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On the union of homogeneous symmetric Cantor set with its translations

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Abstract

Fix a positive integer N and a real number $0 < \beta < 1/(N+1)$. Let Γ be the homogeneous symmetric Cantor set generated by the IFS

$$\left\{\phi_i(x) = \beta x + i\frac{1-\beta}{N} : i = 0, 1, \ldots, N\right\}.$$

For $m \in \mathbb{Z}_+$ we show that there exist infinitely many translation vectors $\mathbf{t} = (t_0, t_1, \ldots, t_m)$ with $0 = t_0 < t_1 < \cdots < t_m$ such that the union $\bigcup_{j=0}^{m}(\Gamma + t_j)$ is a self-similar set. Furthermore, for $0 < \beta < 1/(2N+1)$, we give a finite algorithm to determine whether the union $\bigcup_{j=0}^{m}(\Gamma + t_j)$ is a self-similar set for any given vector $\mathbf{t}$. Our characterization relies on determining whether some related directed graph has no cycles, or whether some related adjacency matrix is nilpotent.

Keywords Self-similar set · Iterated function system · Homogeneous symmetric Cantor set · Union of Cantor sets

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1 Introduction

Self-similar set is a fundamental object in the study of fractal geometry (cf. [4]). A non-empty compact set E in a complete metric space X is called a *self-similar set* if there exists a finite set of contractive similitudes $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ such that $E = \bigcup_{i=1}^n f_i(E)$. The set $\mathcal{F}$ of contractive similitudes is called an *iterated function system* (simply called, IFS) for the self-similar set E (see [7]). In this paper we will study when the union of a self-similar set with its translations is again a self-similar set.

Fix a positive integer N and a real number $0 < \beta < 1/(N+1)$. Let $\Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}$ be the self-similar set in $\mathbb{R}$ generated by the IFS

$$\left\{\phi_i(x) = \beta x + i\frac{1-\beta}{N} : i = 0, 1, \ldots, N\right\}.$$

Then Γ is the unique non-empty compact set satisfying

$$\Gamma = \bigcup_{i=0}^{N} \phi_i(\Gamma),$$

and it can be written as

$$\Gamma = \left\{\frac{1-\beta}{N}\sum_{k=1}^{\infty} j_k\beta^{k-1} : j_k \in \{0, 1, \ldots, N\}\ \forall k \geq 1\right\}. \tag{1.1}$$

Clearly, Γ is symmetric, i.e., $\Gamma = 1 - \Gamma$.

In the literature there is a great interest in the study of intersections of Cantor set with its translations. When $N = 1$, Kraft [9] gave a complete description on when the intersection $\Gamma_{\beta,\{0,1\}} \cap (\Gamma_{\beta,\{0,1\}} + t)$ is a single point, and Li and Xiao [10] calculated the Hausdorff and packing dimensions of the intersection. In [2] Deng, He and Wen studied the self-similarity of the intersection of the middle-third Cantor set with its translation, and gave a necessary and sufficient condition for which the intersection is a self-similar set. This result was later extended by Li, Yao and Zhang in [11] to the homogeneous symmetric Cantor set $\Gamma_{\beta,\{0,1,\ldots,N\}}$ for $0 < \beta \leq 1/(2N+1)$, and by Kong, Li and Dekking [8] to $\Gamma_{\beta,\{0,1,\ldots,N\}}$ for $1/(2N+1) < \beta < 1/(N+1)$.

On the other hand, there are several papers about the self-similarity of a finite union of intervals, see [5] and [12]. As for the union of the Cantor set with its translations, Deng, Liu first investigated the self-similarity of $\Gamma_{\beta,\{0,1\}} \cup (\Gamma_{\beta,\{0,1\}} + t)$ in [3, Theorem 1.1], where $\beta = 1/k$ with $k \in \mathbb{Z}_{\geq 3}$. However, we know very little about the general case of the self-similarity of the union of $\Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}$ with its translations.

In this paper, we are interested in whether the union

$$\Gamma_{\mathbf{t}} := \bigcup_{j=0}^{m} (\Gamma + t_j) \quad \text{with} \quad \mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$$

is a self-similar set, where for a set X and $a, b \in \mathbb{R}$ we write $aX + b := \{ax + b : x \in X\}$.

Note that the self-similarity is invariant under translations. In other words, if $E \subset \mathbb{R}$ is a self-similar set, then so is its translation $E + t$ for any $t \in \mathbb{R}$. Thus, without loss of generality we assume throughout the paper that the translation vector $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ always satisfies $0 = t_0 < t_1 < \cdots < t_m$. Note that $\Gamma = \bigcup_{i=0}^N \phi_i(\Gamma) = \bigcup_{i=0}^N \big(\beta\Gamma + \phi_i(0)\big)$. Then

$$\beta^{-n}\Gamma=\bigcup_{i_1\cdots i_n\in\{0,1,\ldots,N\}^n}\left(\Gamma+\beta^{-n}\phi_{i_1\cdots i_n}(0)\right),$$

where $\phi_{i_1\cdots i_n}=\phi_{i_1}\circ\cdots\circ\phi_{i_n}$ denotes the composition of maps. It follows that if $m=N^n-1$ and the translation vector $\mathbf{t}=(t_0,t_1,\ldots,t_m)$ takes the values $\left\{\beta^{-n}\phi_{\mathbf{i}}(0):\mathbf{i}\in\{0,1,\ldots,N\}^n\right\}$, then the union $\Gamma_{\mathbf{t}}=\beta^{-n}\Gamma$ is a self-similar set. However, for other $m\in\mathbb{Z}_+$ can we find $\mathbf{t}\in\mathbb{R}^{m+1}$ such that $\Gamma_{\mathbf{t}}$ is a self-similar set?

Our first result answers this affirmatively.

Theorem 1.1 *Suppose* $0<\beta<1/(N+1)$. *Then for any* $m\in\mathbb{Z}_+$ *there exist infinitely many translation vectors* $\mathbf{t}=(t_0,t_1,\ldots,t_m)\in\mathbb{R}^{m+1}$ *with* $0=t_0<t_1<\cdots<t_m$ *such that* $\Gamma_{\mathbf{t}}=\bigcup_{j=0}^m(\Gamma+t_j)$ *is a self-similar set.*

Next we consider for which translation vector $\mathbf{t}=(t_0,t_1,\ldots,t_m)\in\mathbb{R}^{m+1}$ the union $\Gamma_{\mathbf{t}}=\bigcup_{j=0}^m(\Gamma+t_j)$ is a self-similar set. Observe that an IFS of $\Gamma_{\mathbf{t}}$ might contain a similitude with negative contraction ratio. This makes our characterization of self-similarity of $\Gamma_{\mathbf{t}}$ more complicated. To describe the self-similarity of $\Gamma_{\mathbf{t}}$ we first introduce the notation of admissible translation vectors (see Definition 1.2 below).

Set

$$T:=\bigcup_{n=1}^{\infty}\left\{\frac{1-\beta}{N}\sum_{k=1}^{n}j_k\beta^{-k}:j_k\in\{0,1,\ldots,N\}\ \forall 1\le k\le n\right\}.$$

For a translation vector $\mathbf{t}=(t_0,t_1,\ldots,t_m)\in T^{m+1}$ let $\tau_{\mathbf{t}}$ be the smallest integer such that each t_j, $0\le j\le m$, can be written as

$$t_j=\frac{1-\beta}{N}\sum_{k=1}^{\tau_{\mathbf{t}}}t_{j,k}\beta^{-k}\quad\text{with}\quad t_{j,k}\in\{0,1,\ldots,N\}. \tag{1.2}$$

Then for $n\ge\tau_{\mathbf{t}}$ we define

$$\Omega_{\mathbf{t}}^n:=\left\{i_1\cdots i_n\in\{0,1,\ldots,N\}^n:i_{n+1-k}\le N-s_k\text{ for }1\le k\le\tau_{\mathbf{t}}\right\}, \tag{1.3}$$

and its conjugate

$$\hat{\Omega}_{\mathbf{t}}^n:=\left\{i_1\cdots i_n\in\{0,1,\ldots,N\}^n:i_{n+1-k}\ge s_k\text{ for }1\le k\le\tau_{\mathbf{t}}\right\}, \tag{1.4}$$

where $s_k=\max_{0\le j\le m}t_{j,k}$ for $1\le k\le\tau_{\mathbf{t}}$. Clearly, $i_1i_2\cdots i_n\in\Omega_{\mathbf{t}}^n$ if and only if $(N-i_1)(N-i_2)\cdots(N-i_n)\in\hat{\Omega}_{\mathbf{t}}^n$. Note that for any $\mathbf{i}=i_1\cdots i_n\in\Omega_{\mathbf{t}}^n$ and $0\le j\le m$ we have

$$\begin{aligned}\phi_{\mathbf{i}}(t_j)=\beta^nt_j+\phi_{\mathbf{i}}(0)&=\frac{1-\beta}{N}\sum_{k=1}^{\tau_{\mathbf{t}}}t_{j,k}\beta^{n-k}+\frac{1-\beta}{N}\sum_{k=1}^{n}i_{n+1-k}\beta^{n-k}\\&=\frac{1-\beta}{N}\sum_{k=1}^{\tau_{\mathbf{t}}}(i_{n+1-k}+t_{j,k})\beta^{n-k}+\frac{1-\beta}{N}\sum_{k=\tau_{\mathbf{t}}+1}^{n}i_{n+1-k}\beta^{n-k}\in\Gamma.\end{aligned}$$

Similarly, for any $\mathbf{i}=i_1\ldots i_n\in\hat{\Omega}_{\mathbf{t}}^n$ and $0\le j\le m$,

$$\begin{aligned}\phi_{\mathbf{i}}(-t_j)=-\beta^nt_j+\phi_{\mathbf{i}}(0)&=\frac{1-\beta}{N}\sum_{k=1}^{\tau_{\mathbf{t}}}(i_{n+1-k}-t_{j,k})\beta^{n-k}\\&\quad+\frac{1-\beta}{N}\sum_{k=\tau_{\mathbf{t}}+1}^{n}i_{n+1-k}\beta^{n-k}\in\Gamma.\end{aligned}$$

Let $\mathcal{A}_{\mathbf{t}}^n$ and $\hat{\mathcal{A}}_{\mathbf{t}}^n$ be the sets of blocks representing the sets $\left\{\phi_{\mathbf{i}}(t_j) : \mathbf{i} \in \Omega_{\mathbf{t}}^n; 0 \le j \le m\right\}$ and $\left\{\phi_{\mathbf{i}}(-t_j) : \mathbf{i} \in \hat{\Omega}_{\mathbf{t}}^n; 0 \le j \le m\right\}$, respectively:

$$\begin{aligned} \mathcal{A}_{\mathbf{t}}^n &= \{i_1 \cdots i_{n-\tau_{\mathbf{t}}}(i_{n+1-\tau_{\mathbf{t}}} + t_{j,\tau_{\mathbf{t}}}) \cdots (i_{n-1} + t_{j,2})(i_n + t_{j,1}) : \mathbf{i} \in \Omega_{\mathbf{t}}^n,\ 0 \le j \le m\}, \\ \hat{\mathcal{A}}_{\mathbf{t}}^n &= \{i_1 \cdots i_{n-\tau_{\mathbf{t}}}(i_{n+1-\tau_{\mathbf{t}}} - t_{j,\tau_{\mathbf{t}}}) \cdots (i_{n-1} - t_{j,2})(i_n - t_{j,1}) : \mathbf{i} \in \hat{\Omega}_{\mathbf{t}}^n,\ 0 \le j \le m\}. \end{aligned} \tag{1.5}$$

