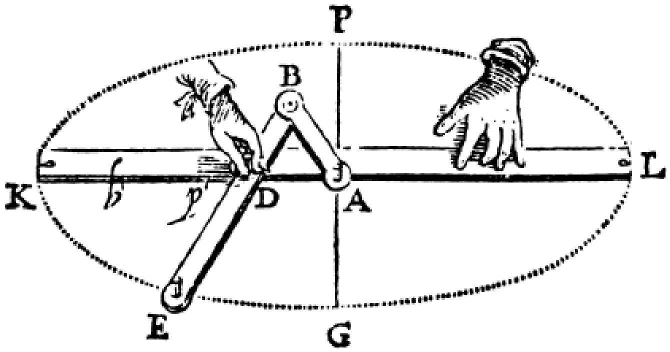


图 1 舒腾的椭圆作图工具

（1）若以 *A* 为原点，*KL* 为 *x* 轴，*BE* 与 *KL* 的夹角为 φ，用 *a*、*b* 和 φ 来表示点 *E* 的直角坐标。

（2）求点 *E* 的轨迹的直角坐标方程。E 的轨迹是否真的椭圆？

（3）舒腾的这一工具是否适合椭圆作图（像用圆规画圆一样）？

类似地，张小明老师利用我国古代数学中的基本立体图形“鳖臑”来编制如下问题（汪晓勤，2012）：

我国古代数学家对立体图形有深刻的研究，著名数学家刘徽在此方面取得了很大的成就，他发现：“邪解堑堵，其一为阳马，一为鳖臑，阳马居二，鳖臑居一，不易之率也”．这个结果被称为“刘徽原理”，其中鳖臑是指四个面都为直角三角形的四面体。在如图所示的鳖臑中，$\angle AOC = \angle BOC = \angle OBA = \angle ABC = 90°$，$AB = OC = 2$，$OB = a$ （$a > 0$），*F* 为线段 *OB* 上的动点，*E* 为 *DB* 的中点，问，当点 *F* 运动到什么位置时，直线 *AF* 与 *OE* 所成的角最小？

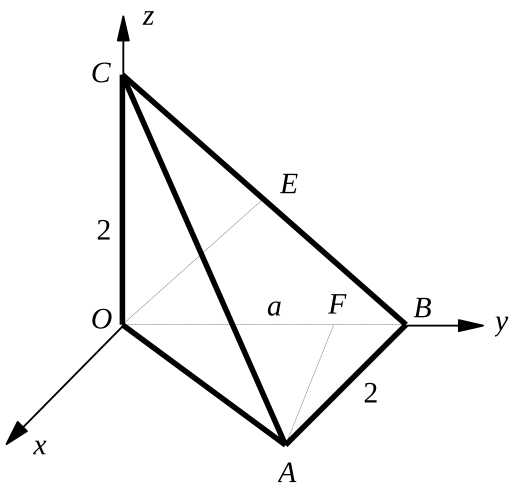


图 2　鳖臑问题

Bagni (2000)采用邦贝利（R. Bombelli, 1526～1572）《代数学》（1572）中的关于 1、-1、*i*、-*i* 的运算法则以及凯莱（A. Caylay, 1821～1895）的乘法表来引入群概念的教学，试图验证利用数学史来引入新概念教学的有效性。

Earnest (1998)让师范生利用数学史料编制作业单，结果，职前教师们设计出丰富多彩的问题，涉及卡洛儿钟表问题、古印度梵天塔问题、芝诺悖论、古埃及分数、拉马努金整数拆分问题、贾宪三角与二项式展开、古代记数制度、欧拉多面体公式、数学与音乐、九宫图、古罗马马赛克中的对称图案等，这些作业单大多为数学史的顺应式应用。

在针对师范生所设计的“倾听学生”工作坊上，Arcavi & Isoda (2007)将莱茵得纸草书上的“猫和老鼠”问题（等比数列问题）进行改编，让学生猜测象形文所表示的数字并复原古埃及的乘法运算。

5.4 重构式

早在 19 世纪，英国数学家德摩根（A. De Morgan, 1806～1871）已经对有关数学知识的演进进行过重构。以符号代数为例。（De Morgan, 1902）德摩根指出，代数学上一般数量关系的发现始于特例。如：两个数的和的一半加上它们的差的一半等于较大的数。首先，取 16 和 10，它们的和的一半为 13，差的一半为 3。13 和 3 相加，得 16，为较大数。上述结果对于其他数对也是成立的，如 27 和 8，15 和 9，等等。利用运算符号，可以发现以下事实：

$$\frac{16+10}{2}+\frac{16-10}{2}=16,$$

$$\frac{27+8}{2}+\frac{27-8}{2}=27,$$

$$\frac{15+9}{2}+\frac{15-9}{2}=15,$$

等等。但如何表达上述结果对于任何一对数都成立呢？我们将较大数称为第一数，较小数称

为第二数，就有

$$\frac{第一数+第二数}{2}+\frac{第一数-第二数}{2}=第一数$$

类似地，我们也可以得到其他等式，如

$$(第一数+第二数)\times(第一数-第二数)=第一数\times第一数-第二数\times第二数$$

但每次写“第一数”、“第二数”很麻烦，若用 x 表示“第一数”，用 y 表示“第二数”，则上面的恒等式可以写成：

$$\frac{x+y}{2}+\frac{x-y}{2}=x,$$

$$(x+y)\cdot(x-y)=x\cdot x-y\cdot y。$$

对于一个代数恒等式，从特例，到文字表达，再到字母表示，正是对符号代数发展过程的重构。

Radford & Guérette (2000)基于古巴比伦的“原始几何”方法，设计了一元二次方程求根公式的教学。整个设计共含五部分。

（1）几何方法之引入

教师先提出如下问题：“已知矩形的半周长等于 20，面积等于 96，问矩形的长和宽各为多少？”让学生分组合作，用任何方法来解决。学生完成后，教师在黑板上用纸版拼图来解释“原始几何”：取边长为 10 的正方形，其面积为 100。从中割去一个边长为 2 的小正方形，余下的图形的面积为 96。沿虚线割去一矩形，并将其竖直补到右边，于是得所求边长为 12 和 8。如图 3 所示。

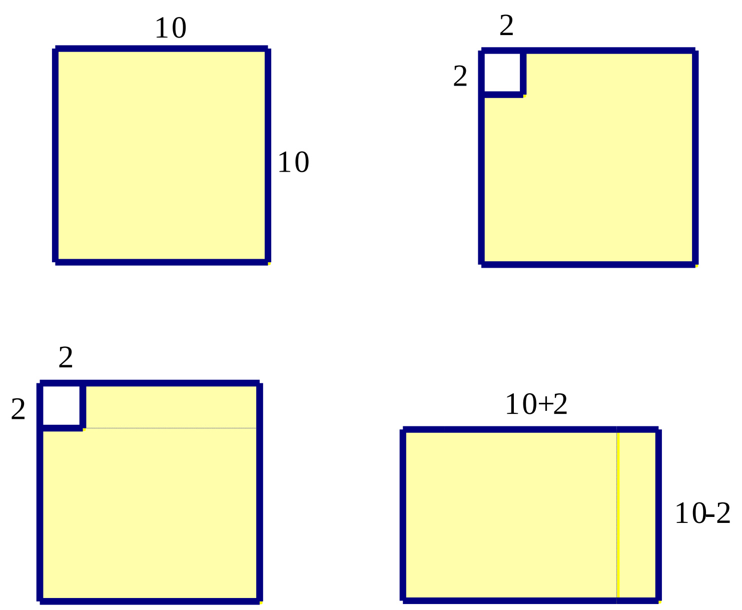


图 3 矩形问题的几何解法

然后，教师给出另一个类似问题："已知矩形面积为 30，半周长为 12。求矩形的长和宽。"（丢番图《算术》卷 1 问题 27）教师要求学生在第二次课上介绍解这类问题的步骤，要求步骤清晰，能让同年级另一个班级的任一个学生听懂。

（2）合作讨论与提出问题

小组讨论解题步骤，派代表向全班介绍本组讨论结果。接着，让学生自己提出类似的问题，要求：所求矩形的边长必须为整数；不一定为整数。

（3）新矩形问题

教师让学生解第二个问题："矩形长为 10，宽未知。在长度未知的边上做一正方形，正方形连同矩形的面积共为 39，求矩形的宽。"（花拉子米问题的几何形式）教师在黑板上用纸版拼图来解释解法：将原矩形一分为二，将其中之一粘到正方形底边上。补一边长为 5 的小正方形，则新的大正方形面积为 39+25=64。其边长为 8。故 x+5=8，x=3。

