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An equivalent definition of packing dimension and its application

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Abstract

An equivalent definition of packing dimension is given for a set in d-dimensional Euclidean space by using its component sets as packings. It is applied to determine the packing dimensions of a class of subsets with prescribed relative group frequencies.

Keywords: Moran set; Component set; Packing dimension; J-type packing

1. Introduction

The packing dimension of a set in d-dimensional Euclidean space plays an important role in fractal geometry. We first recall its definition. For a set $E \subseteq \mathbb{R}^d$ and $\delta > 0$, a δ-packing of E is defined as a collection of at most countable number of disjoint balls of radii at most δ with centers in E. Fixing $s \geq 0$, the s-dimensional packing measure of E is defined by

$$\mathcal{P}^s(E) = \inf\left\{\sum_i \mathcal{P}_0^s(E_i) : E \subseteq \cup_{i=1}^{\infty} E_i\right\},$$

where $\mathcal{P}_0^s(E_i) = \lim_{\delta\to 0+} \sup \sum_j |B_j|^s$ and the sup is taken over all δ-packing $\{B_j\}$ of E_i. The packing dimension of E is then given by

$$\dim_P E = \sup\{s : \mathcal{P}^s(E) = \infty\} = \inf\{s : \mathcal{P}^s(E) = 0\}.$$

Since the packings used for $\mathcal{P}_0^s(E_i)$ are restricted to be disjoint balls centered at E_i, this often makes inconvenient for evaluating the packing dimension of E. Thus, it is wise and necessary to find out an alternative way to relax the restriction on a packing but induce the same value for the packing dimension as that done above. Some results on this topic were given by Taylor and Tricot in [7]. Since the packing dimension is σ-stable, i.e., $\dim_P \bigcup_{i=1}^{\infty} E_i = \sup_i \dim_P E_i$, we may assume that E is bounded. In the present paper, we confine ourselves to the so-called Moran sets defined below as in (1). As one can see, a Moran set is a well-constructed compact set which includes the self-similar sets satisfying open set condition and even any closed cubes. Thus each bounded set E can always be considered as a subset of a Moran set, say F. What we like to do is to make use of the component sets of F as packings for evaluating the packing dimension of E. A Moran set is defined as follows.

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Denote $\Omega = \{1, 2, \ldots, r\}$, where $r \geq 2$. Let $\Omega^* = \bigcup_{m=0}^{\infty} \Omega^m$ with $\Omega^m = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m)) : \sigma(j) \in \Omega\}$ for $m \in \mathbb{N} \cup \{0\}$ (Ω^0 consists of the empty word $\emptyset$) and $\Omega^{\mathbb{N}} = \{\sigma = (\sigma(1), \sigma(2), \ldots) : \sigma(j) \in \Omega\}$. $|\sigma|$ is used to denote the length of word $\sigma \in \Omega^*$. For $\sigma \in \Omega^*$ and $\tau \in \Omega^* \cup \Omega^{\mathbb{N}}$, $\sigma * \tau$ denotes their concatenation. In particular, $\emptyset * \tau = \tau$. For $\sigma \in \Omega^m$, let $C(\sigma) = \{\tau \in \Omega^{\mathbb{N}} : \tau|m = \sigma\}$, called the cylinder set with base σ, where $\tau|m = (\tau(1), \tau(2), \ldots, \tau(m))$.

Fix a constant $0 < c < 1$ and positive real numbers $0 < a_i < 1, \quad i = 1, 2, \ldots, r$. Assume that a collection $(J_\sigma)_{\sigma \in \Omega^*}$ of compact sets in $\mathbb{R}^d$ has the following features:

[A1] nested property: for $\sigma \in \Omega^*$ and $i \in \Omega$, $J_{\sigma * i} \subseteq J_\sigma$;
[A2] nonoverlapping property: all J_σs with $\sigma \in \Omega^m$ are pairwise nonoverlapping in the sense that $J_\sigma \cap J_\tau$ is of zero d-dimensional Lebesgue measure for any distinct $\sigma, \tau \in \Omega^m$;
[A3] regular sizes for J_σs: for $\sigma \in \Omega^*$ and $i \in \Omega$, $|J_{\sigma * i}| = a_i |J_\sigma| > 0$, where, if no confusion occurs, $|J_\sigma|$ denotes the diameter of J_σ;
[A4] regular sizes for the interior of J_σs: each $J_\sigma, \sigma \in \Omega^*$ contains an open ball with diameter $c|J_\sigma|$.