Now we define

$$\mathcal{W}_{\mathbf{t}}^n := \mathcal{A}_{\mathbf{t}}^n \cup \hat{\mathcal{A}}_{\mathbf{t}}^n.$$

By the definitions of $\Omega_{\mathbf{t}}^n$ and $\hat{\Omega}_{\mathbf{t}}^n$ it follows that $\Omega_{\mathbf{t}}^n = \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \Omega_{\mathbf{t}}^{\tau_{\mathbf{t}}}$ and $\hat{\Omega}_{\mathbf{t}}^n = \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \hat{\Omega}_{\mathbf{t}}^{\tau_{\mathbf{t}}}$. This implies that

$$\mathcal{W}_{\mathbf{t}}^n = \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} \quad \forall n \ge \tau_{\mathbf{t}}. \tag{1.6}$$

Definition 1.2 A vector $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$ is called an *admissible translation vector* if $\mathbf{t} \in T^{m+1}$ and there exists $\ell \ge \tau_{\mathbf{t}}$ such that

$$\bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} \times \{0, 1, \ldots, N\}^{\ell-n} = \{0, 1, \ldots, N\}^{\ell}.$$

According to Definition 1.2 it is not easy to verify the admissibility of a translation vector $\mathbf{t}$. In the following we give a more handleable approach by constructing a directed graph.

Given $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in T^{m+1}$, let $G_{\mathbf{t}} = (V_{\mathbf{t}}, E_{\mathbf{t}})$ be the directed graph defined as follows. Let $V_{\mathbf{t}} = \{0, 1, \ldots, N\}^{\tau_{\mathbf{t}}} \backslash \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}}$. For two vertices $\mathbf{i} = i_1 i_2 \cdots i_{\tau_{\mathbf{t}}}, \mathbf{j} = j_1 j_2 \cdots j_{\tau_{\mathbf{t}}} \in V_{\mathbf{t}}$, we draw a directed edge from $\mathbf{i}$ to $\mathbf{j}$ if $i_2 \cdots i_{\tau_{\mathbf{t}}} = j_1 \cdots j_{\tau_{\mathbf{t}}-1}$. Then $E_{\mathbf{t}}$ is the collection of all such directed edges. We say that $G_{\mathbf{t}}$ has a *cycle* if there exists a directed path in $G_{\mathbf{t}}$ starting and ending at the same vertex. For convenience, we say that the empty graph has no cycles. For the directed graph $G_{\mathbf{t}}$ we denote its *adjacency matrix* by $A_{\mathbf{t}}$. Then $A_{\mathbf{t}}$ is a 0-1 matrix with the size $\#V_{\mathbf{t}} \times \#V_{\mathbf{t}}$, and an entry 1 in $A_{\mathbf{t}}$ corresponds to a directed edge in $G_{\mathbf{t}}$. We say that $A_{\mathbf{t}}$ is *nilpotent* if $A_{\mathbf{t}}^{\ell} = 0$ for some power $\ell \in \mathbb{Z}_+$.

Proposition 1.3 *Let* $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in T^{m+1}$. *The following statements are equivalent.*

(i) $\mathbf{t}$ *is admissible;*
(ii) $A_{\mathbf{t}}$ *is nilpotent;*
(iii) $G_{\mathbf{t}}$ *has no cycle.*

Proof (i) $\Rightarrow$ (ii). Suppose $A_{\mathbf{t}}$ is not nilpotent. Then for any $\ell \in \mathbb{Z}_+$ the matrix $A_{\mathbf{t}}^{\ell} \ne 0$. This implies that for any $\ell \ge \tau_{\mathbf{t}}$ there exists a path of length ℓ in the directed graph $G_{\mathbf{t}}$. By the construction of $G_{\mathbf{t}}$ it follows that for any $\ell \ge \tau_{\mathbf{t}}$ there exists a word $\mathbf{i}$ of length ℓ such that each subword of length $\tau_{\mathbf{t}}$ in $\mathbf{i}$ belongs to $V_{\mathbf{t}} = \{0, 1, \ldots, N\}^{\tau_{\mathbf{t}}} \backslash \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}}$. So,

$$\mathbf{i} \in \{0, 1, \ldots, N\}^{\ell} \backslash \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} \times \{0, 1, \ldots, N\}^{\ell-n},$$

which implies that $\mathbf{t}$ is not admissible by Definition 1.2.

(ii) $\Rightarrow$ (iii). This follows directly by observing that an entry in $A_{\mathbf{t}}^{\ell}$ (say, Row $\mathbf{i}$ and Column $\mathbf{j}$) corresponds to the number of length ℓ paths from vertex $\mathbf{i}$ to vertex $\mathbf{j}$.

(iii) $\Rightarrow$ (i). Suppose $G_{\mathbf{t}}$ has no cycles. If $V_{\mathbf{t}} = \emptyset$, then $\mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \{0, 1, \ldots, N\}^{\tau_{\mathbf{t}}}$, and by Definition 1.2 it is clear that $\mathbf{t}$ is admissible. Now for $V_{\mathbf{t}} \neq \emptyset$ let $\ell = \tau_{\mathbf{t}} + \#V_{\mathbf{t}}$. Arbitrarily take a word $\mathbf{i} = i_1 \ldots i_\ell \in \{0, 1, \ldots, N\}^\ell$, it suffices to prove that

$$i_{n_0} i_{n_0+1} \ldots i_{n_0+\tau_{\mathbf{t}}-1} \in \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} \quad \text{for some } 1 \leq n_0 \leq \ell - \tau_{\mathbf{t}} + 1. \tag{1.7}$$

Suppose on the contrary that any block of length $\tau_{\mathbf{t}}$ in $\mathbf{i}$ does not belong to $\mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}}$. Then $i_n i_{n+1} \ldots i_{n+\tau_{\mathbf{t}}-1} \in V_{\mathbf{t}}$ for all $1 \leq n \leq \ell - \tau_{\mathbf{t}} + 1$, and this gives a directed path of length $\ell - \tau_{\mathbf{t}} = \#V_{\mathbf{t}}$ in $G_{\mathbf{t}}$. By the Pigeonhole Principle it follows that $G_{\mathbf{t}}$ contains a cycle, leading to a contradiction with our assumption. This proves (1.7) as desired.

Remark 1.4 We point out that the characterization of admissibility in Proposition 1.3 is more handleable. For example, by using the depth-first search we can detect the existence of cycles in a directed graph (see [1]).

For a translation vector $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$ we define its *conjugate* by $\hat{\mathbf{t}} = (\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_m)$ where $\hat{t}_j = t_m - t_{m-j}$ for $0 \leq j \leq m$. Then the elements in the vector $\hat{\mathbf{t}}$ are also listed in a strictly increasing order. Furthermore, $\mathbf{t}$ and $\hat{\mathbf{t}}$ are conjugate to each other, and by the symmetry of Γ it follows that

$$(1 + t_m) - \Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m} \big((1 - \Gamma) + (t_m - t_j)\big) = \bigcup_{j=0}^{m} \big(\Gamma + \hat{t}_j\big) = \Gamma_{\hat{\mathbf{t}}}. \tag{1.8}$$

Note that $\Gamma_{\mathbf{t}}$ is a self-similar set if and only if $\Gamma_{\hat{\mathbf{t}}}$ is a self-similar set. Based on the definition of admissible translation vectors we give a necessary and sufficient condition for the union $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ to be a self-similar set.

Theorem 1.5 *Let $0 < \beta < 1/(2N+1)$, and $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$. Then $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ is a self-similar set if and only if either $\mathbf{t}$ or its conjugate $\hat{\mathbf{t}}$ is an admissible translation vector.*

As an application, we give an explicit characterization on the self-similarity of $\Gamma \cup (\Gamma + t)$. For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ denote its integer part.

Corollary 1.6 *Let $0 < \beta < 1/(2N+1)$ and $t > 0$. Then $\Gamma \cup (\Gamma + t)$ is a self-similar set if and only if*

$$t = \frac{j(1-\beta)}{N} \beta^{-k}$$

for some $j \in \left\{1, 2, \cdots, \lfloor \frac{N+1}{2} \rfloor \right\}$ and $k \in \mathbb{Z}_+$.

The rest of the paper is arranged as follows. In the next section we give some examples. In Sect. 3 we describe the generating IFSs of $\Gamma_{\mathbf{t}}$. The proofs of Theorem 1.1 and 1.5 will be given in Sect. 4.

2 Examples

In this section we give some examples to illustrate our main results.

Example 2.1 Fix a positive integer N and a real number $0 < \beta < 1/(2N+1)$. For $m \in \mathbb{Z}_+$, let $\mathbf{t} = (t_0, t_1, \cdots, t_m) \in \mathbb{R}^{m+1}$ where $t_0 = 0$ and for $1 \le j \le m$,

$$t_j = \frac{1-\beta}{N} \sum_{k=1}^{j} \beta^{-k}.$$

Clearly, $\mathbf{t} \in T^{m+1}$. By calculation, we have $\tau_{\mathbf{t}} = m$, $\Omega_{\mathbf{t}}^m = \{0, 1, \ldots, N-1\}^m$, and $\hat{\Omega}_{\mathbf{t}}^m = \{1, 2, \ldots, N\}^m$. It follows that

$$\mathcal{A}_{\mathbf{t}}^m = \bigcup_{k=0}^{m} \{0, 1, \ldots, N-1\}^{m-k} \times \{1, 2, \ldots, N\}^k,$$

and

$$\hat{\mathcal{A}}_{\mathbf{t}}^m = \bigcup_{k=0}^{m} \{1, 2, \ldots, N\}^{m-k} \times \{0, 1, \ldots, N-1\}^k.$$

Note that $\mathcal{W}_{\mathbf{t}}^m = \mathcal{A}_{\mathbf{t}}^m \cup \hat{\mathcal{A}}_{\mathbf{t}}^m$ and $V_{\mathbf{t}} = \{0, 1, \ldots, N\}^m \backslash \mathcal{W}_{\mathbf{t}}^m$. The discussion is split into two cases.

Case (i): $m \in \{1, 2\}$. It is easy to check that $\mathcal{W}_{\mathbf{t}}^m = \{0, 1, \ldots, N\}^m$. This implies that $G_{\mathbf{t}}$ is an empty graph. By Proposition 1.3 and Theorem 1.5, we conclude that the set $\Gamma_{\mathbf{t}}$ is a self-similar set.