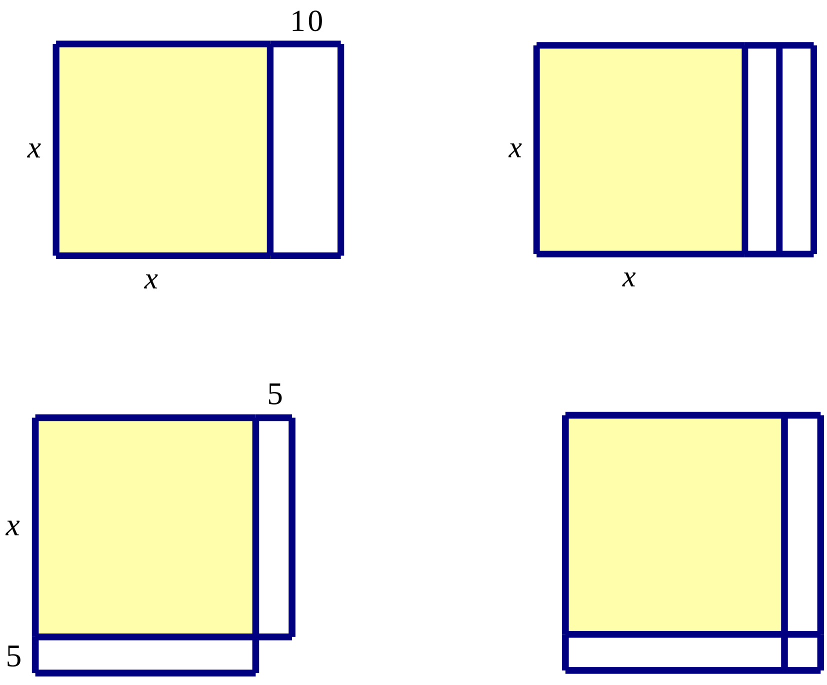


图 4　新矩形问题的几何解法

接着，教师给出类似问题，让学生分组合作解决。并要求他们写出解这类问题的一般步骤。

（4）合作讨论与提出问题

讨论学生所写的解题步骤。并让他们自己提出类似的问题，满足下列条件：（1）矩形的边长必须为整数；（2）矩形的边长必须为分数；（3）矩形的边长必须为无理数。

（5）求根公式之再发现

教师引导学生寻找解上面第三和第四部分中的问题的公式。教师提示：在学生所写的解

题步骤的第四步，用字母来代替文字，用字母“*b*”来表示矩形的底，“*c*”表示总面积。经小组合作讨论，得公式

$$x=\sqrt{c+\left(\frac{b}{2}\right)^2}-\frac{b}{2},$$

接下来，教师将几何问题翻译成代数语言：若未知的一边为“*x*”，则正方形面积为“x^2”，矩形面积为 *bx*；于是两者之和等于 *c*，即：$x^2+bx=c$。为将方程与上述公式联系起来，教师给出一些具体的方程（如 $x^2+8x=9$，$x^2+15x=75$）让他们运用公式来求解。

接着，教师引导学生找出方程 $ax^2+bx=c$ 的求根公式。学生可能会想到：方程两边同除以 *a*，即得前面讨论过的形式。只需在前面的公式中，用 *b*/*a* 代替 *b*，*c*/*a* 代替 *c* 即可：

$$x=\sqrt{\frac{c}{a}+\left(\frac{b}{2a}\right)^2}-\frac{b}{2a};$$

最后让学生考虑一般方程 $ax^2+bx+c=0$：只需用-c 来代替方程 $ax^2+bx=c$ 中的 c，即得。故在上述公式中用-c 代替 c，即得一般公式

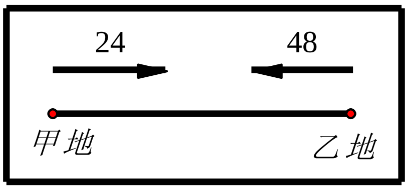
$$x=\sqrt{-\frac{c}{a}+\left(\frac{b}{2a}\right)^2}-\frac{b}{2a}，或 x=\frac{-b+\sqrt{b^2-4ac}}{2a}。$$

为了得到所有的根，需考虑 b^2-4ac 的负根，于是得公式 $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$。

Farmaki 等（2004; 2007）借鉴 14 世纪法国数学家奥雷姆（N. Oresme, 1323～1382）用图像表示运动的方法，设计了一类行程问题的教学。

问题 1：一人以每小时 24 km 的平均速度骑自行车从甲地往乙地。到达乙地后，立即回头，以每小时 18km 的平均速度骑往甲地 。整个旅程共用了 7 小时。则骑车者从甲地到乙地和从乙地到甲地分别用了多长时间？

传统方法：作出如下示意图：



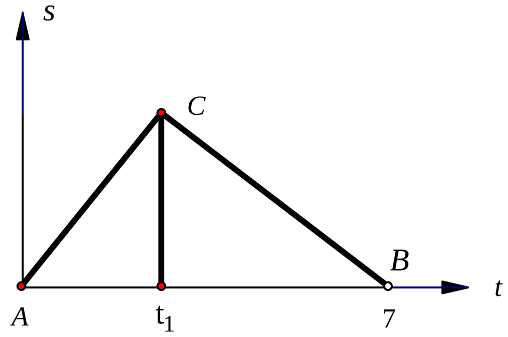
设从甲地往乙地所需时间为 *t*（小时），列出下表：

	时 间	速度	距离
往	t	24	24 t
返	7- t	18	18 (7-t)

于是得 $24t=18(7-t)$，故 $t=3$。

分段函数方法：如图，位移函数为 $s(t)=\begin{cases}24t & (0\le t\le t_1)\\ -18(t-7) & (t_1<t\le 7)\end{cases}$，故由

$24t=-18(t-\ )$ 得 $t=3$。



整体函数方法：

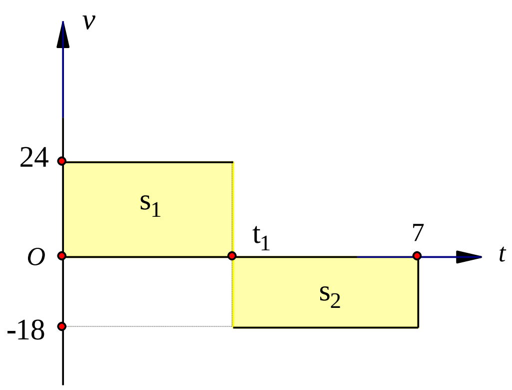
$$s=\begin{cases}24t, & t\in[0,t_1]\\ 24t_1-18(t-t_1), & t\in[t_1,7]\end{cases}$$

$$s(7)=0$$

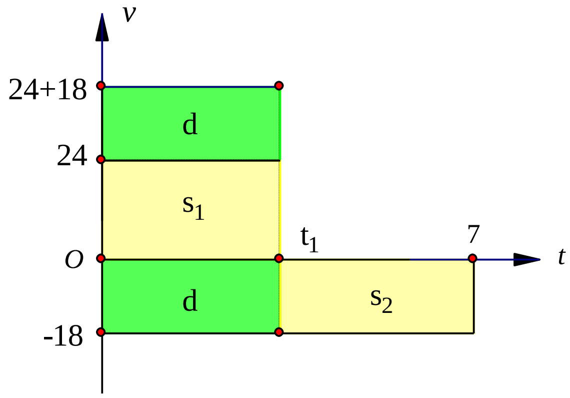
$$\Rightarrow 24t_1=18(7-t_1)$$

$$\Rightarrow t_1=3$$

重构式运用历史的方法：作出速度-时间图像如下：



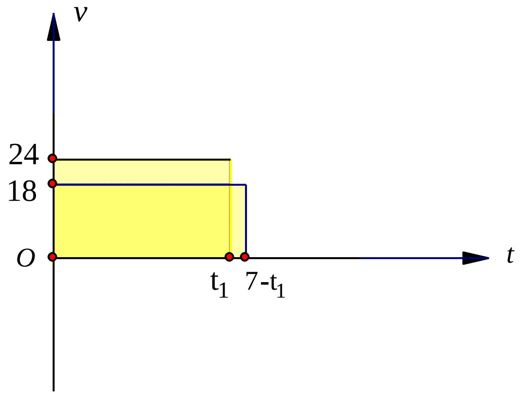
在两个阴影部分矩形上加一个共同的矩形，上图变为：



因 $s_1+d=42t_1, s_2+d=7\times18=126, s_1+d=s_2+d\Rightarrow t_1=3$。

但上述方法还可以简化，例如，将两个阴影部分矩形叠置，既可以得到

$6t_1=18(7-27_1)$，$t_1=3$。



接下来，让学生根据图形来描述运动问题，如根据上图，可提出如下问题：两人同时从甲地出发，分别以每小时 24 km 和 18 km 的平均速度骑自行车从甲地往乙地。甲用时 t_1，乙用时 t_2，若 $t_1+t_2=7$ 小时，则甲、乙两地的距离为多少？

类似的案例还有：Panagiotou(2011)借鉴对数的历史进行对数概念的教学设计，并将其付诸实践。

在台湾，基于数学史的教学设计的主要采用学习单的方式，已有案例涉及圆与圆周率、对数、三角函数、数学归纳法、曲线下的面积等等。

近年来，中国大陆也开始关注 HPM 视角下的数学教学。一个典型的例子是椭圆的概念与方程。椭圆的历史大致可以分成椭圆的发现、截线定义的形成、基本性质的推导、焦半径性质的获得、机械作图的产生、轨迹定义的确立以及椭圆方程的推导等七个重要环节，但教材只截取了最后三个环节，显然，所呈现的椭圆知识并非自然发生。鉴于椭圆历史的复杂性，

我们对椭圆历史进行了重构。从球的影子、建筑、水杯等现实例子出发将椭圆知识建立在生活经验基础之上；利用圆柱中的旦德林双球，推导出椭圆焦半径性质，从而实现了从古希腊截线定义到课本轨迹定义的自然过渡，并创造学生的学习动机。该设计在上海、浙江、新疆等地的实施都取得了理想的效果。（汪晓勤等，2011；陈锋等，2012）

6 结语

以上我们看到，在 HPM 领域，关于"为何在数学教学中运用数学史"和"如何在数学教学中运用数学史"的讨论很多，前者较为成熟，后者尚无定论，相关讨论还将继续下去。在我国，关于"为何"的讨论很多，而关于"如何"的讨论却很少。我们所提出的 "附加式"、"复制式"、"顺应式"和"重构式"四种方式基本上能够涵盖已有的实践案例，但有待于实践的进一步检验。在教育取向的数学史研究上，尽管西方文献丰富多彩，但可用于课堂教学的素材并没有我们想象的多。历史相似性研究案例仍然屈指可数，研究工具有待于开发，研究方法有待于改善。而在 HPM 教学实践方面，尽管西方学者作了很多尝试，但成功案例凤毛麟角，且绝大多数案例也并不适合我们的课堂。

