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Non-differentiability points of Cantor functions

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Let the Cantor set C in $\mathbb{R}$ be defined by $C = \bigcup_{j=0}^{r} h_j(C)$ with a disjoint union, where the h_j's are similitude mappings with ratios $0 < a_j < 1$. Let μ be the self-similar Borel probability measure on C corresponding to the probability vector $(p_0, p_1, \ldots, p_r)$. Let S be the set of points at which the probability distribution function $F(x)$ of μ has no derivative, finite or infinite. For the case where $p_i > a_i$ we determine the packing and box dimensions of S and give an approach to calculate the Hausdorff dimension of S.

1 Introduction

The *Cantor set* C in $\mathbb{R}$ is defined as the unique nonempty compact set invariant under h_j's:

$$C = \bigcup_{j=0}^{r} h_j(C), \tag{1.1}$$

where $h_j(x) = a_j x + b_j$, $j = 0, 1, \ldots, r$, with $0 < a_j < 1$ and $r \geq 1$ being a positive integer. C is also termed as the self-similar set determined by the h_j's. Without loss of generality we shall assume that $b_0 = 0$ and $a_r + b_r = 1$. We furthermore assume that the images $h_j([0, 1])$, $j = 0, 1, \ldots, r$, are pairwise disjoint (i.e., the h_j's satisfy the (strongly) open set condition) and are ordered from left to right. It is well-known that $\dim_H C = \dim_B C = \dim_P C = \xi$ and $0 < \mathcal{H}^\xi(C) < +\infty$ where ξ is given by (ref. [5])

$$\sum_{j=0}^{r} a_j^\xi = 1. \tag{1.2}$$

As usual, the elements of C in (1.1) can be encoded by digits in $\Omega = \{0, 1, \ldots, r\}$ as follows. We write $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \ldots) : \sigma(j) \in \Omega\}$ and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$ with $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k)) : \sigma(j) \in \Omega\}$ for $k \in \mathbb{N}$. $|\sigma|$ is used to denote the length of a word $\sigma \in \Omega^*$. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \ldots, \sigma(|\sigma|), \tau(1), \ldots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \ldots, \tau(|\tau|), \sigma(1), \sigma(2), \ldots)$ for any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$. $\sigma|k = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ for $\sigma \in \Omega^\omega$ and $k \in \mathbb{N}$. Denote $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \Omega^k$ and $x \in \mathbb{R}$. Then for $\sigma \in \Omega^k$, the intervals $h_{\sigma*0}([0, 1]), h_{\sigma*1}([0, 1]), \ldots, h_{\sigma*r}([0, 1])$ are contained in $h_\sigma([0, 1])$ in this order where the left endpoint of $h_{\sigma*0}([0, 1])$ coincides with the left endpoint of $h_\sigma([0, 1])$, and the right endpoint of $h_{\sigma*r}([0, 1])$ coincides with the right endpoint of $h_\sigma([0, 1])$. Moreover the length of interval $h_\sigma([0, 1])$ equals $m(h_\sigma([0, 1])) = \prod_{j=1}^{k} a_{\sigma(j)} =: a_\sigma$ for $\sigma \in \Omega^k$, where $m(\cdot)$ denotes the one-dimensional Lebesgue measure. For $j = 1, 2, \ldots$, we define

$$C_j = \bigcup_{\sigma \in \Omega^j} h_\sigma([0, 1]).$$

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Then $C_j \downarrow C$ as $j \to \infty$ and $x \in C$ can be encoded by a unique $\sigma \in \Omega^\omega$ satisfying $\{x\} = \bigcap_{k=1}^{\infty} h_{\sigma|k}([0,1])$. We usually denote this unique code of x by $\widetilde{x}$ and use $x(k)$ to denote the k-th component of $\widetilde{x}$. The endpoints of $h_\sigma([0,1])$ for a $\sigma \in \Omega^*$ will be called the endpoints of C. So the set of endpoints of C is countable. Obviously any endpoint e of C lies in C and except for a finite number of terms, its coding $\widetilde{e}$ consists of either only the symbol 0 if e is the left endpoint of some $h_\sigma([0,1])$, or only the symbol r if e is the right endpoint of some $h_\sigma([0,1])$.

Let μ be the *self-similar Borel probability measure* on C corresponding to the probability vector $(p_0, p_1, \dots, p_r)$ where each $p_i > 0$ and $\sum_{i=0}^{r} p_i = 1$, i.e., the measure satisfying

$$\mu(A) = \sum_{j=0}^{r} p_j \mu\big(h_j^{-1}(A)\big) \quad \text{for any Borel set} \quad A,$$

and so

$$\mu(h_\sigma([0,1])) = \prod_{j=1}^{k} p_{\sigma(j)} =: p_\sigma, \quad \text{for any} \quad \sigma \in \Omega^k, \quad k \in \mathbb{N}. \tag{1.3}$$

Obviously, μ is atomless. Consider the distribution function of the probability measure μ, also called *Cantor function* or a self-affine “devil's staircase” function:

$$F(x) = \mu([0,x]), \quad x \in [0,1]. \tag{1.4}$$

Then $F(x)$ is a non-decreasing continuous function with $F(0) < F(1)$ that is constant off the support of μ. It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0,1] \setminus C$. In particular, the set S of *points of non-differentiability* of $F(x)$, that is those x where

$$\lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta}$$

does not exist either as a finite number or ∞, has Lebesgue measure 0. Naturally one likes to determine the Hausdorff dimension of S. Some conclusions have been obtained about it.

(**R1**) Let $h_0(x) = \frac{1}{3}x$ and $h_1(x) = \frac{1}{3}x + \frac{2}{3}$ that corresponds the case where $r = 1$ and $a_0 = a_1 = \frac{1}{3}$. Then C defined by (1.1) is the famous *middle-third Cantor set* and has the Hausdorff dimension $\xi = \frac{\log 2}{\log 3}$. Taking $p_0 = p_1 = \frac{1}{2} = \left(\frac{1}{3}\right)^\xi$, Darst [1] showed that

$$\dim_H S = \left(\frac{\log 2}{\log 3}\right)^2 = (\dim_H C)^2.$$

(**R2**) Let $h_j(x) = ax + (1-a)\frac{j}{r}$, $j = 0, 1, \dots, r$, with $0 < a < \frac{1}{r+1}$. Then C defined by (1.1) is a *homogeneous Cantor set* in which all similitude mappings $h_j(x)$ have the same scaling factor $a_i = a$ and the gaps between the images $h_j([0,1])$, $j = 0, 1, \dots, r$, have the same length, where C has the Hausdorff dimension $\xi = -\frac{\log(r+1)}{\log a}$. Taking $p_j = \frac{1}{r+1} = a^\xi$, $j = 0, 1, \dots, r$, Darst [2] showed that