Case (ii): $m \ge 3$. Let $m' = \lfloor m/2 \rfloor$. If m is odd, we have $(0N)^{m'}0$, $N(0N)^{m'} \notin \mathcal{W}_{\mathbf{t}}^m$, and the cycle $(0N)^{m'}0 \to N(0N)^{m'} \to (0N)^{m'}0$ is in $G_{\mathbf{t}}$; if m is even, we have $(0N)^{m'}$, $(N0)^{m'} \notin \mathcal{W}_{\mathbf{t}}^m$, and the cycle $(0N)^{m'} \to (N0)^{m'} \to (0N)^{m'}$ is in $G_{\mathbf{t}}$. Note that the conjugate $\hat{\mathbf{t}} \in T^{m+1}$. We can check that $\mathcal{W}_{\hat{\mathbf{t}}}^m = \mathcal{W}_{\mathbf{t}}^m$ and thus, $G_{\hat{\mathbf{t}}} = G_{\mathbf{t}}$ has a cycle. By Proposition 1.3 and Theorem 1.5, we conclude that $\Gamma_{\mathbf{t}}$ is *not* a self-similar set for all $m \ge 3$.

Example 2.2 Let $N = 1$ and $0 < \beta < 1/3$. Take $\mathbf{t} = (t_0, t_1, t_2, t_3) \in \mathbb{R}^4$ where $t_0 = 0$, and

$$t_1 = (1-\beta)(\beta^{-1} + \beta^{-2}), \quad t_2 = (1-\beta)(\beta^{-1} + \beta^{-3}), \quad t_3 = (1-\beta)(\beta^{-1} + \beta^{-4}).$$

Clearly, $\mathbf{t} \in T^4$. By calculation, we have $\tau_{\mathbf{t}} = 4$, $\Omega_{\mathbf{t}}^4 = \{0000\}$, and $\hat{\Omega}_{\mathbf{t}}^4 = \{1111\}$. It follows that

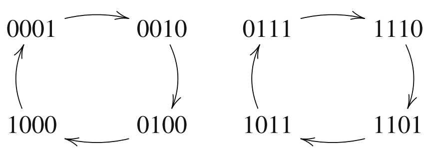
$$\mathcal{A}_{\mathbf{t}}^4 = \{0000, 0011, 0101, 1001\}, \ \hat{\mathcal{A}}_{\mathbf{t}}^4 = \{1111, 1100, 1010, 0110\}.$$

Note that $\mathcal{W}_{\mathbf{t}}^4 = \mathcal{A}_{\mathbf{t}}^4 \cup \hat{\mathcal{A}}_{\mathbf{t}}^4$, and

$$V_{\mathbf{t}} = \{0, 1\}^4 \setminus \mathcal{W}_{\mathbf{t}}^4 = \{0001, 0010, 0100, 0111, 1000, 1011, 1101, 1110\}.$$

The directed graph

$G_{\mathbf{t}}$:



has two cycles. Note that $\hat{\mathbf{t}} \notin T^4$. By Proposition 1.3 and Theorem 1.5, the set $\Gamma_{\mathbf{t}}$ is *not* a self-similar set.

3 Generating IFSs of the union $\Gamma_{\mathbf{t}}$

Given a self-similar set $E \subset \mathbb{R}$, any IFS $\{f_i(x) = r_i x + b_i\}_{i=1}^n$ with $0 < |r_i| < 1$ and $b_i \in \mathbb{R}$ satisfying $E = \bigcup_{i=1}^n f_i(E)$ is called a *generating IFS* of E (cf. [6]). Clearly, a self-similar set has infinitely many generating IFSs. In this section we describe the generating IFSs of $\Gamma_{\mathbf{t}}$.

Proposition 3.1 *Let $0 < \beta < 1/(2N+1)$, and let $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^m (\Gamma + t_j)$ be a self-similar set, where $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$. If $r\Gamma_{\mathbf{t}} + b \subset \Gamma_{\mathbf{t}}$ with $0 < |r| < 1$, then $|r| = \beta^q$ for some $q \in \mathbb{Z}_+$.*

Our strategy to prove Proposition 3.1 is as follows: first we prove that either $\Gamma_{\mathbf{t}}$ or $\Gamma_{\hat{\mathbf{t}}} = 1 + t_m - \Gamma$ has a generating IFS which contains a similitude $g(x) = rx$ with $0 < r < 1$, see Lemma 3.2; next we show that $r = \beta^q$ for some $q \in \mathbb{Z}_+$, and either $\mathbf{t} \in T^{m+1}$ or $\hat{\mathbf{t}} \in T^{m+1}$, see Lemmas 3.5 and 3.6; finally we give a complete characterization of all generating IFSs of $\Gamma_{\mathbf{t}}$, see Lemmas 3.7, 3.8 and 3.9.

Lemma 3.2 *Let $0 < \beta < 1/(N+1)$, and let $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$. If $\Gamma_{\mathbf{t}}$ is a self-similar set, then either $\Gamma_{\mathbf{t}}$ or $\Gamma_{\hat{\mathbf{t}}}$ has a generating IFS containing a similitude $g(x) = rx$ with $0 < r < 1$.*

Proof Suppose that $\{f_i(x) = r_i x + b_i\}_{i=1}^n$ is a generating IFS of $\Gamma_{\mathbf{t}}$. Note that $0 \in \Gamma \subset \Gamma_{\mathbf{t}}$ and $1 + t_m = \max \Gamma_{\mathbf{t}} \in \Gamma_{\mathbf{t}}$. Without loss of generality we assume

$$0 \in f_1(\Gamma_{\mathbf{t}}) = r_1\Gamma_{\mathbf{t}} + b_1 \quad \text{and} \quad 1 + t_m \in f_n(\Gamma_{\mathbf{t}}) = r_n\Gamma_{\mathbf{t}} + b_n. \tag{3.1}$$

If $r_1 > 0$, then (3.1) implies $b_1 = \min f_1(\Gamma_{\mathbf{t}}) = 0$, and thus we are done by taking $g(x) = f_1(x) = r_1 x$. If $r_1 < 0$, then we consider two cases: $r_n > 0$, or $r_n < 0$.

Case (I): $r_1 < 0$ and $r_n > 0$. Then by (3.1) we have $1 + t_m = f_n(1+t_m) = r_n(1+t_m) + b_n$, and thus

$$(1 - r_n)(1 + t_m) - b_n = 0. \tag{3.2}$$

Since $\Gamma_{\mathbf{t}}$ is a self-similar set generated by $\{f_i(x) = r_i x + b_i\}_{i=1}^n$, by (1.8) it follows that

$$\begin{aligned}
\Gamma_{\hat{\mathbf{t}}} = (1 + t_m) - \Gamma_{\mathbf{t}} &= \bigcup_{i=1}^n (1 + t_m - b_i - r_i\Gamma_{\mathbf{t}}) \\
&= \bigcup_{i=1}^n \Big(r_i(1 + t_m - \Gamma_{\mathbf{t}}) + (1 - r_i)(1 + t_m) - b_i \Big) \\
&= \bigcup_{i=1}^n \Big(r_i\Gamma_{\hat{\mathbf{t}}} + (1 - r_i)(1 + t_m) - b_i \Big).
\end{aligned}$$

Then $\Gamma_{\hat{\mathbf{t}}}$ is a self-similar set generated by the IFS

$$\left\{ \hat{f}_i(x) = r_i x + (1 - r_i)(1 + t_m) - b_i \right\}_{i=1}^n.$$

Note by (3.2) that $\hat{f}_n(x) = r_n x + (1 - r_n)(1 + t_m) - b_n = r_n x$. Then we are done by taking $g(x) = \hat{f}_n(x) = r_n x$.

Case (II): $r_1 < 0$ and $r_n < 0$. Then by (3.1) it follows that $0 = f_1(1 + t_m)$ and $1 + t_m = f_n(0)$. This implies that $f_1 \circ f_n(0) = 0$, and thus $f_1 \circ f_n(x) = r_1 r_n x$ with $r_1 r_n > 0$. Note that $\left\{f_i \circ f_j(x) = r_i r_j x + r_i b_j + b_i\right\}_{1 \le i,j \le n}$ is also a generating IFS of $\Gamma_{\mathbf{t}}$. Hence, we are done by taking $g(x) = f_1 \circ f_n(x) = r_1 r_n x$.

For a finite digit set $D \subset \mathbb{Z}$ let

$$\Gamma_{\beta,D} := \left\{ \frac{1-\beta}{N} \sum_{k=1}^{\infty} j_k \beta^{k-1} : j_k \in D \ \forall k \ge 1 \right\}. \tag{3.3}$$

Then each $x \in \Gamma_{\beta,D}$ can be written as $x = \frac{1-\beta}{N} \sum_{k=1}^{\infty} j_k \beta^{k-1}$ with $j_k \in D$, and the infinite sequence $(j_k) = j_1 j_2 \cdots \in D^{\mathbb{Z}_+}$ is called a *D-coding* of x. In general, a point in $\Gamma_{\beta,D}$ may have multiple D-codings.

For the rest of this section we always assume that

$$0 < \beta < \frac{1}{2N+1}.$$

The key in our proof is the following result on unique codings.

Lemma 3.3 *Each* $x \in \Gamma_{\beta,\{0,1,\ldots,N\}} \subset \Gamma_{\beta,\{-N,\cdots,-1,0,1,\cdots,2N\}}$ *has a unique* $\{-N, \cdots, -1, 0, 1, \cdots, 2N\}$*-coding which coincides with its* $\{0, 1, \ldots, N\}$*-coding.*

Proof Let $x \in \Gamma_{\beta,\{0,1,\ldots,N\}}$. Since $0 < \beta < 1/(2N+1)$, x has a unique $\{0, 1, \ldots, N\}$-coding, say (i_k). Note that x also belongs to $\Gamma_{\beta,\{-N,\cdots,-1,0,1,\cdots,2N\}}$. Then x has a $\{-N, \cdots, -1, 0, 1, \cdots, 2N\}$-coding, say (j_k). It suffices to prove that $j_k = i_k$ for all $k \ge 1$.

For $k \ge 1$, we define $j_k' := -\min\{j_k, 0\}$. Then $x + \frac{1-\beta}{N} \sum_{k=1}^{\infty} j_k' \beta^{k-1}$ can be written as

$$\frac{1-\beta}{N} \sum_{k=1}^{\infty} (i_k + j_k') \beta^{k-1} = \frac{1-\beta}{N} \sum_{k=1}^{\infty} (j_k + j_k') \beta^{k-1}. \tag{3.4}$$

Since $i_k, j_k' \in \{0, 1, \ldots, N\}$, we have $i_k + j_k' \in \{0, 1, \ldots, 2N\}$ for all $k \ge 1$. Note that $j_k + j_k' = j_k$ if $j_k \ge 0$; $j_k + j_k' = 0$ if $j_k < 0$. Thus, we also have $j_k + j_k' \in \{0, 1, \ldots, 2N\}$ for all $k \ge 1$. Since $0 < \beta < 1/(2N+1)$, each point in $\Gamma_{\beta,\{0,1,\ldots,2N\}}$ has a unique $\{0, 1, \ldots, 2N\}$-coding. Then (3.4) implies that $i_k + j_k' = j_k + j_k'$ for all $k \ge 1$. So, $j_k = i_k$ for all $k \ge 1$.