因此，HPM 为我们留下了广阔的研究空间，在这里，我们将大有可为。

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数学文化

数学与游戏漫步

——mathematics out of the game

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这个小品文缘自 myself 在华师大的一些个数学文化课堂片段。其最初的目的是想谈谈二进制与数学游戏；待得心之所至，还是不由自主多加了两个小游戏，或许这或多或少破坏了"二进制的魅力"之完整性，但从某种意义上，可以更好地演绎数学与游戏这一主题。

我们的课堂开始于下面的这个简单的游戏。

1 你所"忘却"的那个数

请写下一个多位数，比如你的学号或者你的电话号码，然后把这组数随意组合进而得到一个新的数;其后作如下的运算:大数减去小数,再把所得之差数中的某一个数字随意去掉。如今只要你告诉我剩下的这些数字的和是多少,我即可猜得你去掉的数字是什么？

这个小游戏看似简单，却也折射着一个蛮深刻的数学理念:代数学的不变量理论...说的明确一点，其数学的秘密在于：由同一组数字随意组合而得到的两个数,其差数会是 9 的倍数。其详情可陈述如下:

不妨设最初的时刻有数 $\overline{b_1b_2\ldots b_n}$，由 $b_1,b_2,\ldots,b_n$ 这些数随意组合而得到的新数可记作 $\overline{b_{i_1}b_{i_2}\ldots b_{i_n}}$，则 $\overline{b_1b_2\ldots b_n}$ 和 $\overline{b_{i_1}b_{i_2}\ldots b_{i_n}}$ 的差是 9 的倍数。于是游戏的设计者只要知道余下的那些数的数字和离 9 的倍数多远，即可晓得你所"忘却"的那个数是什么。

一点注释：这是一个很有互动性的游戏,消磨时间在 10 --30 分钟之间，从某种意义上,或许不要详细地告诉此中的秘密，于是会有几许"围城效应"。这个游戏有一个小小的缺憾，若其上某一同学去掉的数字是 9 或 0，则你或许得先察言观色一番，再作计较。

课间的第 2 个游戏或可以有一个有点古怪的名字。

2 好朋友如何可以分离?

这个游戏需要如下的人物和道具: 课堂间的两个小朋友和两枚绳子~

在课上邀请两位同学来参与游戏, 如下图 1 所示... 问在不解开绳子和不剪断绳子的情境下如何让他/她们分离?

图 1

图 2

这个游戏的消磨时间在16到20分钟之间, 课上的一些教学实践表明, 在没有任何提示的情形下, 找寻到最后的"好朋友之分离"是一个"很"有挑战性的问题。掩映在这一游戏中的思维障碍或可通过上图 2 中的提示来打破。此游戏吐露着拓扑学的神奇。

这一课堂的第 3 个游戏画片是

3 一则数学的魔术: 猜电影

魔术道具的设计: 在图 3 中收藏有 31 部非常经典的电影...它们来自各个时代-你我他的梦想和追逐。

《简爱》	《音乐之声》	《老人与海》	《记忆碎片》	《罗马假日》	《莎翁情史》
《七宗罪》	《绿野仙踪》	《廊桥遗梦》	《天使之城》	《潘的迷宫》	《瓶中信》
《这个杀手不太冷》	《千与千寻》	《雨中曲》	《钢琴家》	《战争与和平》	《男人百分百》
《飞屋环游记》	《教父》	《小叛逆》	《摩登时代》	《卡莎布兰卡》	《罗密欧与朱丽叶》
《盗梦空间》	《沉默的羔羊》	《肖申克的救赎》	《辛德勒的名单》	《魔戒》	《西雅图夜未眠》
	《新龙门客栈》				

图 3

加上下面的 5 枚卡片：

《简爱》	《老人与海》	《罗马假日》	《七宗罪》
《廊桥遗梦》	《潘的迷宫》	《这个杀手不太冷》	《雨中曲》
《战争与和平》	《飞屋环游记》	《小叛徒》	《卡莎布兰卡》
《盗梦空间》	《肖申克的救赎》	《魔戒》	《新龙门客栈》

图 3.1

《音乐之声》	《老人与海》	《莎翁秘史》	《七宗罪》
《天使之城》	《潘的迷宫》	《千与千寻》	《雨中曲》
《男人百分百》	《飞屋环游记》	《摩登时代》	《卡莎布兰卡》
《沉默的羔羊》	《肖申克的救赎》	《西雅图夜未眠》	《新龙门客栈》

图 3.2

《记忆碎片》	《罗马假日》	《莎翁情史》	《七宗罪》
《瓶中信》	《这个杀手不太冷》	《千与千寻》	《雨中曲》
《教父》	《小叛逆》	《摩登时代》	《卡莎布兰卡》
《辛德勒的名单》	《魔戒》	《西雅图夜未眠》	《新龙门客栈》

图 3.3

《绿野仙踪》	《廊桥遗梦》	《天使之城》	《潘的迷宫》
《瓶中信》	《这个杀手不太冷》	《千与千寻》	《雨中曲》
《罗密欧与朱丽叶》	《盗梦空间》	《沉默的羔羊》	《肖申克的救赎》
《辛德勒的名单》	《魔戒》	《西雅图夜未眠》	《新龙门客栈》

图 3.4

《钢琴家》	《战争与和平》	《男人百分百》	《飞屋环游记》
《教父》	《小叛逆》	《摩登时代》	《卡莎布兰卡》
《罗密欧与朱丽叶》	《盗梦空间》	《沉默的羔羊》	《肖申克的救赎》
《辛德勒的名单》	《魔戒》	《西雅图夜未眠》	《新龙门客栈》

图 3.5

游戏的程序:- Question 在上面的 31 部电影中，哪一部会是你的最爱？请把其装在你的心间（源于游戏的需要，你只可选一部）~ 问在上面的 5 枚卡片中有没有你最喜爱的电影…你只需回答有或没有即可。然后我们可猜得你心中的所想是什么。

隐藏在这一数学魔术中的原理，简单地说，却是如下的三个字：二进制。

回眸处，这些卡片的设计秘密是这样的：先给图 3 中的 31 部电影各自赋予 1-31 间的一个数字（图 4）；再把 1-31 间的这些数字转化为二进制（图 5）。于是上面的这五枚卡片中的那些电影的取舍分别恰是在二进制下,个位数–十位数–百位数–千位数-万位数都是 1 的那些电影。

《简爱》 1	《音乐之神》 2	《老人与海》 3	《记忆碎片》 4	《罗马假日》 5	《莎翁情史》 6
《七宗罪》 7	《绿野仙踪》 8	《廊桥遗梦》 9	《天使之城》 10	《潘的迷宫》 11	《瓶中信》 12
《这个杀手不太冷》 13	《千与千寻》 14	《雨中曲》 15	《钢琴家》 16	《战争与和平》 17	《男人百分百》 18
《飞屋环游记》 19	《教父》 20	《小叛徒》 21	《摩登时代》 22	《卡莎布兰卡》 23	《罗密欧与朱丽叶》 24
《盗梦空间》 25	《沉默的羔羊》 26	《肖申克的救赎》 27	《辛德勒的名单》 28	《魔戒》 29	《西雅图夜未眠》 30
《新龙门客栈》 31					

图 4

《简爱》 01	《音乐之夜》 10	《老人与海》 11	《记忆碎片》 100	《罗马假日》 101	《莎翁情史》 110
《七宗罪》 111	《绿野仙踪》 1000	《廊桥遗梦》 1001	《天使之城》 1010	《潘的迷宫》 1011	《瓶中信》 1100
《这个杀手不太冷》 1101	《千与千寻》 1110	《雨中曲》 1111	《钢琴家》 10000	《战争与和平》 10001	《男人百分百》 10010
《飞屋环游记》 11001	《教父》 10100	《小叛徒》 10101	《摩登时代》 10110	《卡莎布兰卡》 10111	《罗密欧与朱丽叶》 11000
《盗梦空间》 11001	《沉默的羔羊》 11010	《肖申克的救赎》 11011	《辛德勒的名单》 11100	《魔戒》 11101	《西雅图夜未眠》 11110
《新龙门客栈》 11111					

图 5

魔术之兑现：在于现代数学中一个非常本原的理念即一一对应的运用。比如有一个同学他心中的电影在上面的几张卡片中的呈现分别是：卡片一（图 3.1）中，无；卡片二（图 3.2）中，无；卡片三（图 3.3）中,有；卡片四（图 3.4）中，有；卡片五（图 3.5）中，无。若把"无"记作 "0"，把"有"记作 "1"，则得到一个数 $(1100)_2$，其转化为十进制模式下的数则是 $(1100)_2 = 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 0\times 2^0 = 12$ …因而其相应的电影为瓶中信。一一对应的魅力掩藏于此 ~

这个数学的魔术可以有别样的设计，比如以闪烁在数学历史的星空上的一些著名的数学家作为亮点。如下面的这组图片所示：

特勒斯	毕达哥拉斯	柏拉图	欧几里得	阿基米德	丢番图
希帕蒂娅	祖冲之	奥马海亚姆	斐波那契	笛卡儿	费马
帕斯卡	牛顿	莱布尼茨	伯努利家族	欧拉	拉格朗日
高斯	柯西	罗巴切夫斯基	阿贝尔	狄利克雷	伽罗瓦
魏尔斯特拉斯	黎曼	康托尔	克莱因	科瓦列夫斯卡娅	庞加莱
希尔伯特	哈代	布劳威尔	拉马努金	哥德尔	弗兰克-科尔

特勒斯	柏拉图	阿基米德
希帕蒂娅	奥马海亚姆	笛卡尔
帕斯卡	莱布尼茨	欧拉
高斯	罗巴切夫斯基	狄利克雷
魏尔斯特拉斯	康托尔	科瓦列夫斯卡娅
希尔伯特	布劳戴尔	哥德尔