The *Moran set* F associated with $(J_\sigma)_{\sigma \in \Omega^*}$ is defined as the nonempty compact set

$$F = \bigcap_{m=0}^{\infty} \bigcup_{\sigma \in \Omega^m} J_\sigma. \tag{1}$$

The definition of Moran sets here is close to that in [1,6] with a bit variation, but simpler than that in [9] where a more general structure is discussed (Readers can refer to [9] and more references therein for related results on this kind of structure). Obviously, self-similar sets satisfying open set condition are Moran sets. The latter lose the property of similitude, but keep some typical properties exhibited by the former, e.g., they can be encoded by elements from $\Omega^{\mathbb{N}}$ and

$$\dim_H F = \overline{\dim}_B F = s,\ 0 < \mathcal{H}^s(F) \leq \mathcal{P}^s(F) < \infty,$$

where s is determined by $\sum_{j=1}^{r} a_j^s = 1$. The compact sets $J_\sigma, \sigma \in \Omega^*$ are generally referred to as *component sets* of F. In particular, J_σ is referred to as an mth-level component set of F if $\sigma \in \Omega^m$. Define ϕ: $\Omega^{\mathbb{N}} \longrightarrow \mathbb{R}^d$ by

$$\{\phi(\sigma)\} = \bigcap_{m=0}^{\infty} J_{\sigma|m}. \tag{2}$$

It is easy to see that $\phi(\Omega^{\mathbb{N}}) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (2). But ϕ may not be injective. Let ρ be the metric on $\Omega^{\mathbb{N}}$ such that for any $\sigma, \tau \in \Omega^{\mathbb{N}}$

$$\rho(\sigma, \tau) = 2^{-\min\{i :\, \sigma(i) \neq \tau(i)\}},$$

with the convention $\rho(\sigma, \sigma) = 0$. Let F be equipped with the Euclidean metric. Then ϕ is continuous. Thus each $x \in F$ can be encoded via ϕ: a sequence $\sigma \in \Omega^{\mathbb{N}}$ is called a location code of $x \in F$ if $\phi(\sigma) = x$. Therefore, ϕ is also called the coding map and $\Omega^{\mathbb{N}}$ is called the code space (or symbolic space). As a result, F is a projection of $\Omega^{\mathbb{N}}$ on $\mathbb{R}^d$ via ϕ.

Let $0 < R < a_{\min}|J_\emptyset|$ where $a_{\min} := \min_{1 \leq i \leq r} a_i$. A component set J_σ of F is termed as an R-size component set if

$$|J_\sigma| \leq R \quad \text{and} \quad |J_{\sigma|(|\sigma|-1)}| > R.$$

It is easy to see that for any $0 < R < a_{\min}|J_\emptyset|$, the set of all R-size component sets of F is a nonoverlapping finite R-covering of F. Hence, by means of Lemma 9.2 in [3], the requirement (see [A4]) that J_σ contains an open ball of diameter $c|J_\sigma|$ implies an important fact: there exists a positive integer ϑ, independent of R and $x \in \mathbb{R}^d$, such that any ball $B_R(x)$ with radius R and center at x intersects at most ϑ of the R-size component sets of F. Many analogues of this fact appear in this paper. A direct sequel leads to an important property of ϕ:

$$\sup_{x \in F} \#\{\phi^{-1}(x)\} < \vartheta,$$

where and throughout this paper, by $\#A$ we denote the cardinality of a finite set A. Otherwise, suppose that for some $x \in F$ we have $\#\{\phi^{-1}(x)\} > \vartheta$. Take $\vartheta + 1$ different elements $\sigma_1, \sigma_2, \ldots, \sigma_{\vartheta+1}$ from this set. Let $m \in \mathbb{N}$ be such that $\sigma_1|m, \sigma_2|m, \ldots, \sigma_{\vartheta+1}|m$ differ from each other. Taking $R = \min_{1\le i\le\vartheta+1} |J_{\sigma_i|m}|$, x lies in at least $\vartheta + 1$ R-size component sets of F, which implies that the ball $B_R(x)$ intersects at least $\vartheta + 1$ of the R-size component sets of F.

As we know, the component sets from $(J_\sigma)_{\sigma\in\Omega^*}$ play an important role in determining the Hausdorff measures (correspondingly, the Hausdorff dimensions) of F or its subsets. In fact, when considering the Hausdorff measures of F or its subsets one can restrict their attention to those covers related to $(J_\sigma)_{\sigma\in\Omega^*}$. For $n \ge 0$ let $\mathcal{G}_n$ be the collection of all possible union of the nth-level component sets from $(J_\sigma)_{\sigma\in\Omega^n}$. Then for a nonempty compact subset E of F,

$$\mathcal{H}^s(E) = \mathcal{H}^s_{\mathcal{G}}(E),$$

where $0 \le s < \infty$ and $\mathcal{H}^s_{\mathcal{G}}(E) = \lim_{\delta\to 0} \inf\{\sum |V_i|^s : \bigcup V_i \supseteq E, |V_i| \le \delta \text{ and all } V_i \in \mathcal{G}_n \text{ for some } n\}$.

In the following, we claim that the collection $(J_\sigma)_{\sigma\in\Omega^*}$ of component sets also can be employed as packings to evaluate the packing dimension of any subsets of the Moran set F. In other words, we like to give an equivalent definition of packing dimension for subsets of the Moran set F by taking the component sets from $(J_\sigma)_{\sigma\in\Omega^*}$ as packings instead of disjoint balls in its original definition. As a direct application, we determine packing dimensions of a class of subsets with prescribed relative group frequencies (see Theorem 1.3).