Lemma 3.4

(i) *If* $x + y, 2x + y, \ldots, Nx + y \in \Gamma$ *for some* $y \in \Gamma$*, then* $x \in \Gamma_{\beta,\{-1,0,1\}}$.
(ii) *If* $x, 2x, \cdots, Nx \in \Gamma$*, then* $x \in \Gamma_{\beta,\{0,1\}}$.

Proof (i) Take $y \in \Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}$. Then we can write it as

$$y = \frac{1-\beta}{N} \sum_{k=1}^{\infty} y_k \beta^{k-1} \text{ with each } y_k \in \{0, 1, \ldots, N\}. \tag{3.5}$$

Since $x + y \in \Gamma$, we have $x \in \Gamma - y \subset \Gamma - \Gamma = \Gamma_{\beta,\{-N,\ldots,-1,0,1,\ldots,N\}}$, which can be written as

$$x = \frac{1-\beta}{N} \sum_{k=1}^{\infty} x_k \beta^{k-1} \text{ with each } x_k \in \{-N, \ldots, -1, 0, 1, \ldots, N\}. \tag{3.6}$$

Note that $x + y \in \Gamma_{\beta,\{0,1,\ldots,N\}}$ and

$$x + y = \frac{1-\beta}{N} \sum_{k=1}^{\infty} (x_k + y_k) \beta^{k-1}.$$

It follows by (3.5) and (3.6) that $(x_k + y_k)_{k=1}^{\infty}$ is a $\{-N, \dots, -1, 0, 1, \dots, 2N\}$-coding of $x + y$. So, by Lemma 3.3 it follows that $x_k + y_k \in \{0, 1, \dots, N\}$ for all $k \geq 1$.

Next, observe that

$$2x + y = \frac{1-\beta}{N} \sum_{k=1}^{\infty} (2x_k + y_k)\beta^{k-1} \in \Gamma_{\beta,\{0,1,\dots,N\}}.$$

Since $x_k + y_k \in \{0, 1, \dots, N\}$ for all $k \geq 1$, by (3.6) it follows that $(2x_k + y_k)_{k=1}^{\infty}$ is a $\{-N, \dots, -1, 0, 1, \dots, 2N\}$-coding of $2x + y$. Hence, by Lemma 3.3 we conclude that $2x_k + y_k \in \{0, 1, \dots, N\}$ for all $k \geq 1$.

Proceeding this argument N times we conclude that $Nx_k + y_k \in \{0, 1, \dots, N\}$ for all $k \geq 1$. Note by (3.5) that $y_k \in \{0, 1, \dots, N\}$. Thus, $x_k \in \{-1, 0, 1\}$ for all $k \geq 1$. That is, $x \in \Gamma_{\beta,\{-1,0,1\}}$.

(ii) Taking $y = 0$, by (i) we obtain $x \in \Gamma_{\beta,\{-1,0,1\}}$. Since x also belongs to $\Gamma = \Gamma_{\beta,\{0,1,\dots,N\}}$, by Lemma 3.3 it follows that $x \in \Gamma_{\beta,\{0,1,\dots,N\}} \cap \Gamma_{\beta,\{-1,0,1\}} = \Gamma_{\beta,\{0,1\}}$.

Lemma 3.5 *Let* $\mathbf{t} = (t_0, t_1, \dots, t_m) \in \mathbb{R}^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$*. If* $r\Gamma_{\mathbf{t}} \subset \Gamma_{\mathbf{t}}$ *with* $0 < r < 1$*, then* $r = \beta^q$ *for some* $q \in \mathbb{Z}_+$*.*

Proof Note by (1.1) that $j(1-\beta)\beta^k/N \in \Gamma$ for $k \geq 0$ and $j \in \{1, \dots, N\}$. Take k large enough so that $r(1-\beta)\beta^k < t_1$. Since $r\Gamma_{\mathbf{t}} \subset \Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ with $0 = t_0 < t_1 < \cdots < t_m$, it follows that $jr(1-\beta)\beta^k/N \in \Gamma$ for all $j \in \{1, \dots, N\}$. By Lemma 3.4 (ii), we conclude that $r(1-\beta)\beta^k/N \in \Gamma_{\beta,\{0,1\}}$. Note that $0 < r < 1$. By (3.3) it follows that

$$r = \beta^q + \sum_{k=q+1}^{\infty} j_k\beta^k \quad \text{with} \quad j_k \in \{0, 1\},$$

where $q \in \mathbb{Z}_+$. So, it suffices to prove that $j_k = 0$ for all $k \geq q + 1$.

Suppose on the contrary there exists $q' > q$ such that

$$r = \beta^q + \beta^{q'} + \sum_{k=q'+1}^{\infty} j_k\beta^k. \tag{3.7}$$

Note that $(1-\beta)\beta^n(\beta^q + \beta^{q'}) \in \Gamma$ for any $n \geq 0$. Take n sufficiently large so that $r(1-\beta)\beta^n(\beta^q + \beta^{q'}) < t_1$. Then we obtain

$$y := r(1-\beta)\beta^n(\beta^q + \beta^{q'}) \in \Gamma = \Gamma_{\beta,\{0,1,\dots,N\}}. \tag{3.8}$$

On the other hand, by (3.7) it follows that

$$\begin{aligned} y &= (1-\beta)\beta^n(\beta^q + \beta^{q'})\left(\beta^q + \beta^{q'} + \sum_{k=q'+1}^{\infty} j_k\beta^k\right) \\ &= \frac{(1-\beta)\beta^n}{N}\left(N\beta^{2q} + 2N\beta^{q+q'} + N\beta^{2q'} + \sum_{k=q'+1}^{\infty} j_kN\beta^{k+q} + \sum_{k=q'+1}^{\infty} j_kN\beta^{k+q'}\right), \end{aligned}$$

which has a $\{-N, \dots, -1, 0, 1, \dots, 2N\}$-coding different from its $\{0, 1, \dots, N\}$-coding by (3.8). This leads to a contradiction with Lemma 3.3.

Recall that

$$T = \bigcup_{n=1}^{\infty}\left\{\frac{1-\beta}{N}\sum_{k=1}^{n} j_k\beta^{-k} : j_k \in \{0, 1, \dots, N\}\ \forall 1 \leq k \leq n\right\}.$$

Lemma 3.6 *Let* $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ *be a self-similar set, where* $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$. *Then either* $\mathbf{t} \in T^{m+1}$ *or its conjugate* $\hat{\mathbf{t}} \in T^{m+1}$. *Furthermore,* $t_{j+1} - t_j > 1$ *for all* $0 \le j < m$.

Proof We first assume that $\Gamma_{\mathbf{t}}$ has a generating IFS which contains a similitude rx with $0 < r < 1$. Then by Lemma 3.5 there exists $q \in \mathbb{Z}_+$ such that $\beta^q \Gamma_{\mathbf{t}} \subset \Gamma_{\mathbf{t}}$. Take u sufficiently large so that $\beta^{uq}(1 + t_m) < t_1$. Then we obtain

$$\beta^{uq}\Gamma_{\mathbf{t}} \subset \Gamma. \tag{3.9}$$

Since $t_j \in \Gamma + t_j \subset \Gamma_{\mathbf{t}}$ for each $0 \le j \le m$, by (3.9) we have $\beta^{uq} t_j \in \Gamma$. This together with (1.1) implies that

$$t_j = \frac{1-\beta}{N}\beta^{-uq}\sum_{k=1}^{\infty} t_{j,k}\beta^{k-1} \quad \text{with} \quad t_{j,k} \in \{0, 1, \ldots, N\}. \tag{3.10}$$

So, to prove $t_j \in T$ it suffices to prove that $t_{j,k} = 0$ for all $k > uq$.

Suppose on the contrary that $t_{j,k_1} \neq 0$ for some $k_1 > uq$. Note that

$$y := \frac{1-\beta}{N}(N + 1 - t_{j,k_1})\beta^{k_1 - uq - 1} + t_j \in \Gamma_{\mathbf{t}}.$$

Then by (3.9) we have $\beta^{uq} y \in \Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}$. On the other hand, by (3.10) it follows that

$$\begin{aligned}\beta^{uq} y &= \frac{1-\beta}{N}(N + 1 - t_{j,k_1})\beta^{k_1-1} + \frac{1-\beta}{N}\sum_{k=1}^{\infty} t_{j,k}\beta^{k-1} \\ &= \frac{1-\beta}{N}\Bigg(\sum_{k\ge 1, k\neq k_1} t_{j,k}\beta^{k-1} + (N+1)\beta^{k_1-1}\Bigg),\end{aligned}$$

which has a $\{-N, \cdots, -1, 0, 1, \ldots, 2N\}$-coding different from its $\{0, 1, \ldots, N\}$-coding. This leads to a contradiction with Lemma 3.3. So,

$$t_j = \frac{1-\beta}{N}\sum_{k=1}^{uq} t_{j,k}\beta^{k-uq-1} \in T \quad \text{for all } 0 \le j \le m.$$

Take $j \in \{0, 1, \ldots, m-1\}$, and let $k_2 = \min\left\{1 \le k \le uq : t_{j+1,k} \neq t_{j,k}\right\}$. Since $t_{j+1} > t_j$, by using $0 < \beta < 1/(2N+1)$ it follows that $t_{j+1,k_2} > t_{j,k_2}$. Therefore,

$$t_{j+1} - t_j \ge \frac{1-\beta}{N}\beta^{k_2-uq-1} - \frac{1-\beta}{N}\sum_{k=k_2+1}^{uq} N\beta^{k-uq-1} = 1 + \beta^{k_2-uq}\left(\frac{1-\beta}{N\beta} - 1\right) > 1,$$

as desired.

Next, we assume that $\Gamma_{\hat{\mathbf{t}}}$ has a generating IFS which contains a similitude rx with $0 < r < 1$. By the above argument, we conclude that $\hat{\mathbf{t}} \in T^{m+1}$, and $\hat{t}_{j+1} - \hat{t}_j > 1$ for all $0 \le j < m$. Note that $\hat{t}_j = t_m - t_{m-j}$ for all $0 \le j \le m$. Thus, we also have $t_{j+1} - t_j = \hat{t}_{m-j} - \hat{t}_{m-j-1} > 1$ for all $0 \le j < m$.