图片一

	毕达哥拉斯	柏拉图	丢番图	希帕蒂亚	
	斐波那契	笛卡尔	牛顿	莱布尼茨	
	拉格朗日	高斯	阿贝尔	狄利克雷	
希尔伯特	黎曼	康托尔	拉马努金	哥德尔	庞加莱

图片二

费马	欧几里得	阿基米德	丢番图	
希帕蒂娅	柯西	罗巴切夫斯基	阿贝尔	
帕斯卡	牛顿	莱布尼茨	狄利克雷	
克莱因	科瓦列夫斯卡娅	庞加莱	弗兰克-科尔	希尔伯特

图片三

祖冲之	奥马海娅姆	斐波那契	笛卡尔
费马	帕斯卡	牛顿	莱布尼茨
伽罗瓦	魏尔斯特拉斯	黎曼	康托尔
克莱因	科瓦列夫斯卡娅	庞加莱	希尔伯特

图片四

伯努利家族	欧拉	拉格朗日	高斯
柯西	罗巴切夫斯基	阿贝尔	狄利克雷
伽罗瓦	魏尔斯特拉斯	黎曼	康托尔
克莱因	科瓦列夫斯卡娅	庞加莱	希尔伯特

图片五

哈代	布劳威尔	拉马努金	哥德尔	弗兰克-科尔

图片六

恰如这个游戏的哲思之所在，当你懂得数学的变中含有不变的精神，你可以有万千种 game 设计曲的变化。其主题可以是大众的偶像明星，或是你我心中的音乐之声，抑或是我们似曾相识的陌生的各种名字 … Everything 源自你当初的课堂所需要和期待的情境。

3.1 二进制的校园

1＝1　2＝10　3＝11　4＝100　5＝101　6＝110

7＝111　8＝1000　9＝1001　10＝1010　11＝1011　12＝1100

二进制是一种非常古老的进位制：这里的世界，只有 0 和 1 两个符号 —— 0 代表“零”，1 代表 “一”；经由 “逢二进一”的原则，就可用 0 和 1 两个数码表示一切自然数。在今日二进制已成为漫步现代计算机世界的通行证。

数学上真正运用二进制记数法当归功于德国数学家莱布尼茨（Gottfried Wilhelm Leibniz, 1646-1716）。在德国图灵根著名的郭塔王宫图书馆保存着一份弥足珍贵的手稿——那是莱布尼茨的手迹，其文里有片断如下：“ 1 与 0，一切数字的神奇渊源。这是造物的秘密美妙的典范；因为,一切无非都来自上帝。”

L–先生赋予这个神奇美妙的数字系统以独特的想象：他形象的用 1 表示上帝，用 0 表示虚无；上帝从虚无中创造出所有的实物，恰如在数学中用 0 和 1 可表示所有的（自然）数。

莱布尼茨

莱布尼兹，是一位百科全书式的学者。其研究领域涉及到数学、哲学、逻辑学、力学、地质学、法学、历史、语言、法律及神学…他或许也是西方研究中国文化的第一人，话说从其年轻时候起,莱布尼茨就通过广泛阅读来了解中国传统文化。在 1701 年,Leibniz 有通过到中国来的传教士,看到了《周易》中的所谓伏羲六十四卦图—— 一个伏羲六十四爻排列的木版图。 于此他赞叹道：“ 这恰是二进制算术，而这六十四个图形中可寻找到 哲学的秘密 ”。在他的一篇论文中, Leibniz 阐述了二进制与中国的八卦的联系…

3.2 神奇的八卦

八卦是中国古代人们的一个神奇的创造。在《易经》里记载，八卦相传是伏羲氏所造，他 “仰则观象于天，俯则观法于地 … 近取诸身，远取诸物,于是始作八卦。” 以八种符号包括天地万物的诸种现象，这也成为人类最早的记数符号。在八卦中，每个卦的上，中，下三部分叫“三爻” ——上面的叫“上爻”，中间的叫“中爻”，下面的叫“下爻”。如果把阳爻“—”当作数码 1，阴爻 “— —” 当作数码 0，并且自上而下，把上爻看作是第一位上的数字，中爻和下爻依次看作是第二位和第三位上的数字，我们就可以把八卦转化为二进制中的数了。经由此，二进制最早起源于中国。而这比西方的要早几千年。

然相比中国的八卦，Leibniz 的二进制思想无疑更胜一筹：二进制的简洁和优美或在于，其不单可以进行数的表示和运算,还可以表达集合和逻辑代数，进而构筑一种通用的数理语言。

二进制的思想可以在许多游戏中得到应用。尼姆游戏即是这样的一个。The Game of Nim, 或曰中国二人游戏，源于我国民间，深受大众喜爱……大约在 18 世纪传入欧洲，被称作 Chinese Game of Nim。其游戏布局和相应的规则：有若干堆火柴（或在身边别的事物），每堆数目任意。现由 I,II 两人轮流的拿这些火柴；每人每次可以拿走一堆火柴的几根，但不能不拿,也不许跨堆拿。约定拿到最后的那枚火柴者为胜利者。Question: 你有什么制胜之道么 ？

Mathematics is a game, but it tells the truth…相约数学与游戏，可以是一个非常广阔的天地（请参阅一些相关的著作与书籍，比如 [1-3] ）。想象着在以后，某一天谨经由二进制的主题来演绎数学与游戏的魅力...而隐藏在这一小品文后的数学文化课堂，或在于如下的桥情：这里有代数学的和弦，几何学的灵动和分析学的入微……

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信息技术

制作动态几何课件的关键是什么

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摘要：用一种统一的作法制作了椭圆、双曲线等图形，从中可以看到制作动态几何课件的关键是数学思想方法和知识的灵活运用。制作动态几何软件可以作为一种推动教师专业发展的方式。

关键词：动态几何课件；数学思想方法；教师专业发展

What is the Key to Make Dynamic Geometry Software

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Abstract: Using a unified method to make elliptic, hyperbolic, and other graphics. From this process, we can see what is the key to make dynamic geometry .It is necessary to use the thinking and methods of mathematics flexibly. The way of making dynamic geometry software can promote the development of teacher profession.

Keywords: dynamic geometry software; the thinking and methods of mathematics; the development of teacher profession

1 引言

几何画板、超级画板都是成功的动态几何软件，都有很好的教育价值。在使用动态几何的过程中，**我们深刻地感受到：使用信息技术并不是比拼技术的熟练与否，能否制作有创意的作品源于对数学知识的深刻理解。**一个精当的例子胜过一打说明。我们以制作椭圆、双曲线、阿波罗尼斯圆和卡西尼卵形线为例来说明。

2 问题

我们知道，加、减、乘、除四则运算，可以分别对应于四种不同曲线（设两定点间的距

离为 $2c>0$）：

到两个定点距离之和等于定长 $2a$ （$2a>2c$）的点的轨迹是椭圆；

到两个定点距离之差等于定长 $2a$ （$2a<2c$）的点的轨迹是双曲线；

到两个定点距离之比等于定值 $(\neq 1)$ 的点的轨迹是圆（阿波罗尼斯圆）；

到两个定点的距离之积等于定值 (a^2) 的点的轨迹，叫卡西尼卵形线（双纽线是其特例，此时 $a=c$）。

用动态几何软件可以实现椭圆、双曲线和阿氏圆，这里产生了两个问题：（1）卡西尼卵形线能否用动态几何软件实现，如能实现，又该如何实现？（2）上述四种曲线能否用一种统一的作法实现？熟悉动态几何软件的人都知道：现有的作椭圆、双曲线的方法没有用到椭圆、双曲线的定义，而是其它方法（如几何性质）实现的。概念的定义具有基本的作用，从直觉上看，应该能用定义实现，那么该如何实现呢？

3 探究

椭圆、双曲线的代数定义很容易转化为两条线段的和、差，但卡西尼卵形线却不然，不好从形上考虑。这样，思路就转到了代数上。设其中一个圆的半径是 t，另一个圆的半径是 a^2/t，如果这两个圆有交点，那么交点分别到两个圆心的距离之积不就是 a^2 了吗？这个想法简单而朴实，但不知可不可行。不妨先在几何画板里进行实验。（1）先在 x 轴上作四个动点 $(-c,0), C(c,0)$，$A(a,0), T(t,0)$；（2）分别度量 A,C,T 到原点的距离；（3）以 $(-c,0)$ 为圆心，$|OT|$ 为半径作圆；以 $(c,0)$ 为圆心，$\dfrac{|OA|^2}{|OT|}$ 为半径作圆；（4）这两个圆的相交于 E,F 两点；（5）以 T 为主动点，分别以 E,F 为从动点作轨迹；（6）利用属性，把轨迹的象素设为 3000，图形显得光滑些。这样就生成了卡西尼卵形线。当分别拖动 A,T,C 时，就会出现不同的图形。下面是一些截图，如图 1、2、3。