$$\dim_H S = \left(-\frac{\log(r+1)}{\log a}\right)^2 = (\dim_H C)^2.$$

(**R3**) For a general Cantor set C defined by (1.1), let ξ be the Hausdorff dimension of C which is determined by (1.2). Taking $p_j = a_j^\xi$, $j = 0, 1, \dots, r$, Dekking & Li [3] showed that

$$\dim_H S = \xi^2 = (\dim_H C)^2.$$

It is important to note that the results for all three cases above present a curious property, i.e.,

$$\dim_H S = (\dim_H C)^2. \tag{1.5}$$

In fact, $\dim_H S$ depends on the choice of μ supported by C. For all three cases above the measures μ are constructed in the same way by choosing $p_i = a_i^{\xi},\ i = 0, 1, \ldots, r$, so that the corresponding measures μ are of ξ-*Ahlfors regularity*, i.e., there exist $c_1, c_2 > 0$ such that

$$c_1\delta^{\xi} \le \mu(B(x,\delta)) \le c_2\delta^{\xi}$$

for all $x \in C$ and $0 < \delta < 1$, where $B(x,\delta)$ is the interval of centre x and length 2δ. The curious property shown in (1.5) has been revealed successfully by K. J. Falconer in his recent paper [4] for a very general nonempty compact subset E (unnecessarily restricted to Cantor sets in (1.1)) of $\mathbb{R}$ and the positive finite Borel d-Ahlfors regular measures μ supported by E.

Then a natural question one is interested in is that whether or not the dimension formula shown in (1.5) still holds if the measure μ is not of ξ-Ahlfors regularity. The following (**R4**) gives a negative answer.

(**R4**) Let $r = 1$. Take $p_0 = \frac{a_0}{a_0+a_1}$ and $p_1 = \frac{a_1}{a_0+a_1}$. Then the corresponding probability measure μ defined by (1.3) is a non-Ahlfors regular measure except for $a_0 = a_1$. Let d^+ and d^- be respectively determined by

$$\frac{\log a_1 - \log(a_0+a_1)}{\log a_1}\log\left(a_0^{d^+} + a_1^{d^+}\right) + \left(1 - \frac{\log a_1 - \log(a_0+a_1)}{\log a_1}\right)\log a_1^{d^+} = 0,$$

and

$$\frac{\log a_0 - \log(a_0+a_1)}{\log a_0}\log\left(a_0^{d^-} + a_1^{d^-}\right) + \left(1 - \frac{\log a_0 - \log(a_0+a_1)}{\log a_0}\right)\log a_0^{d^-} = 0.$$

Morris [7] proved that $\dim_H S = \max\{d^+, d^-\}$.

In the present paper we focus on the Cantor sets C in (1.1) and consider a wide variety of self-similar Borel probability measures μ in (1.3) where we only require that $p_i > a_i$, $i = 0, 1, \ldots, r$. Thus these measures μ fail to be Ahlfors regular for general choices of p_i's and include all above cases (**R1–4**) considered in [1, 2, 3, 7]. The requirement that $p_i > a_i$, $i = 0, 1, \ldots, r$, ensures that the upper derivative of $F(x)$ is infinite for all $x \in C$, the proof of which is given in the next section as Proposition 2.1. Thus S now can be characterized as

$$S = N^+ \cup N^- \cup \{\text{endpoints of } C\},$$

where N^+ (N^-) is the set of non-endpoints of C at which the lower right (left) derivative of $F(x)$ is finite. Thus $\dim_H S = \max\{\dim_H N^+, \dim_H N^-\}$. For each $\delta > 0$ we construct sets $C^*(q_1(\delta)), C(q_2(\delta)), C^*(q_3(\delta))$ and $C(q_4(\delta))$, to be defined later by (2.10), (2.11), (2.16), (2.17), (2.19) and (2.20) respectively, to approximate N^+ and N^- by

$$C^*(q_1(\delta)) \subseteq N^+ \subseteq C(q_2(\delta)), \quad C^*(q_3(\delta)) \subseteq N^- \subseteq C(q_4(\delta)).$$

Therefore,

$$\eta(q_1(\delta)) := \dim_H C^*(q_1(\delta)) \le \dim_H N^+ \le \eta(q_2(\delta)) := \dim_H C(q_2(\delta)),$$

and

$$\eta(q_3(\delta)) := \dim_H C^*(q_3(\delta)) \le \dim_H N^- \le \eta(q_4(\delta)) := \dim_H C(q_4(\delta)),$$

where $\eta(q_1(\delta))$ and $\eta(q_2(\delta))$ are determined by (2.23) and (2.24) respectively, and $\eta(q_3(\delta))$ and $\eta(q_4(\delta))$ are determined in the same way just to replace a_δ by $\widetilde{a}_\delta$ there. We prove that $\dim_H N^+ = \lim_{\delta\to+\infty}\eta(q_1(\delta))$ and $\dim_H N^- = \lim_{\delta\to+\infty}\eta(q_3(\delta))$ by showing that $\lim_{\delta\to+\infty}\eta(q_1(\delta)) = \lim_{\delta\to+\infty}\eta(q_2(\delta))$ and $\lim_{\delta\to+\infty}\eta(q_3(\delta)) = \lim_{\delta\to+\infty}\eta(q_4(\delta))$. By means of monotonicity of $\eta(\cdot)$ and (2.18) we obtain

Theorem 1.1 *Suppose that $p_i > a_i$, $i = 0, 1, \ldots, r$. Let $d^+(\delta)$ and $d^-(\delta)$ be respectively given by*

$$\frac{\log p_r}{\log a_r}\log\sum_{\sigma\in\Omega_\delta} a_\sigma^{d^+(\delta)} + \left(1 - \frac{\log p_r}{\log a_r}\right)\log a_\delta^{d^+(\delta)} = 0,$$

and

$$\frac{\log p_r}{\log a_r}\log\sum_{\sigma\in\Omega_\delta} a_\sigma^{d^-(\delta)} + \left(1 - \frac{\log p_r}{\log a_r}\right)\log \widetilde{a}_\delta^{d^-(\delta)} = 0,$$

where for each $\delta > 0$, Ω_δ, a_δ *and* $\widetilde{a}_\delta$ *are defined by* (2.13), (2.14) *and* (2.15) *respectively. Then we have*

[A] $\dim_P S = \dim_B S = \xi$, *where* ξ *is defined in* (1.2)*;*

[B] $\dim_H S = \max\{\dim_H N^+, \dim_H N^-\}$, *where* $\dim_H N^+ = \lim_{\delta\to+\infty} d^+(\delta)$ *and* $\dim_H N^- = \lim_{\delta\to+\infty} d^-(\delta)$.