Let F be a Moran set defined as in (1). Let $E \subseteq F$. A collection $\{J_\sigma : \sigma \in \mathcal{A}\}$ with $\mathcal{A} \subseteq \Omega^*$ is said a *J-type δ-packing* of E if

$$|J_\sigma| \le \delta; \quad \text{and} \quad J_\sigma \cap E \ne \emptyset \text{ for each } \sigma \in \mathcal{A}; \qquad \operatorname{int} J_\sigma \cap \operatorname{int} J_\tau = \emptyset \quad \text{for any distinct } \sigma, \tau \in \mathcal{A}.$$

For a fixed $s \ge 0$, we define

$$\widetilde{\mathcal{P}}^s_\delta(E) = \sup\left\{\sum_{\sigma\in\mathcal{A}} |J_\sigma|^s : \{J_\sigma\}_{\sigma\in\mathcal{A}} \text{ is a } J\text{-type } \delta\text{-packing of } E\right\},$$

and

$$\widetilde{\mathcal{P}}^s_0(E) = \lim_{\delta\to 0} \widetilde{\mathcal{P}}^s_\delta(E). \tag{3}$$

Clearly, $\widetilde{\mathcal{P}}^s_0 : 2^F \longrightarrow [0, +\infty]$ has the following properties:

(I) $\widetilde{\mathcal{P}}^s_0(E_1) \le \widetilde{\mathcal{P}}^s_0(E_2)$ if $E_1 \subseteq E_2 \subseteq F$;
(II) $\widetilde{\mathcal{P}}^s_0(E_1 \cup E_2) \le \widetilde{\mathcal{P}}^s_0(E_1) + \widetilde{\mathcal{P}}^s_0(E_2)$ for $E_1, E_2 \subseteq F$;
(III) $\widetilde{\mathcal{P}}^s_0(E_1 \cup E_2) = \widetilde{\mathcal{P}}^s_0(E_1) + \widetilde{\mathcal{P}}^s_0(E_2)$ if $E_1, E_2 \subseteq F$ are positively separated;

Then, we obtain a packing-type (outer) measure on subsets of F from the pre-measure $\widetilde{\mathcal{P}}^s_0$ (cf. [5, Theorem 4]) by

$$\widetilde{\mathcal{P}}^s(E) := \inf\left\{\sum_{i=1}^{\infty} \widetilde{\mathcal{P}}^s_0(E_i) : E \subseteq \cup_{i=1}^{\infty} E_i,\ E_i \subseteq F\right\} = \inf\left\{\sum_{i=1}^{\infty} \widetilde{\mathcal{P}}^s_0(E_i) : E = \cup_{i=1}^{\infty} E_i,\ E_i \subseteq F\right\}. \tag{4}$$

Note that each ball $B_R(x)$ with $x \in E$ contains a J_σ with $a_{\min}|J_\emptyset|R \le |J_\sigma| = |J_\emptyset|a(\sigma) \le R$ where, and throughout this paper, we denote $a(\sigma) := a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(m)}$ for $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m)) \in \Omega^m$. If $\{B_{R_i}(x) : i = 1, 2, \ldots\}$ is a ball δ-packing of E, then these J_σ's are the J-type δ-packings of E. Then

$$\sum_i (2R_i)^s \le \left(\frac{2}{a_{\min}|J_\emptyset|}\right)^s \sum_\sigma |J_\sigma|^s \le \left(\frac{2}{a_{\min}|J_\emptyset|}\right)^s \widetilde{\mathcal{P}}^s_\delta(E),$$

giving $\mathcal{P}^s_\delta(E) \le \left(\frac{2}{a_{\min}|J_\emptyset|}\right)^s \widetilde{\mathcal{P}}^s_\delta(E)$. So

$$\mathcal{P}^s_0(E) \le \left(\frac{2}{a_{\min}|J_\emptyset|}\right)^s \widetilde{\mathcal{P}}^s_0(E) \quad \text{and} \quad \mathcal{P}^s(E) \le \left(\frac{2}{a_{\min}|J_\emptyset|}\right)^s \widetilde{\mathcal{P}}^s(E).$$

Furthermore, we have

$$\widetilde{\mathcal{P}}^s(E) \ge \inf\left\{\lim_{n\to\infty} \widetilde{\mathcal{P}}^s_0(E_n) : E_n \subseteq F, E_n \uparrow E\right\}. \tag{5}$$

In fact, if $E = \bigcup_{i=1}^{\infty} E_i$ with $E_i \subseteq F$, we have $E_n^* := \bigcup_{i=1}^{n} E_i \uparrow E$ and

$$\widetilde{\mathcal{P}}_0^s(E_n^*) \le \sum_{i=1}^{n} \widetilde{\mathcal{P}}_0^s(E_i) \le \sum_{i=1}^{\infty} \widetilde{\mathcal{P}}_0^s(E_i),$$

leading to $\lim_{n\to\infty} \widetilde{\mathcal{P}}_0^s(E_n^*) \le \sum_{i=1}^{\infty} \widetilde{\mathcal{P}}_0^s(E_i)$. We like to point out that the (outer) measures $\widetilde{\mathcal{P}}^s$ and $\mathcal{P}^s$ may not be equivalent (the equivalence means there are positive constants c_1, c_2 such that $c_1\widetilde{\mathcal{P}}^s(E) \le \mathcal{P}^s(E) \le c_2\widetilde{\mathcal{P}}^s(E)$ for any $E \subseteq F$). However, they do induce the same critical index.