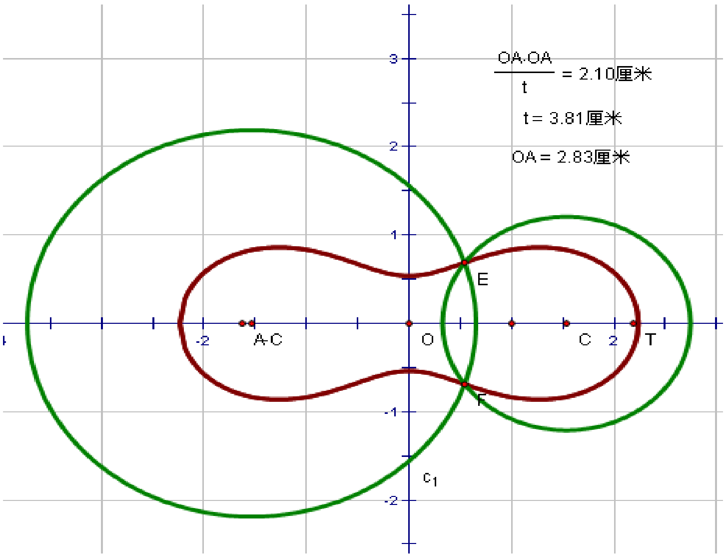

图 1 平面凸闭的卡西尼卵形线

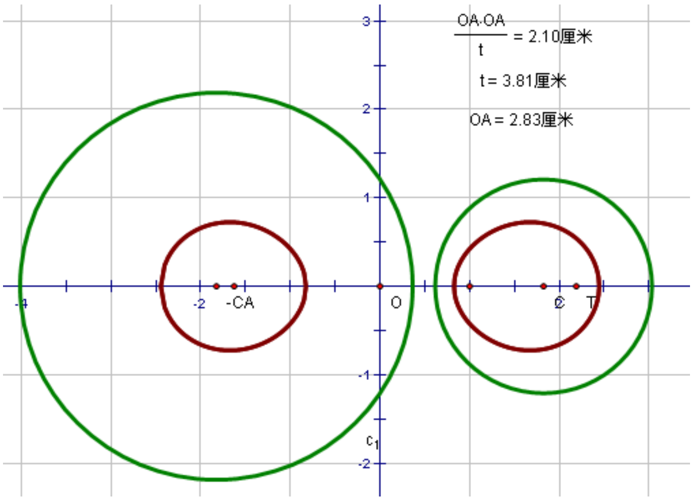

图 2 二分支的卡西尼卵形线

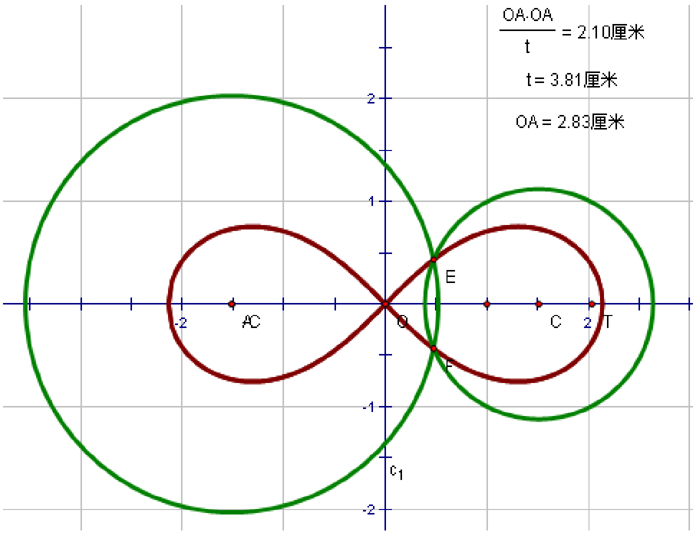

图 3 双纽线：卡西尼卵形线的特例

上面实验的成功激发了进一步探究的激情。

3.1 牛刀初试，阿氏圆中得验证

按上述步骤，设其中一个圆的半径为 t，另一个圆的半径为 $3t$，就作出到两定点的距离之比是 $\dfrac{1}{3}$ 的点的轨迹，是一个圆。如图 4 所示。

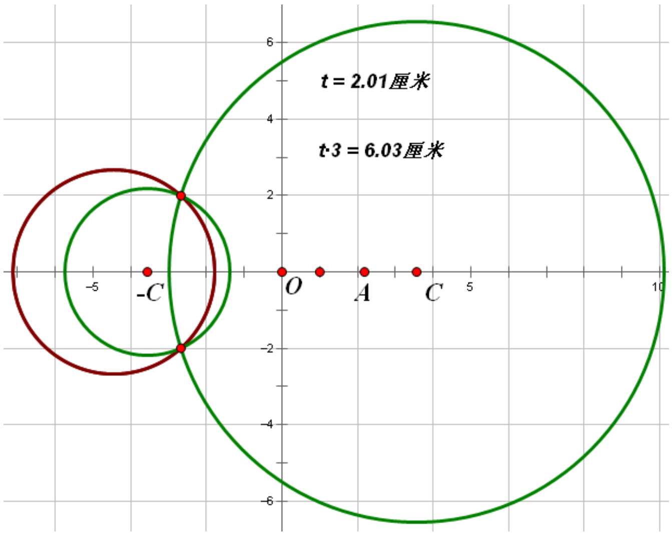

图 4 阿氏圆

3.2 牛刀再试，椭圆中遇小挫折

在上述成功作法的推动下，产生了把这种作法用之于作椭圆与双曲线的念头。这里设其中一个圆的半径是 $a+t$，另一个圆的半径是 $a-t$，那么这两个圆的交点到两个圆心的距离之和为 $2a$。按上述操作步骤在几何画板中进行实验时，只出现了“半椭圆”。如图 5。

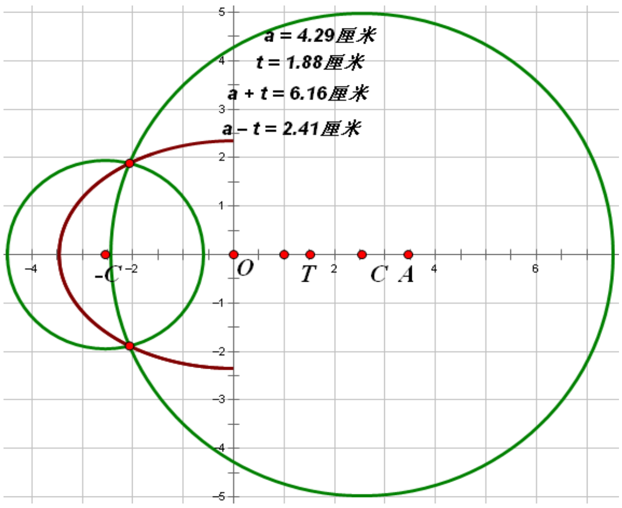

图 5 半椭圆

为什么会出现这种情况呢？原来，在上述操作中 t 只能取正值，所以在作卡西尼卵形线和阿氏圆时，不会有问题。但在椭圆定义中，设一个圆的半径为 $a+t$，另一个圆的半径为 $a-t$，

这样就只有 $a+t>a-t$，只能出现“半椭圆”了！

补救措施是有的。那就要重复上述操作了。比如，第一次，以 $(-c,0)$ 为圆心，$a-t$ 为半径作圆，以 $(c,0)$ 为圆心，$a+t$ 为半径作圆；那么第二次就要以 $(-c,0)$ 为圆心，$a+t$ 为半径作圆，以 $(c,0)$ 为圆心，$a-t$ 为半径作圆；这样才能实现。如图 6 所示。

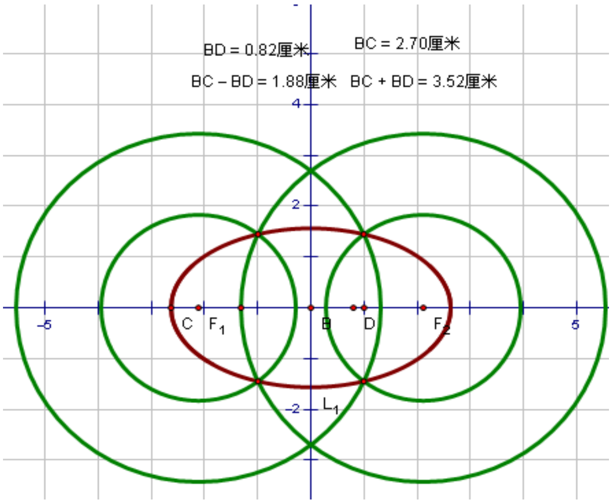

图 6　完整的椭圆

用同样的方法也可作出双曲线。如图 7。

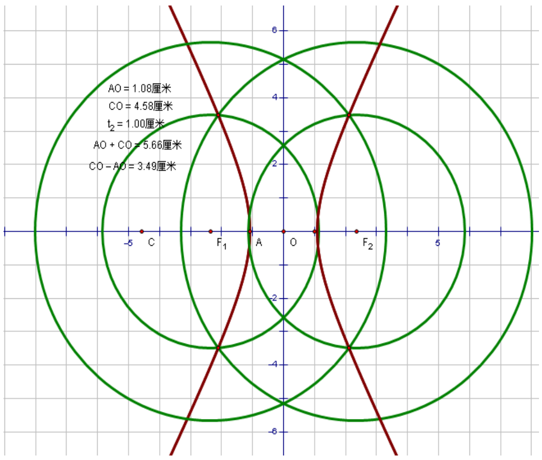

图 7　双曲线

虽然把椭圆、双曲线给作出来了，但总感到有点美中不足。何不用超级画板试试呢？

3.3 牛刀再试，超级画板显威力

超级画板与几何画板虽同为动态几何软件，但超级画板是基于代数原理设计的，以数驭形、以形表数是其突出特点。充分利用超级画板中的变量，就可以弥补几何画板制作椭圆中的美中不足了。

下面是用超级画板制作椭圆的主要步骤：（1）制作两个可以拖动的坐标点 $F_1(-c,0)$，$F_2(c,0)$，拖动参数是 c；（2）作坐标点 $A(a+c+t,0)$，拖动参数是 t；（3）以 F_2 为圆心，过点 A 作圆，所作圆的半径是 $a+t$；（4）作坐标点 $B(t-c-a,0)$，拖动参数是 t；（5）以 F_1 为圆心，过点 B 作圆，所作圆的半径是 $a-t$；（6）作变量 a 的变量尺；（7）两个圆的交点是 C,D；（8）以 A 为主动点，分别以 C,D 为从动点，作点的轨迹；（9）利用属性，把轨迹的象素设为 3000；图形显得光滑些。