A very interesting and simple case occurs if $p_i = a_i \left(\sum_{i=0}^r a_i\right)^{-1}$, $i = 0, 1, \ldots, r$. In this case, all $\frac{p_i}{a_i}, i = 0, 1, \ldots, r$, are equal. Thus for each $\delta > 0$, $\Omega_\delta = \Omega^k$ for some $k = k(\delta)$ by (2.13), and so $a_\delta = a_r^k$ and $\widetilde{a}_\delta = a_0^k$. Theorem 1.1 then leads to (or alternatively from Corollary 2.3, Lemma 2.4 and Lemma 2.5 directly)

Corollary 1.2 *Let* $p_i = a_i \left(\sum_{i=0}^r a_i\right)^{-1}$, $i = 0, 1, \ldots, r$. *Let* d^+ *and* d^- *be respectively determined by*

$$\frac{\log a_r - \log\sum_{i=0}^r a_i}{\log a_r} \log\sum_{j\in\Omega} a_j^{d^+} + \left(1 - \frac{\log a_r - \log\sum_{i=0}^r a_i}{\log a_r}\right)\log a_r^{d^+} = 0$$

and

$$\frac{\log a_0 - \log\sum_{i=0}^r a_i}{\log a_0} \log\sum_{j\in\Omega} a_j^{d^-} + \left(1 - \frac{\log a_0 - \log\sum_{i=0}^r a_i}{\log a_0}\right)\log a_0^{d^-} = 0.$$

Then

[A] $\dim_P S = \dim_B S = \xi$, *where* ξ *is defined in* (1.2)*;*

[B] $\dim_H S = \max\{\dim_H N^+, \dim_H N^-\}$, *where* $\dim_H N^+ = d^+$ *and* $\dim_H N^- = d^-$.

This also was given by Morris [7] *for* $r = 1$ (*see* **R4**).

We like to remark that the case where $p_i = a_i^\xi$, $i = 0, 1, \ldots, r$, is not so simple as the case in Corollary 1.2 except all $a_i = a$ which reduces to the case considered in [2] and satisfies the conditions in Corollary 1.2. A finer treatment in [3] shows that $\dim_H N^+ = \dim_H N^- = \lim_{\delta\to+\infty} d^+(\delta) = \lim_{\delta\to+\infty} d^-(\delta) = \xi^2$ (see **R3**). The proof of Theorem 1.1 is given in the next section.

2 Proofs

In this section, we explore the Hausdorff, packing and box dimensions of S for the case where $p_i > a_i$, $i = 0, 1, \ldots, r$.

Proposition 2.1 *Let* μ *and* $F(x)$ *be given by* (1.3) *and* (1.4) *respectively. Let* $p_i > a_i$, $i = 0, 1, \ldots, r$. *Then the upper right derivative of* $F(x)$ *at a non-right-endpoint* t *of* C *is infinite and the upper left derivative of* $F(x)$ *at a non-left-endpoint* t *of* C *is infinite. Thus the upper derivative of* $F(x)$ *is infinite for all* $x \in C$.

P r o o f. For any t in C, t not a right endpoint, let its code be $\widetilde{t} = (t(1), t(2), \ldots)$. Then $\widetilde{t}$ has infinitely many entries lying in $\Omega \setminus \{r\}$. Suppose $\widetilde{t}$ has an entry from $\Omega \setminus \{r\}$ in position j. Then t lies in the interval $h_{\widetilde{t}|(j-1)}([0,1])$ but is not equal to the right endpoint u of $h_{\widetilde{t}|(j-1)}([0,1])$, where

$$\widetilde{u} = (t(1), \ldots, t(j-1), r, r, \ldots).$$

Note that u is also the right endpoint of $h_{\widetilde{u}|j}([0,1])$ and that $t \notin h_{\widetilde{u}|j}([0,1])$. Thus we have that $t, u \in h_{\widetilde{t}|(j-1)}([0,1])$ and $(t,u] \supseteq h_{\widetilde{u}|j}([0,1])$. Consider the slope of the line segment from the point $P = (t, F(t))$ on the graph of $F(x)$ to the point $Q = (u, F(u))$. We have

$$\frac{F(u) - F(t)}{u - t} = \frac{\mu((t,u])}{u - t} \ge \frac{\mu\big(h_{\widetilde{u}|j}([0,1])\big)}{m\big(h_{\widetilde{t}|(j-1)}([0,1])\big)} = \frac{p_{\widetilde{t}|(j-1)} p_r}{a_{\widetilde{t}|(j-1)}}. \tag{2.1}$$

Since $p_i > a_i, i = 0, 1, \ldots, r$, it is directly derived from (2.1) that the upper right derivative of $F(x)$ at a non-right-endpoint t of C is infinite. Symmetrically the upper left derivative of $F(x)$ at a non-left-endpoint t of C is infinite. □

From Proposition 2.1 it follows that for the case where $p_i > a_i$, $i = 0, 1, \ldots, r$, the set S can be decomposed into

$$S = N^+ \cup N^- \cup \{\text{endpoints of } C\}, \tag{2.2}$$

where N^+ (N^-) is the set of non-endpoints of C at which the lower right (left) derivative of $F(x)$ is finite. The following lemma characterizes the elements of N^+ and N^- by means of property of their codes.

Lemma 2.2 *Suppose that $p_i > a_i$, $i = 0, 1, \dots, r$. Let $\Gamma = \{0, 1, \dots, r-1\}$. Let $t \in C$ be no endpoint of C and let $z(t, n)$ denote the position of the n-th occurrence of elements of Γ in $\widetilde{t}$. Let $\underline{w} = \min_{i\in\Gamma} \frac{p_i}{a_i}$ and let $\overline{w} = \max_{i\in\Gamma} \frac{p_i}{a_i}$. Then*

(I) *if $t \in N^+$, then* $\limsup_{n\to\infty} \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_r}{\log p_r} + \left(\log \underline{w} - \log \frac{p_r}{a_r}\right) \frac{n}{z(t,n)\log p_r}\right) \geq 0$;

(II) *if $t \in C$ satisfies* $\limsup_{n\to\infty} \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_r}{\log p_r} + \left(\log \overline{w} - \log \frac{p_r}{a_r}\right) \frac{n}{z(t,n)\log p_r}\right) > 0$, *then $t \in N^+$.*

Symmetrically if we replace Γ by $\{1, 2, \dots, r\}$, then

(I') *if $t \in N^-$, then* $\limsup_{n\to\infty} \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_0}{\log p_0} + \left(\log \underline{w} - \log \frac{p_0}{a_0}\right) \frac{n}{z(t,n)\log p_0}\right) \geq 0$;

(II') *if $t \in C$ satisfies* $\limsup_{n\to\infty} \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_0}{\log p_0} + \left(\log \overline{w} - \log \frac{p_0}{a_0}\right) \frac{n}{z(t,n)\log p_0}\right) > 0$, *then $t \in N^-$.*

Proof. Since the upper derivative of $F(x)$ is infinite for all $x \in C$ by Proposition 2.1, N^+ (N^-) consists of non-endpoints of C at which the lower right (left) derivative of $F(x)$ is finite.