Theorem 1.1. *Let F be defined as in (1). Let $\widetilde{\mathcal{P}}_0^s$ and $\widetilde{\mathcal{P}}^s$ be defined as in (3) and (4). Then for any $E \subseteq F$*

$$\overline{\dim}_B E = \inf\left\{s : \widetilde{\mathcal{P}}_0^s(E) = 0\right\} = \sup\left\{s : \widetilde{\mathcal{P}}_0^s(E) = +\infty\right\}, \tag{6}$$

and

$$\dim_P E = \inf\left\{s : \widetilde{\mathcal{P}}^s(E) = 0\right\} = \sup\left\{s : \widetilde{\mathcal{P}}^s(E) = +\infty\right\}. \tag{7}$$

We remark that any closed cube in $\mathbb{R}^d$ can be considered as a self-similar set satisfying the open set condition (so it is a Moran set). Thus the above theorem can be applied to any bounded subset of $\mathbb{R}^d$ which, in a sense, amounts to taking subcubes as packings (also see [7] with dyadic cubes as packings). Usually, the Hausdorff and packing dimensions of a set can be estimated in terms of local dimensions of measures. For a finite measure μ on $\mathbb{R}^d$, the lower and upper local dimensions of μ at $x \in \mathbb{R}^d$ (also called the pointwise dimension or Hölder exponent) are defined by

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{R\to 0} \frac{\log \mu(B_R(x))}{\log R}$$

and

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{R\to 0} \frac{\log \mu(B_R(x))}{\log R}.$$

Proposition 2.3 in [2] describes a method to estimate the lower and upper bounds of the Hausdorff and packing dimensions for a set in terms of the local dimensions of measures. The following proposition provides a version for the case of Moran sets where we consider measures on $\Omega^{\mathbb{N}}$ and replace the balls $B_R(x)$ by the cylinder sets $C(\sigma)$.

Proposition 1.2. *Let $\Lambda \subset \Omega^{\mathbb{N}}$ be a Borel set and ϕ be defined as in (2). Let $\hat{\mu}$ be a finite Borel measure on $\Omega^{\mathbb{N}}$.*

(I) *If $\limsup_{m\to+\infty} \frac{\log \hat{\mu}(C(\sigma|m))}{\log a(\sigma|m)} \le s$ for all $\sigma \in \Lambda$, then $\dim_P \phi(\Lambda) \le s$.*
(II) *If $\hat{\mu}(\Lambda) > 0$ and $\limsup_{m\to+\infty} \frac{\log \hat{\mu}(C(\sigma|m))}{\log a(\sigma|m)} \ge s$ for all $\sigma \in \Lambda$, then $\dim_P \phi(\Lambda) \ge s$.*
(III) *if $\liminf_{m\to+\infty} \frac{\log \hat{\mu}(C(\sigma|m))}{\log a(\sigma|m)} \le s$ for all $\sigma \in \Lambda$, then $\dim_H \phi(\Lambda) \le s$.*
(IV) *if $\hat{\mu}(\Lambda) > 0$ and $\liminf_{m\to+\infty} \frac{\log \hat{\mu}(C(\sigma|m))}{\log a(\sigma|m)} \ge s$ for all $\sigma \in \Lambda$, then $\dim_H \phi(\Lambda) \ge s$.*

A direct application of Proposition 1.2 is to determine the packing dimensions of a class of subsets of the Moran set F. Let $2 \le k \le r$. Fix real numbers c_j, $j = 1, 2, \ldots, k$, such that $c_j > 0$ and $\sum_{j=1}^{k} c_j = 1$. Let Γ_j, $j = 1, \ldots, k$, be disjoint nonempty subsets of Ω with $\bigcup_{j=1}^{k} \Gamma_j = \Omega$. We consider sets Λ specified by relative frequencies:

$$\Lambda = \widehat{M}(\Gamma_1, \ldots, \Gamma_k) = \left\{\sigma \in \Omega^{\mathbb{N}} : \lim_{n\to\infty} \frac{\#\{1 \le i \le n : \sigma(i) \in \Gamma_j\}}{n} = c_j,\ 1 \le j \le k\right\}. \tag{8}$$

Let

$$M(\Gamma_1, \ldots, \Gamma_k) = \phi(\widehat{M}(\Gamma_1, \ldots, \Gamma_k)),$$

i.e., the subset of the Moran set F whose elements have their codings lying in Γ_j with a prescribed relative frequency c_j. We remark that $M(\Gamma_1, \ldots, \Gamma_k)$ is dense in F since $\sigma \in \widehat{M}(\Gamma_1, \ldots, \Gamma_k)$ if and only if $i * \sigma \in \widehat{M}(\Gamma_1, \ldots, \Gamma_k)$, $i \in$

Ω; and that $M(\Gamma_1, \ldots, \Gamma_k) = \bigcup_{i=1}^r f_i(M(\Gamma_1, \ldots, \Gamma_k))$ when F is the self-similar set determined by $\{f_i\}_{i=1}^r$. Let

$$Z(t) = \sum_{j=1}^k c_j \log \sum_{i \in \Gamma_j} a_i^t - \sum_{j=1}^k c_j \log c_j.$$

It is easy to see that the function $Z(t)$ has a unique zero in $[0, s]$ where s is defined by $\sum_{i=1}^r a_i^s = 1$, since $Z(t)$ is strictly decreasing with $Z(0) \geq 0$ and $Z(s) \leq 0$.