这里 t 可正可负，故 $a+t$ 并不一定大于 $a-t$，这样就能生成完整的椭圆。如图 8。

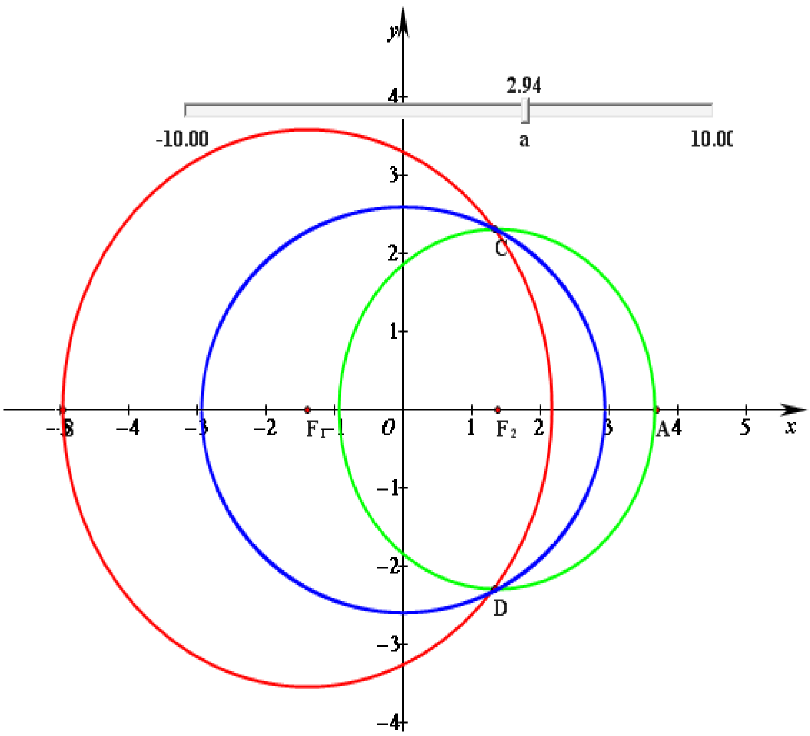

图 8　超级画板作的椭圆

类似的，也可作出双曲线，如图 9。

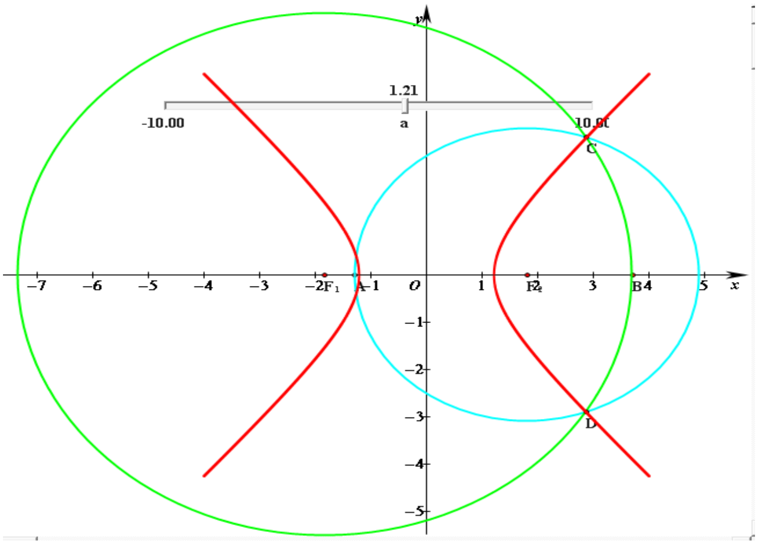

图 9 超级画板作的双曲线

那么，用这种作法，能否作为卡西尼卵形线及阿氏圆呢？如图 10 和图 11。

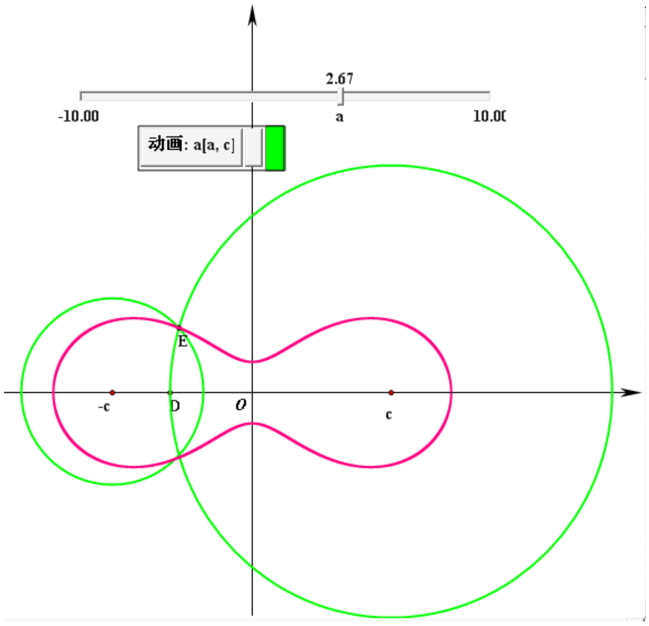

图 10 超级画板作卡西尼卵形线

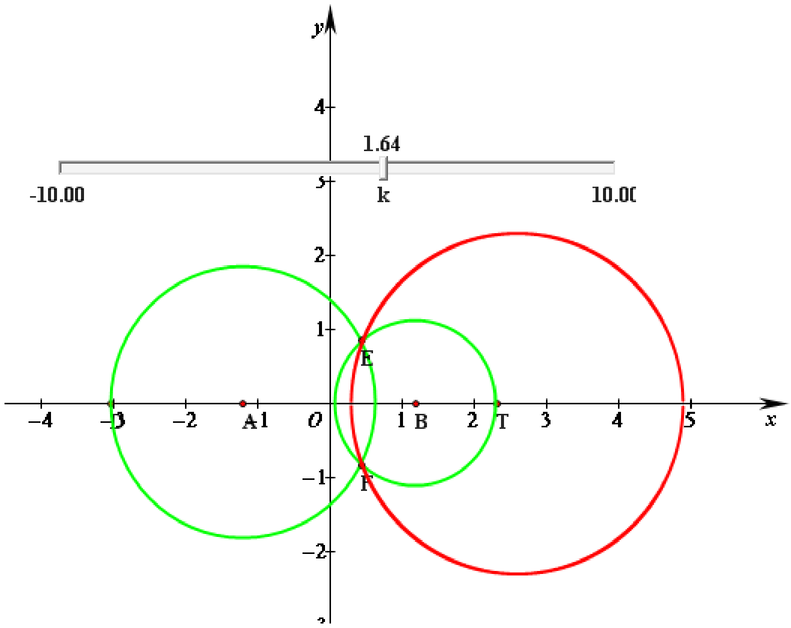

图 11 超级画板实现阿氏圆

由在几何画板没有变量，只有参数，控制参数不如控制变量那么方便，所以的制作椭圆、双曲线、卡西尼卵形线及阿氏圆时，超级画板显得更便捷，虽然两者的原理都一样。

3.4 牛刀回眸，作法小结

乍看起来，阿波罗尼斯圆、卡西尼卵形线不好实现。通过实验、探索之后，发现只要找到合适的作法，困难也就迎刃而解了。回顾、小结一下上述作法：（1）作两个可以拖动的定点；（2）作两个半径合乎一定条件的圆；（3）作两个圆的交点；（4）用主动点带动从动点，用轨迹生成上述曲线。其中，如何确定圆的半径及如何作圆是关键。

4 感想

人们常说数学是统一的，那么数学的统一性体现在哪些方面呢？“数学是统一的”无非是表达了人们对数学的一种体会，作为初学者对这样的命题的理解应**从具体实例中**感知。这里，借助信息技术，制作者对“数学是统一的”这样口号式的命题有了一定的感想。两个数的和为定值 $2a$，一个设为 $a+t$，另一个设为 $a-t$；两个数的差为定值 $2a$，一个设为 $a+t$，另一个设为 $t-a$；两个数的积为定值 a^2，一个设为 t，另一个设为 $\frac{a^2}{t}$；两个数的比为定值 a，那一个设为 t，另一个设 ta；这些是很自然、很平凡的想法，也是一种很统一的想法（比如，两个数的和为定值 $2a$，一个设为 $a+t$，另一个设为 $a-t$，就是历史上的和差术，在解方程中广泛应用），但在常规教学中，**我们仅仅把这样的想法当成一种技法来使用，没有发**

挥“思想”的威力和实用性。那么这种想法究竟有没有实用性呢？信息技术是检验我们的想法是否切实可行的一个很好的工具。在用深入学科的信息技术工具制作课件的过程中，初步体会到了数学建模的想法——如何用数学来描述现象、解释现象，初步体现到了数学的威力。从这个角度而言，深入学科的信息技术工具是检验教师是否具有数学眼光，数学功底是否扎实的一块“试金石”；学习制作动态几何课件的关键是数学思想方法和知识的灵活运用；因此，学习制作动态几何课件可作为教师专业发展的一个抓手。

调查研究

关于数学文化教育价值与运用现状的网上调查

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《普通高中数学课程标准》在“课程基本理念”中指出：“数学是人类文化的重要组成部分。数学课程应反映数学的历史、应用和发展趋势，数学对推动社会发展的作用，数学的社会需求，社会发展对数学发展的推动作用，数学科学的思想体系，数学的美学价值，数学家的创新精神。数学课程应帮助学生了解数学在人类文明发展中的作用，逐步形成正确的数学观。为此，高中数学课程提倡体现数学的文化价值，并在适当的内容中提出对数学文化的学习要求。”[1]从中我们大致可以将数学文化的内涵概括为：数学的历史、数学的思想、数学的精神、数学与人类其他知识领域之间的关联。