By Lemma 3.2, either $\Gamma_{\mathbf{t}}$ or $\Gamma_{\hat{\mathbf{t}}}$ has a generating IFS which contains a similitude rx with $0 < r < 1$. Thus, we conclude that either $\mathbf{t} \in T^{m+1}$ or its conjugate $\hat{\mathbf{t}} \in T^{m+1}$. In any case, we have $t_{j+1} - t_j > 1$ for all $0 \le j < m$.

The following lemma states that $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ is a self-similar set if and only if Γ can be written as a union of similar copies of $\Gamma_{\mathbf{t}}$.

Lemma 3.7 *Let* $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$*. Then* $\Gamma_{\mathbf{t}}$ *is a self-similar set if and only if there exists a finite set* $\mathcal{G}$ *of similitudes such that*

$$\bigcup_{g \in \mathcal{G}} g(\Gamma_{\mathbf{t}}) = \Gamma. \tag{3.11}$$

Proof The sufficiency is easier, because if (3.11) holds for some finite set $\mathcal{G}$, then

$$\bigcup_{j=0}^{m} \left\{ g(x) + t_j : g \in \mathcal{G} \right\}$$

is a generating IFS of $\bigcup_{j=0}^{m}(\Gamma + t_j) = \Gamma_{\mathbf{t}}$.

For the necessity suppose that $\Gamma_{\mathbf{t}}$ is a self-similar set generated by an IFS $\mathcal{F} = \{f_i(x) = r_i x + b_i\}_{i=1}^{n}$. By Lemma 3.6, we have

$$\delta := \min_{0 \le j < m} (t_{j+1} - t_j - 1) = \min_{0 \le j_1 < j_2 \le m} \operatorname{dist}(\Gamma + t_{j_1}, \Gamma + t_{j_2}) > 0.$$

Note that for any $p \ge 1$,

$$\left\{ f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_p}(x) : 1 \le i_1, i_2, \cdots, i_p \le n \right\}$$

is again a generating IFS of $\Gamma_{\mathbf{t}}$. Without loss of generality we may assume that all similarity ratios are sufficiently small so that $|r_i| < \delta/(1 + t_m)$ for all $1 \le i \le n$. Then we have $\operatorname{diam}\big(f_i(\Gamma_{\mathbf{t}})\big) = |r_i| \cdot \operatorname{diam}(\Gamma_{\mathbf{t}}) < \delta$. Therefore, for each $f \in \mathcal{F}$ there exists a unique $j \in \{0, 1, \ldots, m\}$ such that $f(\Gamma_{\mathbf{t}}) \subset \Gamma + t_j$. Set $\mathcal{G} := \{f \in \mathcal{F} : f(\Gamma_{\mathbf{t}}) \subset \Gamma\}$. Then $\bigcup_{g \in \mathcal{G}} g(\Gamma_{\mathbf{t}}) = \Gamma$ as desired.

Recall that for a translation vector $\mathbf{t} \in T^{m+1}$ and $n \ge \tau_{\mathbf{t}}$ the sets $\Omega_{\mathbf{t}}^n$ and $\hat{\Omega}_{\mathbf{t}}^n$ are defined in (1.3) and (1.4), respectively.

Lemma 3.8 *Let* $\mathbf{t} = (t_0, t_1, \cdots, t_m) \in T^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$*, and suppose* $g(x) = rx + b$ *with* $0 < r < 1$*. If* $g(\Gamma_{\mathbf{t}}) \subset \Gamma$*, then we have* $g(x) = \phi_{\mathbf{i}}(x)$ *for some* $\mathbf{i} \in \Omega_{\mathbf{t}}^n$ *with* $n \ge \tau_{\mathbf{t}}$*.*

Proof Note that $r\Gamma_{\mathbf{t}} + b \subset \Gamma$ and $0 \in \Gamma_{\mathbf{t}}$. It follows that $b \in \Gamma$, and we write

$$b = \frac{1-\beta}{N} \sum_{k=1}^{\infty} b_k \beta^{k-1} \text{ with } b_k \in \{0, 1, \ldots, N\}. \tag{3.12}$$

Note that $j(1-\beta)/N \in \Gamma \subset \Gamma_{\mathbf{t}}$ for all $1 \le j \le N$ and $r\Gamma_{\mathbf{t}} + b \subset \Gamma$. So we have $jr(1-\beta)/N + b \in \Gamma$ for all $1 \le j \le N$. By Lemma 3.4 (i), we have $r(1-\beta)/N \in \Gamma_{\beta,\{-1,0,1\}}$.

Note that $r\Gamma_{\mathbf{t}} + b \subset \Gamma$. Then $0 < r \le \operatorname{diam}(\Gamma)/\operatorname{diam}(\Gamma_{\mathbf{t}}) = 1/(1 + t_m) < 1/2$, where the last inequality follows by Lemma 3.6. So, by using $0 < \beta < 1/(2N+1) \le 1/3$ and (3.3) it follows that

$$r = \beta^n + \sum_{k=n+1}^{\infty} r_k \beta^k \text{ with } r_k \in \{-1, 0, 1\}, \tag{3.13}$$

where $n \in \mathbb{Z}_+$. In view of (3.12) and (3.13) we will split our proof into the following three steps: (i) we show in (3.12) that $b_k = 0$ for all $k \ge n+1$, and then $b = \frac{1-\beta}{N} \sum_{k=1}^{n} b_k \beta^{k-1}$; (ii) we show in (3.13) that $r_k = 0$ for all $k \ge n+1$, and then we have $r = \beta^n$ and thus $g(x) = rx + b = \phi_{b_1 b_2 \cdots b_n}(x)$; (iii) we show that $n \ge \tau_{\mathbf{t}}$ and $b_1 b_2 \cdots b_n \in \Omega_{\mathbf{t}}^n$.

Step 1. We will show in (3.12) that $b_k = 0$ for all $k \geq n+1$. Fix $k_1 \geq n+1$. Note that $(1-\beta)\beta^{k_1-n-1} \in \Gamma \subset \Gamma_{\mathbf{t}}$ and $r\Gamma_{\mathbf{t}} + b \subset \Gamma$. It follows that

$$y := r(1-\beta)\beta^{k_1-n-1} + b \in \Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}.$$

By (3.12) and (3.13) we have

$$y = \frac{1-\beta}{N}\left(\sum_{k=1}^{k_1-1} b_k\beta^{k-1} + (N + b_{k_1})\beta^{k_1-1} + \sum_{k=k_1+1}^{\infty}(Nr_{k+n-k_1} + b_k)\beta^{k-1}\right).$$

By Lemma 3.3, we have $N + b_{k_1} \in \{0, 1, \ldots, N\}$. This implies $b_{k_1} = 0$. Since k_1 is arbitrary, we conclude that

$$b = \frac{1-\beta}{N}\sum_{k=1}^{n} b_k\beta^{k-1} \text{ with } b_k \in \{0, 1, \ldots, N\}. \tag{3.14}$$

Step 2. We will show $r = \beta^n$, that is, $r_k = 0$ for all $k \geq n+1$ in (3.13).

First, suppose that there exists $k_2 \geq n+1$ such that $r_{k_2} = -1$. Note that $1-\beta \in \Gamma \subset \Gamma_{\mathbf{t}}$ and $r\Gamma_{\mathbf{t}} + b \subset \Gamma$. It follows that $r(1-\beta) + b \in \Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}$. On the other hand, by (3.13) and (3.14), we have

$$r(1-\beta) + b = \frac{1-\beta}{N}\left(\sum_{k=1}^{n} b_k\beta^{k-1} + N\beta^n + \sum_{k=n+1}^{\infty} Nr_k\beta^k\right),$$

which has a $\{-N, \ldots, -1, 0, 1, \ldots, 2N\}$-coding different from its $\{0, 1, \ldots, N\}$-coding. This leads to a contradiction with Lemma 3.3. Thus we have $r_k \in \{0, 1\}$ for all $k \geq n+1$.

Next, suppose that there exists $k_3 \geq n+1$ such that

$$r = \beta^n + \beta^{k_3} + \sum_{k=k_3+1}^{\infty} r_k\beta^k \quad \text{with} \quad r_k \in \{0, 1\}. \tag{3.15}$$

Note that $(1-\beta)(\beta^n + \beta^{k_3}) \in \Gamma \subset \Gamma_{\mathbf{t}}$ and $r\Gamma_{\mathbf{t}} + b \subset \Gamma$. It follows that $z := r(1-\beta)(\beta^n + \beta^{k_3}) + b \in \Gamma = \Gamma_{\beta,\{0,1,\ldots,N\}}$. On the other hand, by (3.14) and (3.15) we have

$$\begin{aligned} z &= (1-\beta)(\beta^n + \beta^{k_3})\left(\beta^n + \beta^{k_3} + \sum_{k=k_3+1}^{\infty} r_k\beta^k\right) + b \\ &= \frac{1-\beta}{N}\left(\sum_{k=1}^{n} b_k\beta^{k-1} + N\beta^{2n} + 2N\beta^{n+k_3} + N\beta^{2k_3}\right. \\ &\quad \left. + \sum_{k=k_3+1}^{\infty} Nr_k\beta^{k+n} + \sum_{k=k_3+1}^{\infty} Nr_k\beta^{k+k_3}\right), \end{aligned}$$

which has a $\{-N, \ldots, -1, 0, 1, \ldots, 2N\}$-coding different from its $\{0, 1, \ldots, N\}$-coding. This leads to a contradiction with Lemma 3.3. Thus we conclude that $r = \beta^n$.

Step 3. By the definition of $\tau_{\mathbf{t}}$, there exists $1 \leq j_1 \leq m$ such that $t_{j_1,\tau_{\mathbf{t}}} \neq 0$. Then we have

$$t_{j_1} = \frac{1-\beta}{N}\sum_{k=1}^{\tau_{\mathbf{t}}} t_{j_1,k}\beta^{-k} \geq \frac{1-\beta}{N}\beta^{-\tau_{\mathbf{t}}}.$$

Note that $t_{j_1} \in \Gamma_{\mathbf{t}}$ and $r\Gamma_{\mathbf{t}} + b \subset \Gamma$. It follows that $rt_{j_1} + b \in \Gamma \subset [0, 1]$. Thus, $\frac{1-\beta}{N}\beta^{n-\tau_{\mathbf{t}}} \leq rt_{j_1} \leq rt_{j_1} + b \leq 1$. By using $0 < \beta < 1/(2N+1)$ this implies that $n \geq \tau_{\mathbf{t}}$. Note that

$$g(x) = \beta^n x + \frac{1-\beta}{N}\sum_{k=1}^{n} b_k \beta^{k-1} = \phi_{b_1 b_2 \cdots b_n}(x).$$

It remains to show that $b_1 b_2 \cdots b_n \in \Omega_{\mathbf{t}}^n$, that is, $b_{n+1-k} \leq N - s_k$ for all $1 \leq k \leq \tau_{\mathbf{t}}$, where $s_k = \max_{0 \leq j \leq m} t_{j,k}$.