Theorem 1.3. $\dim_P M(\Gamma_1, \ldots, \Gamma_k) = t$ *where* t *is the unique nonnegative real number such that*

$$\sum_{j=1}^k c_j \log c_j = \sum_{j=1}^k c_j \log \sum_{i \in \Gamma_j} a_i^t. \tag{9}$$

Proof. We define the probability vector $(b_1, b_2, \ldots, b_r)$ by taking

$$b_i = (a_i^t c_j) / \sum_{\ell \in \Gamma_j} a_\ell^t, \quad i \in \Gamma_j.$$

Thus we have $\sum_{i \in \Gamma_j} b_j = c_j$, $j = 1, 2, \ldots, k$ and $\sum_{i=1}^r b_i = 1$. Construct a Borel probability measure $\hat{\mu}$ on $\Omega^{\mathbb{N}}$ by defining for any $\sigma \in \Omega^m$

$$\hat{\mu}(C(\sigma)) = \prod_{i=1}^m b_{\sigma(i)},$$

where, as before, $C(\sigma) = \{\tau \in \Omega^{\mathbb{N}} : \tau|m = \sigma\}$ is the cylinder set with base σ. By Birkhoff's Ergodic Theorem (Ref. [8]) we have for $\hat{\mu}$-a.e. $\sigma \in \Omega^{\mathbb{N}}$

$$\lim_{n \to \infty} \frac{\#\{1 \leq \ell \leq n : \sigma(\ell) = i\}}{n} = b_i, \quad 1 \leq i \leq r,$$

and so

$$\lim_{n \to \infty} \frac{\#\{1 \leq \ell \leq n : \sigma(\ell) \in \Gamma_j\}}{n} = c_j, \quad 1 \leq j \leq k.$$

Therefore

$$\hat{\mu}(\widehat{M}(\Gamma_1, \ldots, \Gamma_k)) = \mu(M(\Gamma_1, \ldots, \Gamma_k)) = 1,$$

where μ on F is the image measure of $\hat{\mu}$ under ϕ. Now for each $\sigma \in \widehat{M}(\Gamma_1, \ldots, \Gamma_k)$ defined as in (8) and each $m \in \mathbb{N}$, we have

$$\log \hat{\mu}(C(\sigma|m)) = \log \prod_{i=1}^m b_{\sigma(i)} = \sum_{j=1}^k N(\sigma, m, \Gamma_j) \left(\log c_j - \log \sum_{\ell \in \Gamma_j} a_\ell^t \right) + t \log \prod_{i=1}^m a_{\sigma(i)},$$

where $N(\sigma, m, \Gamma_j) := \#\{1 \leq \ell \leq m : \sigma(\ell) \in \Gamma_j\}$ for $\sigma \in \Omega^{\mathbb{N}}$. Thus by (9)

$$\begin{aligned}
\lim_{m \to +\infty} \frac{\log \hat{\mu}(C(\sigma|m))}{\log a(\sigma|m)} &= \lim_{m \to +\infty} \frac{\log \prod_{i=1}^m b_{\sigma(i)}}{\log \prod_{i=1}^m a_{\sigma(i)}} \\
&= \lim_{m \to +\infty} \left(\frac{\sum_{j=1}^k m^{-1} N(\sigma, m, \Gamma_j)(\log c_j - \log \sum_{\ell \in \Gamma_j} a_\ell^t)}{\log(\prod_{i=1}^m a_{\sigma(i)})^{1/m}} + t \right) = t,
\end{aligned}$$

for each $\sigma \in \widehat{M}(\Gamma_1, \ldots, \Gamma_k)$. Thus we have $\dim_P M(\Gamma_1, \ldots, \Gamma_k) = t$ by (I) and (II) in Proposition 1.2. $\square$

We remark that the above proof also gives $\dim_H M(\Gamma_1, \ldots, \Gamma_k) = t$ which is included in [4, Theorem 1] where c_i's are allowed to be zeros.

The proofs of Theorem 1.1 and Proposition 1.2 are arranged in Section 2.

2. Proofs

In this section, we give the proofs of Theorem 1.1 and Proposition 1.2.

Proof of Theorem 1.1. Note that for any given $s \geq 0$ and $t > 0$

$$\begin{aligned} \widetilde{\mathcal{P}}_\delta^{s+t}(E) &= \sup\left\{\sum_{\sigma\in\mathcal{A}} |J_\sigma|^{s+t} : \{J_\sigma\}_{\sigma\in\mathcal{A}} \text{ is a } J\text{-type } \delta\text{-packing of } E\right\} \\ &\leq \delta^t \sup\left\{\sum_{\sigma\in\mathcal{A}} |J_\sigma|^{s} : \{J_\sigma\}_{\sigma\in\mathcal{A}} \text{ is a } J\text{-type } \delta\text{-packing of } E\right\} \\ &= \delta^t \widetilde{\mathcal{P}}_\delta^{s}(E). \end{aligned}$$

Thus $\widetilde{\mathcal{P}}_0^{s+t}(E) = 0$ if $\widetilde{\mathcal{P}}_0^{s}(E) < \infty$, and that $\widetilde{\mathcal{P}}_0^{s}(E) = \infty$ if $\widetilde{\mathcal{P}}_0^{s+t}(E) = \infty$. This verifies the second equality of (6). Now denote $\beta_*(E) = \inf\left\{s : \widetilde{\mathcal{P}}_0^{s}(E) = 0\right\} = \sup\left\{s : \widetilde{\mathcal{P}}_0^{s}(E) = +\infty\right\}$. We first prove that

$$\beta_*(E) = \limsup_{r\to 0} \frac{\log M_r}{-\log r}, \tag{10}$$

where, and below, M_r denotes the largest number of the nonoverlapping component sets J_σ with $J_\sigma \cap E \neq \emptyset$ and diameter between $\frac{r}{2}$ and r.