近年来，数学文化日益受到国内数学教育工作者的重视，高校数学文化选修课方兴未艾，中学数学文化校本课程悄然兴起，数学文化融入数学教学业已成为一个重要而富有生命力的学术研究领域。那么，在中学数学教师眼里，数学文化究竟有哪些教育价值？数学文化在中学数学课堂中的运用现状如何？中学数学教师运用数学文化的困难是什么？在高中数学新课程实施十年、课程标准即将修订之际，对上述问题进行研究，无疑是有现实意义的。

为了回答上述研究问题，我们在上海市十二五市级共享课程“数学史与数学文化”的在线 BBS 讨论区发起了主题为数学文化教育价值的讨论，得到了教师的积极响应。从 10 月 26 日起截止到 11 月 6 日，共产生 778 个帖子。

1 教师眼中数学文化的教育价值

经过对在线教师所发帖子的逐条整理分析，我们得到关于数学文化教育价值的 344 个有效观点，涉及对学生和教师自身两方面的影响，涵盖数学文化的德育功能、理性精神、学习兴趣、数学知识和能力培养方面。

在所有观点中，所占比例最高的是“激发学生兴趣，形成积极的数学情感态度”。在教师看来，相当多的学生对数学缺乏兴趣，普通中学的情况更是不容乐观。一位教师写道：

我是普通完中的高中老师，最切身的体会就是：老师教得非常累，学生学得非常累，简单的知识教了就忘，公式背了就忘，抄作业现象很普遍，学生的厌学情绪很浓，没有丝毫学习兴趣，还有什么情感、态度、价值观？

数学文化有着丰富多彩的内涵，生动有趣的故事、穿越时空的智慧、贴近生活的问题、赏心悦目的艺术等等，都可以充分调动学生的积极性，使他们形成良好的数学情感态度。在讨论中，一位教师写道：

兴趣是最好的老师，好奇心是研究的源泉。数学教学就是要培养学生的好奇心和兴趣，有了兴趣和好奇心，学生才能更加主动地学习数学。但是，我们在数学教学中往往把"火热"的数学教成"冰冷的美丽"的"骨干美人"，数学的枯燥无味埋没了多少学习数学的热情，让多少学生望而却步！这不得不说是我们数学教学的悲哀。如果我们数学教师能更好地了解数学史，就能深度理解数学的教学内容，从而做好数学教学演绎，把数学课上成让人回味无穷的艺术性和科学性并存的好课！

居于第二位的是"加强爱国主义教育，增强民族自豪感，培养社会责任感"。数学文化带有强烈的时代和民族特色。将数学文化应用于日常教学，向学生介绍中国古代具有世界意义的数学成就（如《周髀算经》中的勾股定理、刘徽的割圆术、祖冲之的圆周率等），让他们了解祖先的伟大智慧和创造力，可以增强他们的民族自豪感，激发他们的爱国热情。当然，笔者之一发帖提出了"爱国爱错"现象：勾股定理常常在课堂上被错误地说成是"中国人最早发现的"，但实际上更早的古巴比伦人已熟练运用过该定理。对此，一些老师认为自己确实对历史知识不甚了了，今后应多多学习；但也有部分教师认为：谁最早发现并不重要，重要的是我们的祖先确实做出过独立的发现。

居于第三位的教育功能是"汲取榜样的力量，锤炼坚强的意志，养成良好的心理品质"。数学史上，很多数学家都是逆境成才、永不言弃的典范；很多数学家都是勤学不怠、百折不挠的楷模；很多数学成都是历经挫折、失败甚至付出生命代价后取得的。一位教师写道：

泰勒斯因天文观测而掉入阴沟，希帕索斯因发现无理数而葬身大海，阿那克萨格拉身陷囹圄而探索不止，阿基米德因沉迷数学而被罗马士兵杀害，希帕蒂亚因追求真理而死于基督徒之手，索菲·热尔曼在墨水结冰的冬夜仍勤奋学习，斯坦纳家境贫寒而自强不息……这些优秀历史人物的事迹可用来激励学生努力学习、坚忍不拔。

位居第四的观点是"体会数学与现实生活之间的普遍联系，经历良好的数学体验，树立正确的数学观"。相当多的学生持有十分消极的数学观，一位教师引用文献中的说法[2]：

什么是数学？对于这个问题，有些学生的回答很有意思："数学是一些居心叵测的成年人为青年学生挖的陷阱"；"数学问题是一些仅仅出现在课本和试卷上的、让某些老师看着学生崴脚而感到窃喜的东西。"听到学生这样的回答，开始还觉得挺新鲜，可是仔细一想，就感到一阵心酸。我们这些用尽心血努力教学的教师在学生心目中无非就是一些布雷高手，而数学成为教师惩治学生的工具，与学生的个人发展并无太大关系，这样的结果无疑是数学教学的一种悲哀。

因此，有教师指出，课堂上为学生提供数学与现实生活密切联系的例子，让学生体会数学的价值，从中获得良好的数学体验，从而形成正确的数学观。

教师眼中数学文化的其他教育价值有：拓宽国际视野，培养包容并蓄的民族文化观；提高美学修养，欣赏数学美；培养理性的科学态度和严谨的思维方式；树立创新意识，发展探索能力；学习解决问题的思想和方法；理清知识结构，理解知识本质。

图 1 给出了各种观点的分布情况。

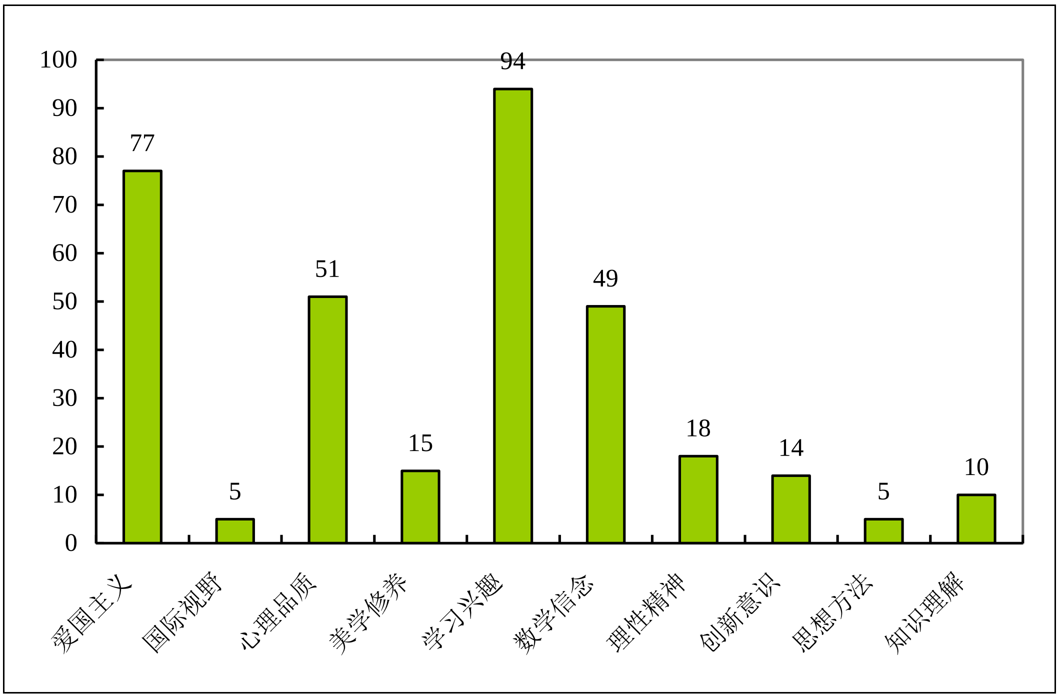

图 1 教师眼中数学文化的教育价值

从图 1 可见，在中学数学教师眼中，数学文化的教育价值主要体现在三维教学目标中的"情感态度价值观"上，数学文化很少能与"知识与技能"、"过程和方法"建立关联。

此外，少数教师在讨论中还提到数学文化对于教师自身专业发展的作用，认为数学文化知识的学习有助于提高教师数学素养，丰富自己的知识储备；同时，历史上曾经困扰过古人的问题，根据历史相似性原理，也很可能是今日学生学习上的困惑点，所以加强数学文化的

学习，可以帮助教师更好地把握教学重、难点，从而选择恰当的教学方法与策略。

2 课堂上的数学文化例子

在线讨论中，许多教师分享了自己在课堂上运用过的具体的数学文化例子。使用频率最高的例子是勾股定理，其次是圆周率，接下来依次为“无理数的由来”、“黄金分割”和“国际象棋棋盘问题”等。笔者对讨论中出现三次以上的数学文化例子进行了统计，详见图 2。

此外，还有一些数学文化例子也为个别教师所用，如：

（1）数学史上的名人轶事，如希帕索斯与无理数、纳皮尔与对数、高斯与等差数列求和公式、韦达与他的定理、刘徽与极限概念，杨辉与多项式乘法、加菲尔德与勾股定理、陈景润与哥德巴赫猜想等。

（2）历史上的数学名题或著名公式，如芝诺悖论（阿喀琉斯追龟问题）、丢番图墓志铭、泰勒斯测量金字塔高度、鸡兔同笼问题、理发师悖论、欧拉公式、赌金分配问题（“点数”问题）、格尼斯堡七桥问题等。