For the demonstration of the statement (I), it suffices to show that the lower-right derivative of $F(x)$ is infinite at a non-endpoint t of C when

$$\limsup_{n\to\infty} \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_r}{\log p_r} + \left(\log \underline{w} - \log \frac{p_r}{a_r}\right) \frac{n}{z(t,n)\log p_r}\right) < 0. \tag{2.3}$$

Consider such a point t with $\widetilde{t} = (t(1), t(2), \dots)$. By (2.3) let k be a positive integer such that

$$\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_r}{\log p_r} + \left(\log \underline{w} - \log \frac{p_r}{a_r}\right) \frac{n}{z(t,n)\log p_r} < q, \tag{2.4}$$

for some negative real number q whenever $n \geq k$. Let u be a positive number such that u is smaller than the distance between t and $[0,1] \setminus C_l$ with $l = z(t,k)$. Let x be a point in the segment $(t, t+u)$. Then $t, x \in h_{\widetilde{t}|l}([0,1])$. We will see that $(F(x) - F(t))/(x-t)$ is large relative to k, so t is not in N^+. Let i denote the level at which $x \notin h_{\widetilde{t}|i}([0,1])$ but $x \in h_{\widetilde{t}|(i-1)}([0,1])$. Note that also $t \in h_{\widetilde{t}|(i-1)}([0,1])$. Thus $x - t \leq m\big(h_{\widetilde{t}|(i-1)}([0,1])\big) = a_{\widetilde{t}|(i-1)}$; also $i = z(t,n)$ for some $n > k$. Put $j = z(t, n+1) - 1$, and by v we denote the right endpoint of $h_{\widetilde{t}|j}([0,1])$, which implies that $\widetilde{v} = (t(1), \dots, t(j), r, r, \dots)$ and $(t, v] \supseteq h_{\widetilde{v}|(j+1)}([0,1])$. Then we have $t < v < x$ and $F(v) - F(t) = \mu((t,v]) \geq \mu\big(h_{\widetilde{v}|(j+1)}([0,1])\big) = p_{\widetilde{t}|j} p_r$. Therefore, we have

$$\begin{aligned} \frac{F(x)-F(t)}{x-t} &\geq \frac{p_{\widetilde{t}|j} p_r}{a_{\widetilde{t}|(i-1)}} \\ &= \frac{p_r \prod_{i=1}^{z(t,n+1)-1} p_{t(i)}}{\prod_{i=1}^{z(t,n)-1} a_{t(i)}} \\ &= a_{t(z(t,n))} p_r^{z(t,n+1)-z(t,n)} \prod_{i=1}^{z(t,n)} \frac{p_{t(i)}}{a_{t(i)}} \\ &\geq \left(\min_{0\leq i\leq r} a_i\right) \left(p_r^{\frac{z(t,n+1)}{z(t,n)}-1} \left(\prod_{i=1}^{z(t,n)} \frac{p_{t(i)}}{a_{t(i)}}\right)^{\frac{1}{z(t,n)}}\right)^{z(t,n)}. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} Q &= p_r^{\frac{z(t,n+1)}{z(t,n)}-1}\left(\prod_{i=1}^{z(t,n)}\frac{p_{t(i)}}{a_{t(i)}}\right)^{\frac{1}{z(t,n)}} \\ &= p_r^{\frac{z(t,n+1)}{z(t,n)}-1}\left(\frac{p_r}{a_r}\right)^{\frac{z(t,n)-n}{z(t,n)}}\left(\prod_{i=1,t(i)\neq r}^{z(t,n)}\frac{p_{t(i)}}{a_{t(i)}}\right)^{\frac{1}{z(t,n)}} \\ &\geq p_r^{\frac{z(t,n+1)}{z(t,n)}-1}\left(\frac{p_r}{a_r}\right)^{\frac{z(t,n)-n}{z(t,n)}}\underline{w}^{\frac{n}{z(t,n)}}. \end{aligned}$$

Taking logs and by (2.4) we have

$$\log Q \geq \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_r}{\log p_r} + \left(\log \underline{w} - \log\frac{p_r}{a_r}\right)\frac{n}{z(t,n)\log p_r}\right)\log p_r > q\log p_r > 0. \quad (2.6)$$

Since t is a non-endpoint, $z(t,n) \to \infty$ as $n \to \infty$ and the lower-right derivative of $F(x)$ is infinite at t by (2.5) and (2.6).

For the proof of statement (II) let $t \in C$ be such that

$$\limsup_{n\to\infty}\left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_r}{\log p_r} + \left(\log \overline{w} - \log\frac{p_r}{a_r}\right)\frac{n}{z(t,n)\log p_r}\right) > 0.$$

Then there exists a sequence $\{n_k\}$ of positive integers such that

$$\frac{z(t,n_k+1)}{z(t,n_k)} - \frac{\log a_r}{\log p_r} + \left(\log \overline{w} - \log\frac{p_r}{a_r}\right)\frac{n_k}{z(t,n_k)\log p_r} > c \quad (2.7)$$

for some positive constant c. Let x_k be the left endpoint of $h_{(\widetilde{t}|j_k)*(t(j_k+1)+1)}([0,1])$, where $j_k = z(t,n_k)-1$. Thus we have $\widetilde{x}_k = (t(1),\dots,t(j_k),t(j_k+1)+1,0,\dots,0,\dots)$. Let u_k be the right endpoint of $h_{\widetilde{t}|(j_k+1)}([0,1])$. Then $\widetilde{u}_k = (t(1),\dots,t(j_k),t(j_k+1),r,r,r,\dots)$. Thus (u_k,x_k) is the gap on the right side of $h_{\widetilde{t}|(j_k+1)}([0,1])$ and $m((u_k,x_k)) = x_k - u_k = a_{\widetilde{t}|j_k}\beta_{t(j_k+1)}$ where by β_j, $j=0,1,\dots,r-1$, we denote the length of the gap between the images $h_j([0,1])$ and $h_{j+1}([0,1])$. Note that $[t,x_k] \supseteq [u_k,x_k]$ and $\mu((t,x_k]) = \mu((t,u_k]) + \mu((u_k,x_k]) = \mu((t,u_k]) \leq \mu\big(h_{\widetilde{t}|(z(t,n_k+1)-1)}([0,1])\big)$ since $\widetilde{t}\,\big|(z(t,n_k+1)-1) = \widetilde{u}_k|(z(t,n_k+1)-1)$. Therefore we have

$$F(x_k) - F(t) = \mu((t,x_k]) \leq \mu\big(h_{\widetilde{t}|(z(t,n_k+1)-1)}([0,1])\big) = p_{\widetilde{t}|(z(t,n_k+1)-1)},$$

and

$$x_k - t \geq m([u_k,x_k]) = a_{\widetilde{t}|(z(t,n_k)-1)}\beta_{t(z(t,n_k))}.$$