Suppose on the contrary that $b_{n+1-k_4} + s_{k_4} \geq N+1$ for some $1 \leq k_4 \leq \tau_{\mathbf{t}}$. By the definition of s_{k_4}, there exists $1 \leq j_2 \leq m$ such that $t_{j_2,k_4} = s_{k_4}$. Then we have $b_{n+1-k_4} + t_{j_2,k_4} \geq N+1$.

Note that

$$\begin{aligned} g(\Gamma + t_{j_2}) &= \beta^n \Gamma + \beta^n t_{j_2} + b \\ &= \beta^n \Gamma + \frac{1-\beta}{N}\left(\sum_{k=1}^{n-\tau_{\mathbf{t}}} b_k \beta^{k-1} + \sum_{k=1}^{\tau_t}(b_{n+1-k} + t_{j_2,k})\beta^{n-k}\right) \\ &= \beta^n \Gamma + \frac{1-\beta}{N}\left(\sum_{k=1}^{n-\tau_{\mathbf{t}}} b_k \beta^{k-1} + \sum_{k=n+1-\tau_t}^{n}(b_k + t_{j_2,n+1-k})\beta^{k-1}\right). \end{aligned}$$

By Lemma 3.3, we have $g(\Gamma + t_{j_2}) \cap \Gamma = \emptyset$. This contradicts with $g(\Gamma_{\mathbf{t}}) \subset \Gamma$. Therefore, we conclude that $b_1 b_2 \cdots b_n \in \Omega_{\mathbf{t}}^n$, as desired.

Lemma 3.9 *Let* $\mathbf{t} = (t_0, t_1, \cdots, t_m) \in T^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$, *and suppose* $g(x) = -rx + b$ *with* $0 < r < 1$. *If* $g(\Gamma_{\mathbf{t}}) \subset \Gamma$, *then we have* $g(x) = \phi_{\mathbf{i}}(1-x)$ *for some* $\mathbf{i} \in \hat{\Omega}_{\mathbf{t}}^n$ *with* $n \geq \tau_{\mathbf{t}}$.

Proof Let $b' = 1 - b$. Then we have $g(\Gamma_{\mathbf{t}}) = -r\Gamma_{\mathbf{t}} + b = 1 - (r\Gamma_{\mathbf{t}} + b') \subset \Gamma$. Note that Γ is symmetric. It follows that

$$r\Gamma_{\mathbf{t}} + b' \subset 1 - \Gamma = \Gamma.$$

By Lemma 3.8, we have

$$r = \beta^n, \; b' = \phi_{\mathbf{i}'}(0),$$

where $n \geq \tau_{\mathbf{t}}$ and $\mathbf{i}' = i_1' i_2' \cdots i_n' \in \Omega_{\mathbf{t}}^n$.

Let $\mathbf{i} = i_1 i_2 \cdots i_n$ with $i_k = N - i_k'$ for all $1 \leq k \leq n$. Clearly, we have $\mathbf{i} \in \hat{\Omega}_{\mathbf{t}}^n$, and

$$\begin{aligned} g(x) &= -\beta^n x + 1 - \phi_{\mathbf{i}'}(0) \\ &= -\beta^n x + 1 - \frac{1-\beta}{N}\sum_{k=1}^{n} i_k' \beta^{k-1} \\ &= \beta^n(1-x) + \frac{1-\beta}{N}\sum_{k=1}^{n}(N - i_k')\beta^{k-1} \\ &= \beta^n(1-x) + \phi_{\mathbf{i}}(0) \\ &= \phi_{\mathbf{i}}(1-x), \end{aligned}$$

as desired.

Proof of Proposition 3.1 Suppose that $\Gamma_{\mathbf{t}}$ is a self-similar set. By Lemma 3.6, we have either $\mathbf{t} \in T^{m+1}$ or $\hat{\mathbf{t}} \in T^{m+1}$. Furthermore, we have

$$\delta := \min_{0 \le j < m} (t_{j+1} - t_j - 1) = \min_{0 \le j_1 < j_2 \le m} \operatorname{dist}(\Gamma + t_{j_1}, \Gamma + t_{j_2}) > 0.$$

Write $g(x) = rx + b$. Then by $g(\Gamma_{\mathbf{t}}) \subset \Gamma_{\mathbf{t}}$ we have $g^n(\Gamma_{\mathbf{t}}) \subset \Gamma_{\mathbf{t}}$ for all $n \ge 1$. Take n large enough so that $|r|^n \cdot \operatorname{diam}(\Gamma_{\mathbf{t}}) < \delta$. Then there exist $j_1, j_2 \in \{0, 1, \ldots, m\}$ such that

$$g^n(\Gamma_{\mathbf{t}}) \subset \Gamma + t_{j_1}, \quad g^{n+1}(\Gamma_{\mathbf{t}}) \subset \Gamma + t_{j_2}. \tag{3.16}$$

If $\mathbf{t} \in T^{m+1}$, let $g_1(x) = r^n x + g^n(0) - t_{j_1}$ and $g_2(x) = r^{n+1}x + g^{n+1}(0) - t_{j_2}$. Then by (3.16) we have $g_1(\Gamma_{\mathbf{t}}) = g^n(\Gamma_{\mathbf{t}}) - t_{j_1} \subset \Gamma$ and $g_2(\Gamma_{\mathbf{t}}) = g^{n+1}(\Gamma_{\mathbf{t}}) - t_{j_2} \subset \Gamma$. By Lemmas 3.8 and 3.9, we conclude that $|r|^n = \beta^{q_1}$ and $|r|^{n+1} = \beta^{q_2}$ for some $q_1, q_2 \in \mathbb{Z}_+$. It follows that $|r| = \beta^q$ for some $q \in \mathbb{Z}_+$.

If $\hat{\mathbf{t}} \in T^{m+1}$, let $g_3(x) = -r^n x + g^n(1+t_m) - t_{j_1}$ and $g_4(x) = -r^{n+1}x + g^{n+1}(1+t_m) - t_{j_2}$. Note that $\Gamma_{\mathbf{t}} = 1 + t_m - \Gamma_{\hat{\mathbf{t}}}$. By (3.16) we have $g_3(\Gamma_{\hat{\mathbf{t}}}) = g^n(1+t_m-\Gamma_{\hat{\mathbf{t}}}) - t_{j_1} = g^n(\Gamma_{\mathbf{t}}) - t_{j_1} \subset \Gamma$ and $g_4(\Gamma_{\hat{\mathbf{t}}}) = g^{n+1}(1 + t_m - \Gamma_{\hat{\mathbf{t}}}) - t_{j_2} = g^{n+1}(\Gamma_{\mathbf{t}}) - t_{j_2} \subset \Gamma$. By Lemmas 3.8 and 3.9, we conclude that $|r|^n = \beta^{q_3}$ and $|r|^{n+1} = \beta^{q_4}$ for some $q_3, q_4 \in \mathbb{Z}_+$. From this we deduce that $|r| = \beta^q$ for some $q \in \mathbb{Z}_+$.

4 Proofs of Theorem 1.1 and 1.5

In this section we will prove our main theorems. First we prove Theorem 1.5. Recall from Definition 1.2 the definition of the admissible translation vectors. In the following we add an equivalent condition for admissible translation vectors to Proposition 1.3.

Lemma 4.1 *Suppose that* $0 < \beta < 1/(N+1)$. *A vector* $\mathbf{t} = (t_0, t_1, \cdots, t_m) \in \mathbb{R}^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$ *is an admissible translation vector if and only if* $\mathbf{t} \in T^{m+1}$, *and there exist finite sets* $\mathcal{I}_1 \subset \bigcup_{n \ge \tau_{\mathbf{t}}} \Omega_{\mathbf{t}}^n$ *and* $\mathcal{I}_2 \subset \bigcup_{n \ge \tau_{\mathbf{t}}} \hat{\Omega}_{\mathbf{t}}^n$ *such that*

$$\bigcup_{\mathbf{i} \in \mathcal{I}_1} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i} \in \mathcal{I}_2} \phi_{\mathbf{i}}(1 - \Gamma_{\mathbf{t}}) = \Gamma.$$

Proof Suppose that $\mathbf{t} \in T^{m+1}$. Recall from (1.2) that for $0 \le j \le m$, we have

$$t_j = \beta^{-\tau_{\mathbf{t}}} \phi_{t_{j,\tau_{\mathbf{t}}} \cdots t_{j,2} t_{j,1}}(0).$$

Take $\mathbf{i} = i_1 i_2 \cdots i_n \in \Omega_{\mathbf{t}}^n$ for $n \ge \tau_{\mathbf{t}}$. By the definition of $\Omega_{\mathbf{t}}^n$ in (1.3), we have

$$i_1 i_2 \cdots i_{n-\tau_{\mathbf{t}}} (i_{n+1-\tau_{\mathbf{t}}} + t_{j,\tau_{\mathbf{t}}}) \cdots (i_{n-1} + t_{j,2})(i_n + t_{j,1}) \in \{0, 1, \ldots, N\}^n$$

for all $0 \le j \le m$. It follows that

$$\begin{aligned}\phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) &= \bigcup_{j=0}^{m} \phi_{\mathbf{i}}(\Gamma + t_j) = \bigcup_{j=0}^{m} \big(\beta^n \Gamma + \beta^n t_j + \phi_{\mathbf{i}}(0)\big) \\ &= \bigcup_{j=0}^{m} \big(\beta^n \Gamma + \beta^{n-\tau_{\mathbf{t}}} \phi_{t_{j,\tau_{\mathbf{t}}} \cdots t_{j,2} t_{j,1}}(0) + \phi_{\mathbf{i}}(0)\big) \\ &= \bigcup_{j=0}^{m} \big(\beta^n \Gamma + \phi_{i_1 i_2 \cdots i_{n-\tau_{\mathbf{t}}} (i_{n+1-\tau_{\mathbf{t}}} + t_{j,\tau_{\mathbf{t}}}) \cdots (i_{n-1}+t_{j,2})(i_n + t_{j,1})}(0)\big) \\ &= \bigcup_{j=0}^{m} \phi_{i_1 i_2 \cdots i_{n-\tau_{\mathbf{t}}} (i_{n+1-\tau_{\mathbf{t}}} + t_{j,\tau_{\mathbf{t}}}) \cdots (i_{n-1}+t_{j,2})(i_n + t_{j,1})}(\Gamma).\end{aligned}$$

Recall the definition of $\mathcal{A}_{\mathbf{t}}^n$ in (1.5), and so we have

$$\bigcup_{\mathbf{i} \in \Omega_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) = \bigcup_{\mathbf{i} \in \mathcal{A}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma).$$