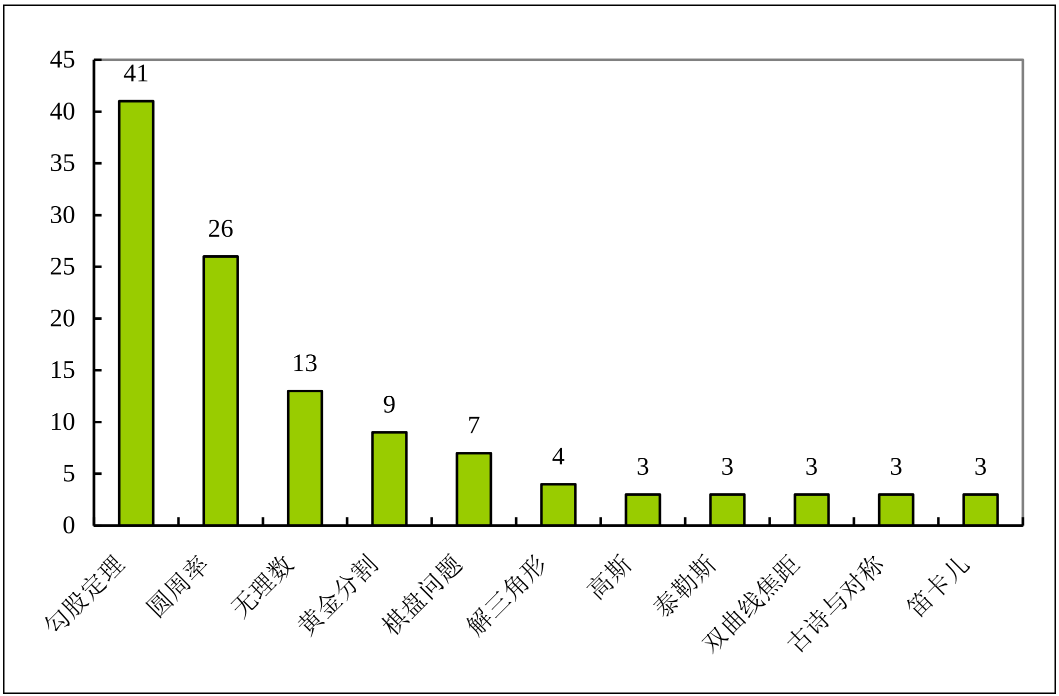

图 2 教师在课堂教学中使用过的数学文化例子

（3）生活中的数学问题，如地铁发车时间安排与最小公倍数、天平平衡与等式的性质、遮阳棚的设计与三角形稳定性和铺地砖与平面镶嵌问题、风车、陀螺与旋转、电影院座位与坐标、体操运动员花样动作平衡点与向量、点燃鞭炮与数学归纳法、上海市出租车计价问题与分段函数、经贸大厦与东方明珠的距离与解斜三角形、牙膏口径与圆柱体积、双曲线与导

航技术、引体向上与向量运算等。

（4）趣味数学，如“抢 31”问题、“24 点”游戏、“蜗牛爬井”问题等。

（5）动物生活中的数学原理，如蜂房与平面镶嵌问题、蜘蛛网与图形对称问题、猫的睡姿与最小表面积问题等。

（6）数学概念的起源与发展，如函数概念的起源、负数概念的历史等等，帮同学理清概念的来龙去脉，更好地理解知识内容。

（7）文学作品中的数学思想，如惠施命题“一尺之棰，日取其半，万世不竭”蕴含了无穷思想，李白名句“孤帆远影碧空尽，唯见长江天际流”形象地描写出极限的意境，杜甫名句“两个黄鹂鸣翠柳，一行白鹭上青天”展示了优美的对称性；陈子昂名篇“前不见古人，后不见来者；念天地之悠悠，独怆然而涕下”记录了作者对时间和空间的感知；华罗庚诗词阐明数形结合思想的重要性；王蒙利用黄金分割来阐述人的能力与他所得回报之间的理想关系，等等。

（8）建筑中的数学问题，如黄金分割在建筑中的应用，胡夫金字塔中所蕴含的数学。

（9）天文学、物理学上的数学，如天王星的发现、做功与数量积。

（10）音乐中的数学问题，如旋律的节奏与数学的周期。

（11）绘画技巧中的数学原理。

上述统计表明，大部分教师所了解的数学文化例子都很少，只有极少数教师知道较多的例子。从数学史料、文化现象、实际应用到教学设计、实证研究，关于勾股定理与圆周率这两个案例的文献很多，网上资源铺天盖地，且两者也是中国古代典型的数学成就，因而成为教师运用最多的数学文化例子。

教师主要采用以下几种方式来运用数学文化：

（1）在序言课中介绍数学史；

（2）用故事引入新课；

（3）将数学史用作课外阅读材料；

（4）开设数学史讲座；

（5）开展数学知识小竞赛；

（6）开设以数学文化为主题的拓展课；

（7）借助现代技术来呈现数学文化。

其中，教师采用得最多的是用故事引入新课。

3 教师运用数学文化的困难

尽管多数教师已经认识到数学文化的某些教育价值，但在实际教学中，运用数学文化存在很多困难。

3.1 课上无时间

在线讨论中，教师普遍反映没有充足的时间讲授数学文化。

T1：说说容易，其实做起来就不是那回事情了。你想，现在各校都在赶进度，赶进度的后果就是砍掉了数学史在课堂中的教学。

T2：我经常纠结于课时不足，现有的课时数仅够课本内容的讲解，若要拓展提高，时间就紧巴巴。课本以及所用的导学都有一些数学史知识，如雪花曲线，课本上的阅读材料介绍得很详细，苦于没有时间利用这一内容。

3.2 手中无资料

巧妇难为无米之炊。几乎所有的在线教师都感觉到手头缺乏适合于课堂教学的数学文化资料，他们一致建议，高校教师能够编写有关参考书。

T3：我浏览过很多有关数学史和数学文化的书，很少有针对中学数学教学来写的，但这往往是中学数学老师迫切需要的。所以很多老师在做教学设计时，由于时间、精力等限制，无法查阅大量的文献或资料。而我觉得，在数学课尤其是一些重要的概念课（比如函数的概念，曲线与方程、数学归纳法等）中，渗透数学史与数学文化是最适合不过的了。很希望能有一本针对中学数学教学的数学史与数学文化的书，选择几个重要的概念，追本溯源，也可附上几篇精彩的教学设计，方便数学老师借鉴和参考。

一些教师则强烈希望建立“数学典故资源库”。

3.3 考试无要求

无疑，升学率是评价学校和教师工作绩效的主要标准，在这种情况下，不少教师认为，只要考试无要求，数学文化就很难受到重视。

T4：在高考的压力下，讲数学史有点“奢侈”，不知道该怎么处理?

T5：要是哪次考试卷里包含数学史内容，数学史就会引起教师和社会的重视。

T6：有了数学文化，必然促进数学观的正确形成，可惜，目前的考试制度和形式让这些都成为浮云啊！

3.4 学生无基础

部分教师认为，经过年复一年的各种考试，“会解题才是王道”的思想已经深植于学生内心，除了训练解题技巧方法以外的其他内容，学生参与度不高，高中尤为明显。

T7：数学文化的教学应该从小学开始，不然都到了高中，你想搞个情境引入什么的，学生早就没心情听了。他们都觉得是浪费时间，还不如做题有用。

T8：二期课改的教材中引进的数学史以及数学家的故事，的确比过去多了很多，但是在实际教学中，存在着两大问题：一方面是教学的老师可能忽略关于数学史的教学；另一方面是，在教学中，很多同学感觉你是在浪费时间，不如直接讲教学内容。我想，这是长期的应试教学带来的后果。

一些教师担心，学生的基础本来就差，加入数学文化内容反而会增加学生的学习负担，最后本末倒置，该掌握的反而没掌握。

T8：有些学生连最基本的知识都弄不清楚，学数学史也是无用的。

T9：对有些学生来说，数学文化无形中会不会又加重了学习负担？的确，有些女生本来学数学就很辛苦，要不要教给学生[数学文化]，我觉得是要看对象的。

此外，教师对数学文化的理解还存在许多误区。如，不少教师误认为数学文化就是数学史，或者数学文化就是数学趣题或者名人故事。

4 结论与启示

通过对教师在线讨论的整理和分析，我们可以得到如下结论：

（1）中学数学教师眼中的数学文化教育价值较为片面，主要局限于情感态度价值观方面，特别是在激发兴趣和爱国主义教育方面，而对于数学文化在认知方面的价值不甚了了。他们的认识多源于网络文献。

（2）中学数学教师自身的数学文化知识比较欠缺，他们所知道的有关例子十分单一，且绝大多数来自教材；他们因为所阅读的相关书籍很少而过于依赖网上资源；他们深感缺乏数学文化资料，期待实用的数学文化参考读本的出版。

（3）中学数学教师主要采用附加式来运用数学文化，因而部分教师认为数学文化会过多占用教学时间、甚至增加学生学习负担。

（4）数学文化“高评价、低应用”的现状并未改变。升学压力大、教学进度快、教师自身文化素养低、数学文化资料严重缺乏、数学文化认识片面、学生基础薄弱、视野狭隘等原

因造成了数学文化教育的巨大困难。

根据上述结论，我们获得如下启示。

（1）数学教学呼唤教育取向的数学文化研究。开发与中学教学内容相契合的数学文化案例，编著数学文化实用参考书，为中学数学教学提供有关文化素材，将是 HPM 研究的重要内容之一。

（2）中学数学教师数学文化在职培训势在必行。教师所反映的课上无时间、考试无要求、学生无基础等问题，都源于教师对于数学文化运用方式的无知。附加式地运用数学文化，为文化而文化，势必会加重课堂负担，挤占课堂时间，同时，也未能在数学文化与数学知识之间建立起内在的联系，难以促进学生对知识的理解，从而也就无法体现数学文化在认知上的价值和在应试上的优势。事实上，数学文化的运用除了附加式，还有复制式、顺应式和重构式等方式[3]，高层次的运用方式，并不以教学时间为代价，也不与考试成绩相冲突，数学文化在其中将起到润物细无声的教育作用。教师培训应采用案例教学的方式。

（3）高校数学文化研究者与教研员、一线教师的合作不可或缺。只有通过这种合作，学术形态的数学史材料才能真正转化为教学材料，而融入数学文化的数学教学设计才能付诸实施，实施的效果才能得到检验，从而，成功的 HPM 案例的开发才变得可期可待，HPM 研究也就能够越走越远。

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