Denote $\beta_* = \min_{j\in\{0,1,\dots,r-1\}}\beta_j$ and $a^* = \max_{j\in\{0,1,\dots,r\}} a_j$. Then we obtain with a similar reasoning which led to (2.5)

$$\begin{aligned} \frac{F(x_k)-F(t)}{x_k-t} &\leq \frac{p_{\widetilde{t}|(z(t,n_k+1)-1)}}{a_{\widetilde{t}|(z(t,n_k)-1)}\beta_{t(z(t,n_k))}} \\ &= \frac{a_{t(z(t,n_k))}}{\beta_{t(z(t,n_k))}p_r}\, p_r^{z(t,n_k+1)-z(t,n_k)}\prod_{i=1}^{z(t,n_k)}\frac{p_{t(i)}}{a_{t(i)}} \\ &\leq \frac{a^*}{\beta_* p_r}\left(p_r^{\frac{z(t,n_k+1)}{z(t,n_k)}-1}\left(\prod_{i=1}^{z(t,n_k)}\frac{p_{t(i)}}{a_{t(i)}}\right)^{\frac{1}{z(t,n_k)}}\right)^{z(t,n_k)}. \end{aligned} \quad (2.8)$$

Let

$$\begin{aligned} Q &= p_r^{\frac{z(t,n_k+1)}{z(t,n_k)}-1}\left(\prod_{i=1}^{z(t,n_k)}\frac{p_{t(i)}}{a_{t(i)}}\right)^{\frac{1}{z(t,n_k)}} \\ &= p_r^{\frac{z(t,n_k+1)}{z(t,n_k)}-1}\left(\frac{p_r}{a_r}\right)^{\frac{z(t,n_k)-n_k}{z(t,n_k)}}\left(\prod_{i=1,t(i)\neq r}^{z(t,n_k)}\frac{p_{t(i)}}{a_{t(i)}}\right)^{\frac{1}{z(t,n_k)}} \\ &\le p_r^{\frac{z(t,n_k+1)}{z(t,n_k)}-1}\left(\frac{p_r}{a_r}\right)^{\frac{z(t,n_k)-n_k}{z(t,n_k)}}\overline{w}^{\frac{n_k}{z(t,n_k)}}. \end{aligned}$$

Taking logs and using (2.7) we obtain

$$\log Q \;\le\; \left(\frac{z(t,n_k+1)}{z(t,n_k)}-\frac{\log a_r}{\log p_r}+\left(\log\overline{w}-\log\frac{p_r}{a_r}\right)\frac{n_k}{z(t,n_k)\log p_r}\right)\log p_r \;<\; c\log p_r \;<\; 0. \tag{2.9}$$

From (2.8) and (2.9) it follows that the lower-right derivative of $F(x)$ at t is finite by letting $k\to\infty$. Finally the (I') and (II') can be proved similarly. □

In particular, when $p_i = a_i\left(\sum_{i=0}^r a_i\right)^{-1}$, $i = 0,1,\dots,r$, the sets of N^+ and N^- can be characterized in a simpler way since we have now that $\overline{w} = \underline{w} = \frac{p_r}{a_r} = \frac{p_0}{a_0} = \left(\sum_{i=0}^r a_i\right)^{-1}$ in Lemma 2.2. Thus it gives the following

Corollary 2.3 *Take $p_i = a_i\left(\sum_{i=0}^r a_i\right)^{-1}$, $i = 0,1,\dots,r$. Let $\Gamma = \{0,1,\dots,r-1\}$. Let $t\in C$ be no endpoint of C and let $z(t,n)$ denote the position of the n-th occurrence of elements of Γ in $\widetilde{t}$, then*

(I) *if $t\in N^+$, then* $\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} \ge \frac{\log a_r}{\log p_r} = \frac{\log a_r}{\log a_r-\log\sum_{i=0}^r a_i}$;

(II) *if* $\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} > \frac{\log a_r}{\log p_r} = \frac{\log a_r}{\log a_r-\log\sum_{i=0}^r a_i}$, *then $t\in N^+$.*

Symmetrically if we replace Γ by $\{1,2,\dots,r\}$, then

(I') *if $t\in N^-$, then* $\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} \ge \frac{\log a_0}{\log p_0} = \frac{\log a_0}{\log a_0-\log\sum_{i=0}^r a_i}$;

(II') *if* $\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} > \frac{\log a_0}{\log p_0} = \frac{\log a_0}{\log a_0-\log\sum_{i=0}^r a_i}$, *then $t\in N^-$.*

To estimate the Hausdorff dimension of N^+ (N^-) we construct, in the following Lemma 2.5, two sets approximating N^+ (N^-) from below and above. The Hausdorff dimensions of the approximating sets can be determined by the following Lemma 2.4 which is a special case of a result in [6].

Lemma 2.4 *Let Γ be a nonempty subset of Ω with $\Gamma^c\neq\emptyset$. Let $z(t,n)$ denote the position of the n-th occurrence of elements of Γ in $\widetilde{t}$. For given $0<q\le 1$, let*

$$C(q) = \left\{t\in C\setminus\{\textit{endpoints of } C\} : \limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} \ge q^{-1}\right\}. \tag{2.10}$$

Let $\eta=\eta(q)$ be such that

$$q\log\sum_{j\in\Omega}a_j^\eta + (1-q)\log\sum_{j\in\Gamma^c}a_j^\eta = 0.$$

Then we have $\dim_H C(q) = \eta$ and $\dim_P C(q) = \dim_B C(q) = \dim_H C = \xi$ where ξ is defined in (1.2).

It is easy to verify that $\eta(q)$ is strictly increasing and continuous in q and that $\eta(0+) < \eta(q) \le \eta(1) = \xi$. We also consider for $0<q\le 1$ and with the same Γ

$$C^*(q) = \left\{t\in C\setminus\{\text{endpoints of } C\} : \limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} > q^{-1}\right\}. \tag{2.11}$$

Directly from Lemma 2.4 it follows that

$$\dim_P C^*(q) = \dim_B C^*(q) = \dim_H C = \xi \quad\text{and}\quad \dim_H C^*(q) = \eta(q). \tag{2.12}$$

To see that the last equality follows from Lemma 2.4, approximate $C^*(q)$ by a union of $C(q_k)$'s, where $q_k\uparrow q$.

Lemma 2.5 *Suppose that* $p_i > a_i, i = 0, 1, \dots, r$. *Let* $\Gamma = \{0, 1, \dots, r-1\}$. *Let* $\underline{w} = \min_{i\in\Gamma} \frac{p_i}{a_i}$ *and let* $\overline{w} = \max_{i\in\Gamma} \frac{p_i}{a_i}$. *Let*

$$q_1^{-1} = \max\left\{\frac{\log a_r}{\log p_r}, \frac{\log a_r}{\log p_r} - \left(\log \overline{w} - \log \frac{p_r}{a_r}\right)\frac{1}{\log p_r}\right\},$$

and

$$q_2^{-1} = \min\left\{\frac{\log a_r}{\log p_r}, \frac{\log a_r}{\log p_r} - \left(\log \underline{w} - \log \frac{p_r}{a_r}\right)\frac{1}{\log p_r}\right\}.$$

Then $C^*(q_1) \subseteq N^+ \subseteq C(q_2)$ *and* $\eta(q_1) \le \dim_H N^+ \le \eta(q_2)$.