At first, we have $\beta_*(E) \geq \limsup_{r\to 0} \frac{\log M_r}{-\log r}$ since $\widetilde{\mathcal{P}}_r^{s}(E) \geq \left(\frac{r}{2}\right)^s M_r$.

On the other hand, without loss of generality we suppose that $\beta_*(E) > 0$ and take real numbers β and γ such that $0 < \beta < \gamma < \beta_*(E)$. For every $0 < r < 1$ there is a sequence $(J_\sigma)_{j=1}^\infty$ of nonoverlapping component sets with $J_\sigma \cap E \neq \emptyset$ and diameter $\leq r$ such that $\sum_{j=1}^\infty |J_\sigma|^\gamma > 1$. For $n \geq 0$, let k_n be the number of elements J_σ in $(J_\sigma)_{j=1}^\infty$ with $2^{-(n+1)} < |J_\sigma| \leq 2^{-n}$. Then $\sum_{n=0}^\infty k_n 2^{-n\gamma} > 1$. Hence there is a positive integer N satisfying

$$M_{2^{-N}} \geq k_N > 2^{N\beta}(1 - 2^{\beta-\gamma}).$$

So $\beta \leq \limsup_{r\to 0} \frac{\log M_r}{-\log r}$, implying $\beta_*(E) \leq \limsup_{r\to 0} \frac{\log M_r}{-\log r}$ by the arbitrariness of β.

Denote $E^r = \{x \in \mathbb{R}^d : \|x - y\| \leq r \text{ for some } y \in E\}$. Note that

$$M_r \left(\frac{cr}{2}\right)^d \mathcal{L}^d(B_{2^{-1}}(0)) \leq \mathcal{L}^d(E^r) \leq M_r (6r)^d \mathcal{L}^d(B_{2^{-1}}(0)),$$

where $\mathcal{L}^d$ stands for the d-dimensional Lebesgue measure. Therefore,

$$\limsup_{r\to 0} \frac{\log M_r}{-\log r} = \limsup_{r\to 0} \left(d - \frac{\log \mathcal{L}^d(E^r)}{\log r}\right). \tag{11}$$

From (10) and (11) and [3, Proposition 3.2] it follows that $\overline{\dim}_B E = \beta_*(E)$.

In the following we turn to prove (7). To do it, we first show the second equality in (7) which is done in the following (I) and (II).

(I) $\widetilde{\mathcal{P}}^{s+t}(E) = 0$ if $\widetilde{\mathcal{P}}^{s}(E) < \infty$.

Since $\widetilde{\mathcal{P}}^{s}(E) < \infty$, there exist E_i, $i = 1, 2, \ldots$, such that

$$\sum_{i=1}^\infty \widetilde{\mathcal{P}}_0^{s}(E_i) < \widetilde{\mathcal{P}}^{s}(E) + 1 < \infty.$$

Thus each $\widetilde{\mathcal{P}}_0^{s}(E_i) < \infty$ and so each $\widetilde{\mathcal{P}}_0^{s+t}(E_i) = 0$, giving that $\widetilde{\mathcal{P}}^{s+t}(E) = 0$.

(II) $\widetilde{\mathcal{P}}^s(E)=\infty$ if $\widetilde{\mathcal{P}}^{s+t}(E)=\infty$.

Since $\widetilde{\mathcal{P}}^{s+t}(E)=\infty$, for each choice of E_i, $i=1,2,\dots$ with $E=\cup_{i=1}^{\infty}E_i$ we have

$$\sum_{i=1}^{\infty}\widetilde{\mathcal{P}}_0^{s+t}(E_i)=\infty.$$

Without loss of generality, suppose that $\widetilde{\mathcal{P}}_0^{s+t}(E_1)=\infty$. So $\widetilde{\mathcal{P}}_0^{s}(E_1)=\infty$, leading to $\widetilde{\mathcal{P}}^s(E)=\infty$.

Denote $\beta^*(E)=\inf\left\{s:\widetilde{\mathcal{P}}^s(E)=0\right\}=\sup\left\{s:\widetilde{\mathcal{P}}^s(E)=+\infty\right\}$. A well-known result shows that (cf. [3, Proposition 3.8]) $\dim_P E=\overline{\dim}_{MB}E$ where $\overline{\dim}_{MB}E$ is the modified upper box-counting dimension of E, defined by

$$\overline{\dim}_{MB}E=\inf\{\sup_i\overline{\dim}_B E_i : E=\cup_{i=1}^{\infty}E_i\}.$$

Thus, we only need to show

$$\beta^*(E)=\inf\{\sup_i\beta_*(E_i) : E=\cup_{i=1}^{\infty}E_i\}.$$

This is done by the following two steps.