Similarly, for $\mathbf{i} = i_1 i_2 \cdots i_n \in \hat{\Omega}_{\mathbf{t}}^n$ with $n \ge \tau_{\mathbf{t}}$, by the symmetry of Γ we have

$$\begin{aligned}\phi_{\mathbf{i}}(1 - \Gamma_{\mathbf{t}}) &= \bigcup_{j=0}^{m} \phi_{\mathbf{i}}(1 - \Gamma - t_j) = \bigcup_{j=0}^{m} \phi_{\mathbf{i}}(\Gamma - t_j) \\ &= \bigcup_{j=0}^{m} \big(\beta^n \Gamma - \beta^n t_j + \phi_{\mathbf{i}}(0)\big) \\ &= \bigcup_{j=0}^{m} \phi_{i_1 i_2 \cdots i_{n-\tau_{\mathbf{t}}} (i_{n+1-\tau_{\mathbf{t}}} - t_{j,\tau_{\mathbf{t}}}) \cdots (i_{n-1}-t_{j,2})(i_n - t_{j,1})}(\Gamma).\end{aligned}$$

Recall the definition of $\hat{\mathcal{A}}_{\mathbf{t}}^n$ in (1.5), and so we obtain

$$\bigcup_{\mathbf{i} \in \hat{\Omega}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) = \bigcup_{\mathbf{i} \in \hat{\mathcal{A}}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma).$$

Note that we set $\mathcal{W}_{\mathbf{t}}^n = \mathcal{A}_{\mathbf{t}}^n \cup \hat{\mathcal{A}}_{\mathbf{t}}^n$. Therefore, we conclude that for $n \ge \tau_{\mathbf{t}}$,

$$\bigcup_{\mathbf{i} \in \Omega_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i} \in \hat{\Omega}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(1 - \Gamma_{\mathbf{t}}) = \bigcup_{\mathbf{i} \in \mathcal{W}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma) \subset \Gamma. \tag{4.1}$$

For the necessity, suppose $\mathbf{t}$ is an admissible translation vector, and let $\ell \ge \tau_{\mathbf{t}}$ be defined as in Definition 1.2. Take $\mathcal{I}_1 = \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \Omega_{\mathbf{t}}^n$ and $\mathcal{I}_2 = \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \hat{\Omega}_{\mathbf{t}}^n$. Recall that

$\mathcal{W}_{\mathbf{t}}^n = \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}}$ in (1.6). By (4.1), we have

$$\begin{aligned}\bigcup_{\mathbf{i}\in\mathcal{I}_1} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\mathcal{I}_2} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}) &= \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \bigcup_{\mathbf{i}\in\mathcal{W}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma) \\ &= \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \bigcup_{\mathbf{i}\in\mathcal{W}_{\mathbf{t}}^n\times\{0,1,\ldots,N\}^{\ell-n}} \phi_{\mathbf{i}}(\Gamma) \\ &= \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \bigcup_{\mathbf{i}\in\{0,1,\ldots,N\}^{n-\tau_{\mathbf{t}}}\times\mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}}\times\{0,1,\ldots,N\}^{\ell-n}} \phi_{\mathbf{i}}(\Gamma) \\ &= \bigcup_{\mathbf{i}\in\{0,1,\ldots,N\}^{\ell}} \phi_{\mathbf{i}}(\Gamma) \\ &= \Gamma,\end{aligned}$$

as desired.

Next, we prove the sufficiency. Suppose $\mathbf{t} \in T^{m+1}$, and let $\mathcal{I}_1 \subset \bigcup_{n\geq\tau_{\mathbf{t}}} \Omega_{\mathbf{t}}^n$ and $\mathcal{I}_2 \subset \bigcup_{n\geq\tau_{\mathbf{t}}} \hat{\Omega}_{\mathbf{t}}^n$ be finite sets so that $\bigcup_{\mathbf{i}\in\mathcal{I}_1} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\mathcal{I}_2} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}) = \Gamma$. Let ℓ be the largest length of words in $\mathcal{I}_1 \cup \mathcal{I}_2$. Clearly, we have $\mathcal{I}_1 \subset \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \Omega_{\mathbf{t}}^n$ and $\mathcal{I}_2 \subset \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \hat{\Omega}_{\mathbf{t}}^n$. It follows that

$$\Gamma = \bigcup_{\mathbf{i}\in\mathcal{I}_1} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\mathcal{I}_2} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}) \subset \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \Big(\bigcup_{\mathbf{i}\in\Omega_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\hat{\Omega}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}) \Big).$$

Together with (4.1), we have

$$\begin{aligned}\Gamma &= \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \Big(\bigcup_{\mathbf{i}\in\Omega_{\mathbf{t}}^n} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\hat{\Omega}_{\mathbf{t}}^n} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}) \Big) \\ &= \bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \bigcup_{\mathbf{i}\in\{0,1,\ldots,N\}^{n-\tau_{\mathbf{t}}}\times\mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}}\times\{0,1,\ldots,N\}^{\ell-n}} \phi_{\mathbf{i}}(\Gamma). \end{aligned} \tag{4.2}$$

Since $0 < \beta < 1/(N+1)$, the generating IFS of Γ,

$$\Big\{ \phi_i(x) = \beta x + i\frac{1-\beta}{N} : i = 0, 1, \ldots, N \Big\}$$

satisfies the strong separation condition. Therefore, the equality (4.2) implies

$$\bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} \times \{0, 1, \ldots, N\}^{\ell-n} = \{0, 1, \ldots, N\}^{\ell}.$$

Hence, $\mathbf{t}$ is an admissible translation vector. We complete the proof.

First, we prove the sufficiency in Theorem 1.5.

Lemma 4.2 *Suppose that* $0 < \beta < 1/(N+1)$. *If* $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ *with* $0 = t_0 < t_1 < \cdots < t_m$ *is an admissible translation vector, then* $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ *is a self-similar set.*

Proof By Lemma 4.1, there exist finite sets $\mathcal{I}_1 \subset \bigcup_{n\geq\tau_{\mathbf{t}}} \Omega_{\mathbf{t}}^n$ and $\mathcal{I}_2 \subset \bigcup_{n\geq\tau_{\mathbf{t}}} \hat{\Omega}_{\mathbf{t}}^n$ such that

$$\bigcup_{\mathbf{i}\in\mathcal{I}_1} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\mathcal{I}_2} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}) = \Gamma.$$

Set $f_{\mathbf{i}}(x) := \phi_{\mathbf{i}}(x)$ for $\mathbf{i} \in \mathcal{I}_1$, and set $g_{\mathbf{i}}(x) := \phi_{\mathbf{i}}(1-x)$ for $\mathbf{i} \in \mathcal{I}_2$. Then we have

$$\bigcup_{\mathbf{i}\in\mathcal{I}_1} f_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\mathcal{I}_2} g_{\mathbf{i}}(\Gamma_{\mathbf{t}}) = \Gamma.$$

This implies that $\Gamma_{\mathbf{t}} = \bigcup_{j=0}^{m}(\Gamma + t_j)$ is a self-similar set generated by the IFS

$$\left\{f_{\mathbf{i}}(x) + t_j : \mathbf{i} \in \mathcal{I}_1, \ 0 \leq j \leq m\right\} \cup \left\{g_{\mathbf{i}}(x) + t_j : \mathbf{i} \in \mathcal{I}_2, \ 0 \leq j \leq m\right\}.$$

The proof is completed.

Proof of Theorem 1.5 Note that $\Gamma_{\mathbf{t}}$ is a self-similar set if and only if $\Gamma_{\hat{\mathbf{t}}}$ is a self-similar set. The sufficiency follows from Lemma 4.2. In the following we prove the necessity.

Suppose that $\Gamma_{\mathbf{t}}$ is a self-similar set. Then by Lemma 3.6 we have either $\mathbf{t} \in T^{m+1}$ or its conjugate $\hat{\mathbf{t}} \in T^{m+1}$. Without loss of generality we may assume $\mathbf{t} \in T^{m+1}$. By Lemma 3.7 there exists a finite set $\mathcal{G}$ of similitudes such that

$$\bigcup_{g\in\mathcal{G}} g(\Gamma_{\mathbf{t}}) = \Gamma.$$

By Lemmas 3.8 and 3.9, for each $g \in \mathcal{G}$ we have $g(x) = \phi_{\mathbf{i}}(x)$ for some $\mathbf{i} \in \bigcup_{n=\tau_{\mathbf{t}}}^{\infty} \Omega_{\mathbf{t}}^n$, or $g(x) = \phi_{\mathbf{i}}(1-x)$ for some $\mathbf{i} \in \bigcup_{n=\tau_{\mathbf{t}}}^{\infty} \hat{\Omega}_{\mathbf{t}}^n$. Set $\mathcal{I}_1 = \{\mathbf{i} : g(x) = \phi_{\mathbf{i}}(x) \in \mathcal{G}\}$ and $\mathcal{I}_2 = \{\mathbf{i} : g(x) = \phi_{\mathbf{i}}(1-x) \in \mathcal{G}\}$. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are finite subsets of $\bigcup_{n=\tau_{\mathbf{t}}}^{\infty} \Omega_{\mathbf{t}}^n$ and $\bigcup_{n=\tau_{\mathbf{t}}}^{\infty} \hat{\Omega}_{\mathbf{t}}^n$, respectively. Furthermore, we have

$$\Gamma = \bigcup_{g\in\mathcal{G}} g(\Gamma_{\mathbf{t}}) = \bigcup_{\mathbf{i}\in\mathcal{I}_1} \phi_{\mathbf{i}}(\Gamma_{\mathbf{t}}) \cup \bigcup_{\mathbf{i}\in\mathcal{I}_2} \phi_{\mathbf{i}}(1-\Gamma_{\mathbf{t}}).$$

By Lemma 4.1, $\mathbf{t} = (t_0, t_1, \ldots, t_m)$ is an admissible translation vector.

Next we prove Theorem 1.1.

Lemma 4.3 *Suppose that $0 < \beta < 1/(N+1)$, and let $\mathbf{t} = (t_0, t_1, \cdots, t_m) \in T^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$. If $\beta^q\mathbf{t} \in T^{m+1}$ for some $q \in \mathbb{Z}_+$, then $\mathbf{t}$ is an admissible translation vector if and only if $\beta^q\mathbf{t}$ is an admissible translation vector.*

Proof Write $\mathbf{t}' = \beta^q\mathbf{t}$, that is, $\mathbf{t} = \beta^{-q}\mathbf{t}'$. Note that $\mathbf{t}, \mathbf{t}' \in T^{m+1}$. We have $\tau_{\mathbf{t}} = \tau_{\mathbf{t}'} + q$. It's easy to check that $\Omega_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \Omega_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^q$, and $\hat{\Omega}_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \hat{\Omega}_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^q$. Then we also have $\mathcal{A}_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \mathcal{A}_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^q$ and $\hat{\mathcal{A}}_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \hat{\mathcal{A}}_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^q$. It follows that

$$\mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \mathcal{W}_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^q.$$

Therefore, for $\ell \geq \tau_{\mathbf{t}}$ the following three equalities are equivalent:

- $\displaystyle\bigcup_{n=\tau_{\mathbf{t}}}^{\ell} \{0, 1, \ldots, N\}^{n-\tau_{\mathbf{t}}} \times \mathcal{W}_{\mathbf{t}}^{\tau_{\mathbf{t}}} \times \{0, 1, \ldots, N\}^{\ell-n} = \{0, 1, \ldots, N\}^{\ell};$
- $\displaystyle\bigcup_{n'=\tau_{\mathbf{t}'}}^{\ell-q} \{0, 1, \ldots, N\}^{n'-\tau_{\mathbf{t}'}} \times \mathcal{W}_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^{\ell-n'} = \{0, 1, \ldots, N\}^{\ell};$

- $\bigcup_{n'=\tau_{\mathbf{t}'}}^{\ell-q} \{0, 1, \ldots, N\}^{n'-\tau_{\mathbf{t}'}} \times \mathcal{W}_{\mathbf{t}'}^{\tau_{\mathbf{t}'}} \times \{0, 1, \ldots, N\}^{\ell-q-n'} = \{0, 1, \ldots, N\}^{\ell-q}$.