Symmetrically if we replace Γ *by* $\{1, 2, \dots, r\}$, *then* $C^*(q_3) \subseteq N^- \subseteq C(q_4)$ *and* $\eta(q_3) \le \dim_H N^- \le \eta(q_4)$ *where*

$$q_3^{-1} = \max\left\{\frac{\log a_0}{\log p_0}, \frac{\log a_0}{\log p_0} - \left(\log \overline{w} - \log \frac{p_0}{a_0}\right)\frac{1}{\log p_0}\right\},$$

and

$$q_4^{-1} = \min\left\{\frac{\log a_0}{\log p_0}, \frac{\log a_0}{\log p_0} - \left(\log \underline{w} - \log \frac{p_0}{a_0}\right)\frac{1}{\log p_0}\right\}.$$

P r o o f. We only prove the result on N^+ since the other result on N^- can be obtained in the same way. Thus it suffices to prove $C^*(q_1) \subseteq N^+ \subseteq C(q_2)$. Let $t \in N^+$. By Lemma 2.2 (I) we have

$$\limsup_{n\to\infty}\left(\frac{z(t, n+1)}{z(t, n)} - \frac{\log a_r}{\log p_r} + \left(\log \underline{w} - \log \frac{p_r}{a_r}\right)\frac{n}{z(t, n)\log p_r}\right) \ge 0.$$

Now if $\underline{w} \ge \frac{p_r}{a_r}$, then $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \ge \frac{\log a_r}{\log p_r}$. If not, we use that $\frac{n}{z(t,n)} \le 1$ and so $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \ge \frac{\log a_r}{\log p_r} - \left(\log \underline{w} - \log \frac{p_r}{a_r}\right)\frac{1}{\log p_r}$. So we find that

$$\limsup_{n\to\infty} \frac{z(t, n+1)}{z(t, n)} \ge q_2^{-1},$$

i.e., $t \in C(q_2)$. On the other hand, $t \in C^*(q_1)$ implies in a similar way that $t \in N^+$. □

Note that for the special case where $p_i = a_i\left(\sum_{i=0}^r a_i\right)^{-1}$, $i = 0, 1, \dots, r$, we have $\overline{w} = \underline{w} = \frac{p_r}{a_r} = \frac{p_0}{a_0} = \left(\sum_{i=0}^r a_i\right)^{-1}$ so that $q_1 = q_2 = \frac{\log p_r}{\log a_r}$ and $q_3 = q_4 = \frac{\log p_0}{\log a_0}$. Thus Lemma 2.5 gives the exact Hausdorff dimensions of N^+ and N^-. But for general cases we have $q_1 \ne q_2$ and $q_3 \ne q_4$. So Lemma 2.5 fails to determine the exact Hausdorff dimensions of N^+ and N^-. On the other hand, note that both $|q_1 - q_2|$ and $|q_3 - q_4|$ will be very small if $\left(\log \max_{i\in\Omega} \frac{p_i}{a_i} - \log \min_{i\in\Omega} \frac{p_i}{a_i}\right) / \min\{|\log p_0|, \log p_r|\}$ is small enough, so that $\eta(q_1)$ and $\eta(q_3)$ will be close to $\eta(q_2)$ and $\eta(q_4)$, respectively, since $\eta(q)$ is continuous in q. To realize this idea we review C as the invariant set under a class of similitude mappings each of which is a composition of certain h_j's, so that Lemma 2.5 could give $\eta(q_1)$ and $\eta(q_2)$ ($\eta(q_3)$ and $\eta(q_4)$) as close as expected.

Let $w_i = \frac{p_i}{a_i}$, $i \in \Omega$, and let $w = \max_{i\in\Omega} w_i$. Let $\delta > w^2$, and let

$$\Omega_\delta = \{\sigma \in \Omega^* : w_\sigma \ge \delta \text{ and } w_{\sigma|(|\sigma|-1)} < \delta\}, \tag{2.13}$$

where $w_\sigma = \prod_{j=1}^{|\sigma|} w_{\sigma(j)}$ for $\sigma \in \Omega^*$. There exist unique $\sigma, \widetilde{\sigma} \in \Omega_\delta$ with $\sigma(j) = r$ for $j = 1, 2, \dots, |\sigma|$ and $\widetilde{\sigma}(j) = 0$ for $j = 1, 2, \dots, |\widetilde{\sigma}|$ respectively. We denote these two special elements by σ_δ and $\widetilde{\sigma}_\delta$ respectively. Note that σ_δ plays the same role in Ω_δ as r in Ω, in the sense that $h_{\sigma_\delta}([0,1])$ is the most right interval in $[0,1]$ of the intervals $(h_\sigma([0,1]))_{\sigma\in\Omega_\delta}$, while $\widetilde{\sigma}_\delta$ plays the same role in Ω_δ as 0 in Ω, in the sense that $h_{\widetilde{\sigma}_\delta}([0,1])$ is the most left interval in $[0,1]$ of the intervals $(h_\sigma([0,1]))_{\sigma\in\Omega_\delta}$. Let

$$\Gamma_\delta := \Omega_\delta \setminus \{\sigma_\delta\}; \quad p_\delta := p_{\sigma_\delta} = p_r^{|\sigma_\delta|}; \quad w_\delta := w_{\sigma_\delta} = \frac{p_r^{|\sigma_\delta|}}{a_r^{|\sigma_\delta|}},$$

and

$$a_\delta := a_{\sigma_\delta} = a_r^{|\sigma_\delta|} \quad \text{where} \quad |\sigma_\delta| = \min\left\{k \in \mathbf{N} : \left(\frac{p_r}{a_r}\right)^k \geq \delta\right\}. \tag{2.14}$$

Symmetrically let

$$\widetilde{\Gamma}_\delta := \Omega_\delta \setminus \{\widetilde{\sigma}_\delta\}; \quad \widetilde{p}_\delta := p_{\widetilde{\sigma}_\delta} = p_0^{|\widetilde{\sigma}_\delta|}; \quad \widetilde{w}_\delta := w_{\widetilde{\sigma}_\delta} = \frac{p_0^{|\widetilde{\sigma}_\delta|}}{a_0^{|\widetilde{\sigma}_\delta|}}$$

and

$$\widetilde{a}_\delta := a_{\widetilde{\sigma}_\delta} = a_0^{|\widetilde{\sigma}_\delta|} \quad \text{where} \quad |\sigma_\delta| = \min\left\{k \in \mathbf{N} : \left(\frac{p_0}{a_0}\right)^k \geq \delta\right\}. \tag{2.15}$$