Step 1 (cf. [3, Lemma 3.7]). $\beta^*(E)\le\beta_*(E)$.

Without loss of generality, we assume that $\beta^*(E)>0$ and take $0<t<s<\beta^*(E)$. Then $\widetilde{\mathcal{P}}^s(E)=+\infty$ and so $\widetilde{\mathcal{P}}_0^s(E)=+\infty$. Thus, for any $0<\delta\le 1$ there exists J-type δ-packing $\{J_\sigma\}_{\sigma\in\mathcal{A}}$ of E such that $\sum_{\sigma\in\mathcal{A}}|J_\sigma|^s>1$. For $k\ge 0$, let n_k be the number of elements J_σ in $(J_\sigma)_{j=1}^{\infty}$ with $2^{-(k+1)}<|J_\sigma|\le 2^{-k}$. Then $\sum_{k=0}^{\infty}n_k 2^{-ks}>1$. Hence there is a positive integer N satisfying

$$M_{2^{-N}}\ge n_N>2^{Nt}(1-2^{t-s}).$$

So $\beta_*(E)\ge t$ by (10) and $\beta_*(E)\ge\beta^*(E)$ by the arbitrariness of t.

Step 2 (cf. [3, Proposition 3.8]). $\beta^*(E)=\inf\{\sup_i\beta_*(E_i) : E=\cup_{i=1}^{\infty}E_i\}$.

At first, when $E=\cup_{i=1}^{\infty}E_i$ we have

$$\beta^*(E)=\sup_i\beta^*(E_i)\le\sup_i\beta_*(E_i)$$

by the σ-stability of the index $\beta^*(E)$. Thus, $\beta^*(E)\le\inf\{\sup_i\beta_*(E_i) : E=\cup_{i=1}^{\infty}E_i\}$. On the other hand, if $s>\beta^*(E)$ then $\widetilde{\mathcal{P}}^s(E)=0$. So there exist $(E_i)_{i=1}^{\infty}$ with $E=\cup_i E_i$ such that $\widetilde{\mathcal{P}}_0^s(E_i)<\infty$ for all i. Thus $\beta_*(E_i)\le s$ for all i, leading to $\inf\{\sup_i\beta_*(E_i) : E=\cup_{i=1}^{\infty}E_i\}\le s$. Therefore, $\inf\{\sup_i\beta_*(E_i) : E=\cup_{i=1}^{\infty}E_i\}\le\beta^*(E)$ by the arbitrariness of s. □

Now we turn to give the proof of Proposition 1.2. We take μ on F as the image measure of $\hat{\mu}$ under ϕ, i.e., $\mu(B)=\hat{\mu}(\phi^{-1}(B))$ for Borel set $B\subseteq F$, and make use of Proposition 2.3 in [2]. The proofs of (I), (III) and (IV) are direct in a routine way. However, the proof of (II) is done by applying Theorem 1.1.

Proof of Proposition 1.2. We first prove (I) and (III). For each $0<R<a_{\min}$ and each $\tau\in\Lambda$ there exists a unique positive integer, denoted by $h(\tau,R)$, such that

$$a(\tau|h(\sigma,R))|J_\emptyset|\le R\quad\text{and}\quad a(\tau|(h(\tau,R)-1))|J_\emptyset|>R.$$

Thus

$$a_{\min}R<a(\tau|h(\tau,R))|J_\emptyset|=|J_{\tau|h(\tau,R)}|\le R. \tag{12}$$

Let $x=\phi(\tau)$. Then $B_R(x)\supseteq J_{\tau|h(\tau,R)}$ and $\mu(B_R(x))\ge\mu(J_{\tau|h(\tau,R)})\ge\hat{\mu}(C(\tau|h(\tau,R)))$. From (12) it follows

$$\begin{aligned}\limsup_{R\to 0}\frac{\log\mu(B_R(x))}{\log R}&\le\limsup_{R\to 0}\frac{\log\hat{\mu}(C(\tau|h(\tau,R)))}{\log R}\\&=\limsup_{R\to 0}\left(\frac{\log\hat{\mu}(C(\tau|h(\tau,R)))}{\log a(\tau|h(\tau,R))}\cdot\frac{\log a(\tau|h(\tau,R))}{\log R}\right)\le s,\end{aligned}$$

and

$$\liminf_{R\to 0}\frac{\log\mu(B_R(x))}{\log R}\le\liminf_{R\to 0}\frac{\log\hat{\mu}(C(\tau|h(\tau,R)))}{\log R}$$
$$=\liminf_{R\to 0}\left(\frac{\log\hat{\mu}(C(\tau|h(\tau,R)))}{\log a(\tau|h(\tau,R))}\cdot\frac{\log a(\tau|h(\tau,R))}{\log R}\right)\le s.$$

This completes the proofs of (I) and (III) by Proposition 2.3 (b) and (d) in [2].