We denote $\underline{w}_\delta = \min_{\sigma\in\Gamma_\delta} w_\sigma$, $\overline{w}_\delta = \max_{\sigma\in\Gamma_\delta} w_\sigma$, and set

$$\begin{aligned} q_1(\delta) &= \left(\max\left\{\frac{\log a_\delta}{\log p_\delta}, \frac{\log a_\delta}{\log p_\delta} - \left(\log \overline{w}_\delta - \log\frac{p_\delta}{a_\delta}\right)\frac{1}{\log p_\delta}\right\}\right)^{-1} \\ &= \left(\max\left\{\frac{\log a_r}{\log p_r}, \frac{\log a_r}{\log p_r} - \left(\log \overline{w}_\delta - \log\frac{p_\delta}{a_\delta}\right)\frac{1}{\log p_\delta}\right\}\right)^{-1}, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} q_2(\delta) &= \left(\min\left\{\frac{\log a_\delta}{\log p_\delta}, \frac{\log a_\delta}{\log p_\delta} - \left(\log \underline{w}_\delta - \log\frac{p_\delta}{a_\delta}\right)\frac{1}{\log p_\delta}\right\}\right)^{-1} \\ &= \left(\min\left\{\frac{\log a_r}{\log p_r}, \frac{\log a_r}{\log p_r} - \left(\log \underline{w}_\delta - \log\frac{p_\delta}{a_\delta}\right)\frac{1}{\log p_\delta}\right\}\right)^{-1}, \end{aligned} \tag{2.17}$$

since $\frac{\log a_\delta}{\log p_\delta} = \frac{\log a_r}{\log p_r}$. It is easy to check that

$$q_1(\delta) \leq \frac{\log p_r}{\log a_r} \leq q_2(\delta). \tag{2.18}$$

Symmetrically we denote $\underline{\widetilde{w}}_\delta = \min_{\sigma\in\widetilde{\Gamma}_\delta} w_\sigma$, $\overline{\widetilde{w}}_\delta = \max_{\sigma\in\widetilde{\Gamma}_\delta} w_\sigma$, and set

$$\begin{aligned} q_3(\delta) &= \left(\max\left\{\frac{\log \widetilde{a}_\delta}{\log \widetilde{p}_\delta}, \frac{\log \widetilde{a}_\delta}{\log \widetilde{p}_\delta} - \left(\log \overline{\widetilde{w}}_\delta - \log\frac{\widetilde{p}_\delta}{\widetilde{a}_\delta}\right)\frac{1}{\log \widetilde{p}_\delta}\right\}\right)^{-1} \\ &= \left(\max\left\{\frac{\log a_0}{\log p_0}, \frac{\log a_0}{\log p_0} - \left(\log \overline{\widetilde{w}}_\delta - \log\frac{\widetilde{p}_\delta}{\widetilde{a}_\delta}\right)\frac{1}{\log \widetilde{p}_\delta}\right\}\right)^{-1}. \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} q_4(\delta) &= \left(\min\left\{\frac{\log \widetilde{a}_\delta}{\log \widetilde{p}_\delta}, \frac{\log \widetilde{a}_\delta}{\log \widetilde{p}_\delta} - \left(\log \underline{\widetilde{w}}_\delta - \log\frac{\widetilde{p}_\delta}{\widetilde{a}_\delta}\right)\frac{1}{\log \widetilde{p}_\delta}\right\}\right)^{-1} \\ &= \left(\min\left\{\frac{\log a_0}{\log p_0}, \frac{\log a_0}{\log p_0} - \left(\log \underline{\widetilde{w}}_\delta - \log\frac{\widetilde{p}_\delta}{\widetilde{a}_\delta}\right)\frac{1}{\log \widetilde{p}_\delta}\right\}\right)^{-1}, \end{aligned} \tag{2.20}$$

since $\frac{\log \widetilde{a}_\delta}{\log \widetilde{p}_\delta} = \frac{\log a_0}{\log p_0}$. Also we have $q_3(\delta) \leq \frac{\log p_r}{\log a_r} \leq q_4(\delta)$. Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 2.4 and 2.5, [A] is trivial since we have $\dim_B N^+ = \dim_P N^+ = \dim_B N^- = \dim_P N^- = \dim_H C = \xi$. Since $S = N^+ \bigcup N^- \bigcup\{\text{the endpoints of } C\}$ in our case, we have $\dim_H S = \max\{\dim_H N^+, \dim_H N^-\}$. In the following we only determine the Hausdorff dimension of N^+ because of the symmetry between N^+ and N^-.

Note that for each $\sigma \in \Omega_\delta$ we have $\delta \leq w_\sigma < w\delta$. Thus for any $\sigma, \tau \in \Omega_\delta$

$$\frac{\log \delta}{\log w + \log \delta} \leq \frac{\log w_\sigma}{\log w_\tau} \leq \frac{\log w + \log \delta}{\log \delta},$$

and

$$|\log w_\sigma - \log w_\tau| \leq \log w. \tag{2.21}$$

Now let $\mathcal{H}_\delta = \{h_\sigma : \sigma \in \Omega_\delta\}$. Note that h_σ is a similitude mapping with ratio $0 < a_\sigma < 1$ for each $\sigma \in \mathcal{H}_\delta$, that the family $\mathcal{H}_\delta$ of similitude mappings still satisfies the (strongly) open set condition, and that the unique self-similar set determined by $\mathcal{H}_\delta$ equals C:

$$C = \bigcup_{\sigma \in \mathcal{H}_\delta} h_\sigma(C).$$

We also have that ξ defined in (1.2) satisfies $\sum_{\sigma \in \Omega_\delta} a_\sigma^\xi = 1$. If we denote μ_δ the self-similar probability measure on C corresponding to the probability vector $(p_\sigma : \sigma \in \Omega_\delta)$, then $\mu_\delta = \mu$ since $\mu_\delta(h_\tau([0,1])) = \mu(h_\tau([0,1]))$ for any $\tau \in \Omega_\delta^k$ and $k \in \mathbb{N}$. Hence for the corresponding sets N_δ^+ and N_δ^- of non-differentiability points we have $N_\delta^+ = N^+$ and $N_\delta^- = N^-$ for all $\delta > 0$. Then by Lemmas 2.4 and 2.5 with Ω replaced by Ω_δ, Γ by Γ_δ, a_r by a_δ, p_r by p_δ, $\underline{w}$ by $\underline{w}_\delta$, $\overline{w}$ by $\overline{w}_\delta$, and q_1, q_2 by $q_1(\delta)$, $q_2(\delta)$ respectively, we have

$$\eta(q_1(\delta)) \leq \dim_H N_\delta^+ = \dim_H N^+ \leq \eta(q_2(\delta)), \tag{2.22}$$

where by means of (2.11), (2.12) and Lemma 2.4 $\eta(q_1(\delta))$ and $\eta(q_2(\delta))$ are given by

$$q_1(\delta) \log \sum_{\sigma \in \Omega_\delta} a_\sigma^{\eta(q_1(\delta))} + (1 - q_1(\delta)) \log a_\delta^{\eta(q_1(\delta))} = 0 \tag{2.23}$$

and

$$q_2(\delta) \log \sum_{\sigma \in \Omega_\delta} a_\sigma^{\eta(q_2(\delta))} + (1 - q_2(\delta)) \log a_\delta^{\eta(q_2(\delta))} = 0. \tag{2.24}$$

From the analysis above it is expected that $\lim_{\delta \to +\infty} \eta(q_1(\delta)) = \lim_{\delta \to +\infty} \eta(q_2(\delta)) = \dim_H N_\delta^+$ since

$$\lim_{\delta \to +\infty} \left(\log \overline{w}_\delta - \log \frac{p_\delta}{a_\delta} \right) \frac{1}{\log p_\delta} = \lim_{\delta \to +\infty} \left(\log \underline{w}_\delta - \log \frac{p_\delta}{a_\delta} \right) \frac{1}{\log p_\delta} = 0,$$

by (2.21), leading to

$$\lim_{\delta \to +\infty} q_1(\delta) = \lim_{\delta \to +\infty} q_2(\delta) = \frac{\log p_r}{\log a_r} \in (0,1). \tag{2.25}$$

Now we rewrite (2.23) and (2.24) as

$$\eta(q_1(\delta)) = \frac{q_1(\delta)}{q_1(\delta) - 1} \cdot \frac{\log \sum_{\sigma \in \Omega_\delta} a_\sigma^{\eta(q_1(\delta))}}{|\sigma_\delta| \log a_r} \tag{2.26}$$

and

$$\eta(q_2(\delta)) = \frac{q_2(\delta)}{q_2(\delta) - 1} \cdot \frac{\log \sum_{\sigma \in \Omega_\delta} a_\sigma^{\eta(q_2(\delta))}}{|\sigma_\delta| \log a_r}. \tag{2.27}$$

Suppose that $\lim_{\delta \to +\infty}(\eta(q_2(\delta)) - \eta(q_1(\delta))) = 0$ doesn't hold. Then there exist a positive real number ϵ_0 and a sequence of positive real numbers $\delta_i \uparrow +\infty$ such that

$$\eta(q_2(\delta_i)) - \eta(q_1(\delta_i)) > \epsilon_0. \tag{2.28}$$

Note that

$$\begin{aligned}
&\frac{\log \sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_2(\delta_i))}}{|\sigma_{\delta_i}|\log a_r} - \frac{\log \sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}{|\sigma_{\delta_i}|\log a_r}\\
&= \frac{1}{|\sigma_{\delta_i}|\log a_r}\log\frac{\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_2(\delta_i))}}{\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}\\
&= \frac{1}{|\sigma_{\delta_i}|\log a_r}\log\frac{\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))} a_\sigma^{\eta(q_2(\delta_i))-\eta(q_1(\delta_i))}}{\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}\\
&\ge \frac{\log(\max_{\sigma\in\Omega_{\delta_i}} a_\sigma)^{\epsilon_0}}{|\sigma_{\delta_i}|\log a_r}\\
&= \frac{\epsilon_0 \log\max_{\sigma\in\Omega_{\delta_i}} a_\sigma}{\log a_{\delta_i}}.
\end{aligned} \tag{2.29}$$

Hence by (2.26), (2.27), (2.28) and (2.29) we have

$$\begin{aligned}
\epsilon_0 &< \eta(q_2(\delta_i)) - \eta(q_1(\delta_i))\\
&= \frac{q_2(\delta_i)}{q_2(\delta_i)-1}\cdot\frac{\log\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_2(\delta_i))}}{|\sigma_{\delta_i}|\log a_r} - \frac{q_1(\delta_i)}{q_1(\delta_i)-1}\cdot\frac{\log\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}{|\sigma_{\delta_i}|\log a_r}\\
&= \frac{q_2(\delta_i)}{q_2(\delta_i)-1}\left(\frac{\log\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_2(\delta_i))}}{|\sigma_{\delta_i}|\log a_r} - \frac{\log\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}{|\sigma_{\delta_i}|\log a_r}\right)\\
&\quad + \frac{\log\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}{|\sigma_{\delta_i}|\log a_r}\left(\frac{q_2(\delta_i)}{q_2(\delta_i)-1} - \frac{q_1(\delta_i)}{q_1(\delta_i)-1}\right)\\
&\le \frac{q_2(\delta_i)}{q_2(\delta_i)-1}\cdot\frac{\epsilon_0\log\max_{\sigma\in\Omega_{\delta_i}} a_\sigma}{\log a_{\delta_i}} + \frac{\log\sum_{\sigma\in\Omega_{\delta_i}} a_\sigma^{\eta(q_1(\delta_i))}}{|\sigma_{\delta_i}|\log a_r}\left(\frac{q_2(\delta_i)}{q_2(\delta_i)-1} - \frac{q_1(\delta_i)}{q_1(\delta_i)-1}\right)\\
&= \frac{q_2(\delta_i)}{q_2(\delta_i)-1}\cdot\frac{\epsilon_0\log\max_{\sigma\in\Omega_{\delta_i}} a_\sigma}{\log a_{\delta_i}} + \frac{\eta(q_1(\delta_i))(q_1(\delta_i)-1)}{q_1(\delta_i)}\left(\frac{q_2(\delta_i)}{q_2(\delta_i)-1} - \frac{q_1(\delta_i)}{q_1(\delta_i)-1}\right).
\end{aligned}$$

Note that in the last line above the first term is negative and the second term tends to zero when i is large enough by (2.25), yielding a contradiction! Therefore we have $\lim_{\delta\to+\infty}(\eta(q_2(\delta)) - \eta(q_1(\delta))) = 0$. Thus $\lim_{\delta\to+\infty}(\eta(q_2(\delta)) = \lim_{\delta\to+\infty}\eta(q_1(\delta))) = \dim_H N_\delta^+ = \dim_H N^+$ by (2.22). Symmetrically we have $\lim_{\delta\to+\infty}(\eta(q_3(\delta)) = \dim_H N_\delta^- = \dim_H N^-$. Finally, by means of monotonicity of $\eta(\cdot)$ we have $\eta(q_1(\delta)) \le \eta\left(\frac{\log p_r}{\log a_r}\right) = d^+(\delta) \le \eta(q_2(\delta))$ by (2.18). This gives $\dim_H N^+ = \lim_{\delta\to+\infty} d^+(\delta)$, and $\dim_H N^- = \lim_{\delta\to+\infty} d^-(\delta)$ in the same way. □

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