The following is devoted to the proof of (IV). It is trivial if $s=0$. Suppose $s>0$. Fix $0<\epsilon<s$. Let

$$\Lambda_k=\left\{\sigma\in\Lambda:\frac{\log\hat{\mu}(C(\sigma|m))}{\log a(\sigma|m)}>s-\epsilon \text{ for all } m\ge k\right\}. \tag{13}$$

Then $\Lambda_k\uparrow\Lambda$ as $k\to+\infty$, i.e., $\Lambda=\cup_{k=1}^{\infty}\Lambda_k$. Thus $\hat{\mu}(\Lambda)=\lim_{k\to\infty}\hat{\mu}(\Lambda_k)$.

Now fix k such that $\hat{\mu}(\Lambda_k)>0$. Let $\hat{\mu}_k$ be the restriction on Λ_k of $\hat{\mu}$. Let μ_k be given by

$$\mu_k(A)=\hat{\mu}_k(\phi^{-1}(A))=\hat{\mu}(\phi^{-1}(A)\cap\Lambda_k)$$

for any Borel set $A\subseteq F$. For each $0<R<a_{\min}^k$ and each $\sigma\in\Lambda_k$ there exists a positive integer $h(\sigma,R)$ such that

$$a_{\min}R\le a(\sigma|h(\sigma,R))\le R. \tag{14}$$

Note that $h(\sigma,R)>k$ and write $W=\{\sigma|h(\sigma,R):\sigma\in\Lambda_k\}$. For any fixed $x\in\phi(\Lambda_k)$ let $W^*=\{\tau\in W: J_\tau\cap B_R(x)\ne\emptyset\}$. Then there exists a finite positive constant ϑ independent of the R and x such that $\#\, W^*\le\vartheta$ by the Lemma 9.2 in [3](also see discussion in Section 1). So

$$\mu_k(B_R(x))\le\hat{\mu}_k(\cup_{\tau\in W^*}C(\tau))\le\sum_{\tau\in W^*}\hat{\mu}(C(\tau))\le\vartheta\cdot R^{s-\epsilon},$$

by (13) and (14). So we get

$$\liminf_{R\to 0}\frac{\log\mu_k(B_R(x))}{\log R}\ge s-\epsilon.$$

Proposition 2.3 (a) in [2] tells

$$\dim_H\phi(\Lambda)\ge\dim_H\phi(\Lambda_k)\ge s-\epsilon,$$

which implies that $\dim_H\phi(\Lambda)\ge s$ if letting $\epsilon\to 0$.

Finally, we prove (II). Without loss of generality, suppose $s>0$. For any increasing sequence $E_n\uparrow\phi(\Lambda)$ let $\Lambda_n=\Lambda\cap\phi^{-1}(E_n)$. Then $\Lambda_n\uparrow\Lambda$. We fix some n such that $\hat{\mu}(\Lambda_n)>\frac{1}{2}\hat{\mu}(\Lambda)$ and denote $\Lambda^*=\Lambda_n$.

Fix $0<\epsilon<s$. For a fixed n_0, let $n(\sigma)$, for each $\sigma\in\Lambda^*$, be the smallest integer $\ge n_0$ such that

$$\frac{\log\hat{\mu}(C(\sigma|n(\sigma)))}{\log a(\sigma|n(\sigma))}>s-\epsilon,\quad\text{i.e., }\hat{\mu}(C(\sigma|n(\sigma)))<a(\sigma|n(\sigma))^{s-\epsilon}.$$

Let $\mathcal{G}=\{\sigma|n(\sigma):\sigma\in\Lambda^*\}$. Then $\Lambda^*\subseteq\cup_{\tau\in\mathcal{G}}C(\tau)$ with disjoint union on the right hand. Thus

$$\sum_{\tau\in\mathcal{G}}|J_\tau|^{s-\epsilon}=|J_\emptyset|^{s-\epsilon}\sum_{\tau\in\mathcal{G}}a(\tau)^{s-\epsilon}>|J_\emptyset|^{s-\epsilon}\sum_{\tau\in\mathcal{G}}\hat{\mu}(C(\tau))$$
$$=|J_\emptyset|^{s-\epsilon}\hat{\mu}(\cup_{\tau\in\mathcal{G}}C(\tau))>|J_\emptyset|^{s-\epsilon}\hat{\mu}(\Lambda^*)>\frac{1}{2}|J_\emptyset|^{s-\epsilon}\hat{\mu}(\Lambda)>0. \tag{15}$$

Thus for any small $\delta>0$, $(J_\sigma)_{\sigma\in\mathcal{G}}$ is a J-type δ-packing of $\phi(\Lambda^*)$ when n_0 is taken big enough. Therefore, $\widetilde{P}_\delta^{s-\epsilon}(\phi(\Lambda^*))>\frac{1}{2}|J_\emptyset|^{s-\epsilon}\hat{\mu}(\Lambda)$ by (15), giving $\widetilde{P}_0^{s-\epsilon}(\phi(\Lambda^*))>\frac{1}{2}|J_\emptyset|^{s-\epsilon}\hat{\mu}(\Lambda)>0$. By means of (5) we have $\widetilde{\mathcal{P}}^{s-\epsilon}(\phi(\Lambda))>0$ leading to $\dim_P\phi(\Lambda)\ge s-\epsilon$ by Theorem 1.1. Therefore, $\dim_P\phi(\Lambda)\ge s$ by letting $\epsilon\to 0$. $\square$

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