We conclude that $\mathbf{t}$ is an admissible translation vector if and only if $\mathbf{t}' = \beta^q \mathbf{t}$ is an admissible translation vector.

Proof of Theorem 1.1 By Lemma 4.2, it suffices to show that for $m \in \mathbb{Z}_+$ there exist infinitely many admissible translation vectors $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$. By Lemma 4.3, if $\mathbf{t} \in \mathbb{R}^{m+1}$ is an admissible translation vector, then $\beta^{-k}\mathbf{t} \in \mathbb{R}^{m+1}$ is also an admissible translation vector for all $k \geq 1$. Thus, we only need to show that for $m \in \mathbb{Z}_+$ there exists an admissible translation vectors $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$ with $0 = t_0 < t_1 < \cdots < t_m$.

For $m = 1$, take $\mathbf{t} = (0, \frac{1-\beta}{N\beta}) \in T^2$ and then we have $\Gamma_{\mathbf{t}} = \Gamma \cup (\Gamma + \frac{1-\beta}{N\beta})$. It's easy to calculate that $\tau_{\mathbf{t}} = 1$ and $\Omega_{\mathbf{t}}^{\tau_{\mathbf{t}}} = \{0, 1, \ldots, N-1\}$. Then we have

$$\bigcup_{i \in \Omega_{\mathbf{t}}^{\tau_{\mathbf{t}}}} \phi_i(\Gamma_{\mathbf{t}}) = \bigcup_{i \in \Omega_{\mathbf{t}}^{\tau_{\mathbf{t}}}} \left(\beta\Gamma + i\frac{1-\beta}{N}\right) \cup \left(\beta\Gamma + (i+1)\frac{1-\beta}{N}\right) = \bigcup_{i=0}^{N} \phi_i(\Gamma) = \Gamma.$$

By Lemma 4.1, $\mathbf{t} \in T^2$ is an admissible translation vector.

For $m \geq 2$, there exists $\ell \in \mathbb{Z}_+$ such that $2^\ell \leq m < 2^{\ell+1}$. Note that $2^\ell + 1 \leq m+1 \leq 2^{\ell+1}$. We can find a subset $S \subset \{0, 1\}^{\ell+1}$ with $\#S = m+1$ such that $0^{\ell+1}, 1^{\ell+1} \in S$, and for any $i_1 i_2 \cdots i_{\ell+1} \in \{0, 1\}^{\ell+1}$ we have either $i_1 i_2 \cdots i_{\ell+1} \in S$ or $(1-i_1)(1-i_2)\cdots(1-i_{\ell+1}) \in S$. Let $\mathbf{t} = (t_0, t_1, \ldots, t_m)$ with $0 = t_0 < t_1 < \cdots < t_m$ taking the values

$$\left\{\frac{1-\beta}{N}\sum_{k=1}^{\ell+1} i_k \beta^{-k} : i_1 i_2 \cdots i_{\ell+1} \in S\right\}.$$

Clearly, we have $\mathbf{t} \in T^{m+1}$. Note that $1^{\ell+1} \in S$. We have $\tau_{\mathbf{t}} = \ell + 1$ and $s_k = 1$ for all $1 \leq k \leq \ell + 1$. It follows that $\Omega_{\mathbf{t}}^{\ell+1} = \{0, 1, \ldots, N-1\}^{\ell+1}$ and $\hat{\Omega}_{\mathbf{t}}^{\ell+1} = \{1, 2, \ldots, N\}^{\ell+1}$. Thus, we have

$$\mathcal{A}_{\mathbf{t}}^{\ell+1} = \left\{(j_1 + i_{\ell+1})\cdots(j_\ell + i_2)(j_{\ell+1} + i_1) : j_1 j_2 \cdots j_{\ell+1} \in \Omega_{\mathbf{t}}^{\ell+1}, i_1 i_2 \cdots i_{\ell+1} \in S\right\},$$

and

$$\hat{\mathcal{A}}_{\mathbf{t}}^{\ell+1} = \left\{(j_1 - i_{\ell+1})\cdots(j_\ell - i_2)(j_{\ell+1} - i_1) : j_1 j_2 \cdots j_{\ell+1} \in \hat{\Omega}_{\mathbf{t}}^{\ell+1}, i_1 i_2 \cdots i_{\ell+1} \in S\right\}.$$

Take $r_1 r_2 \cdots r_{\ell+1} \in \{0, 1, \ldots, N\}^{\ell+1}$. Then there exist $j_1 j_2 \cdots j_{\ell+1} \in \{0, 1, \ldots, N-1\}^{\ell+1}$ and $i_1 i_2 \cdots i_{\ell+1} \in \{0, 1\}^{\ell+1}$ such that

$$r_1 r_2 \cdots r_{\ell+1} = (j_1 + i_{\ell+1})\cdots(j_\ell + i_2)(j_{\ell+1} + i_1).$$

If $i_1 i_2 \cdots i_{\ell+1} \in S$, then we have $r_1 r_2 \cdots r_{\ell+1} \in \mathcal{A}_{\mathbf{t}}^{\ell+1}$; if $(1-i_1)(1-i_2)\cdots(1-i_{\ell+1}) \in S$, then $i_1 i_2 \cdots i_{\ell+1} = (1-i_1')(1-i_2')\cdots(1-i_{\ell+1}')$ for some $i_1' i_2' \cdots i_{\ell+1}' \in S$, and we have

$$r_1 r_2 \cdots r_{\ell+1} = (j_1 + 1 - i_{\ell+1}')\cdots(j_\ell + 1 - i_2')(j_{\ell+1} + 1 - i_1') \in \hat{\mathcal{A}}_{\mathbf{t}}^{\ell+1}.$$

Thus we conclude that $\mathcal{W}_{\mathbf{t}}^{\ell+1} = \mathcal{A}_{\mathbf{t}}^{\ell+1} \cup \hat{\mathcal{A}}_{\mathbf{t}}^{\ell+1} = \{0, 1, \ldots, N\}^{\ell+1}$. This implies that $G_{\mathbf{t}}$ is an empty graph, which has no cycles. By Proposition 1.3, $\mathbf{t} \in \mathbb{R}^{m+1}$ is an admissible translation vector. We complete the proof.

Finally we prove Corollary 1.6.

Proof of Corollary 1.6 Let $\mathbf{t} = (t_0, t_1)$ with $0 = t_0 < t_1$. Note that $\hat{\mathbf{t}} = \mathbf{t}$. By Theorem 1.5, $\Gamma_{\mathbf{t}} = \Gamma \cup (\Gamma + t_1)$ is a self-similar set if and only if $\mathbf{t}$ is an admissible translation vector. We assume $t_1 \in T$ and write

$$t_1 = \frac{1-\beta}{N} \sum_{k=1}^{\tau} s_k \beta^{-k} \text{ with } s_k \in \{0, 1, \dots, N\},$$

where $\tau \in \mathbb{Z}_+$ and $s_\tau \neq 0$. By calculation, we have $\tau_{\mathbf{t}} = \tau$, and

$$\mathcal{W}_{\mathbf{t}}^{\tau} = \left\{i_1 i_2 \cdots i_\tau \in \{0, 1, \dots, N\}^\tau : i_{\tau+1-k} \leq N - s_k \text{ for } 1 \leq k \leq \tau\right\}$$
$$\cup \left\{i_1 i_2 \cdots i_\tau \in \{0, 1, \dots, N\}^\tau : i_{\tau+1-k} \geq s_k \text{ for } 1 \leq k \leq \tau\right\}.$$

The discussion whether $\mathbf{t}$ is an admissible translation vector is split into three cases.

Case I: $\tau = 1$. Then we have $\mathcal{W}_{\mathbf{t}}^{\tau} = \{0, 1, \dots, N - s_1\} \cup \{s_1, s_1 + 1, \cdots, N\}$. In this case, $G_{\mathbf{t}}$ has no cycles if and only if $\mathcal{W}_{\mathbf{t}}^{\tau} = \{0, 1, \dots, N\}$. Thus, by Proposition 1.3, $\mathbf{t}$ is an admissible translation vector if and only if $s_1 \in \left\{1, \dots, \lfloor \frac{N+1}{2} \rfloor\right\}$.

Case II: $\tau \geq 2$, and $s_k = 0$ for all $1 \leq k \leq \tau - 1$. Note that $\beta^{\tau-1} t_1 = \frac{1-\beta}{N} s_\tau \beta^{-1} \in T$. By Lemma 4.3, $\mathbf{t}$ is an admissible translation vector if and only if $\beta^{\tau-1}\mathbf{t}$ is. It follows from **Case I** that $\mathbf{t}$ is an admissible translation vector if and only if $s_\tau \in \left\{1, \dots, \lfloor \frac{N+1}{2} \rfloor\right\}$.

Case III: $\tau \geq 2$, and there exists $1 \leq k \leq \tau - 1$ such that $s_k \neq 0$. Let $q = \min\{1 \leq k \leq \tau - 1 : s_k \neq 0\}$. Note that

$$\beta^{q-1} t_1 = \frac{1-\beta}{N} \sum_{k=q}^{\tau} s_k \beta^{q-k-1} \in T.$$

By Lemma 4.3, $\mathbf{t}$ is an admissible translation vector if and only if $\beta^{q-1}\mathbf{t}$ is. Thus we can assume that $s_1 \neq 0$.

For $i_1 i_2 \cdots i_\tau \in \mathcal{W}_{\mathbf{t}}^{\tau}$, we have either $i_1 \leq N - s_\tau, i_\tau \leq N - s_1$, or $i_1 \geq s_\tau, i_\tau \geq s_1$. It follows that either $i_1, i_\tau \leq N-1$, or $i_1, i_\tau \geq 1$. Thus, if $(i_1, i_\tau) = (0, N)$, or $(i_1, i_\tau) = (N, 0)$, then we have $i_1 i_2 \cdots i_\tau \notin \mathcal{W}_{\mathbf{t}}^{\tau}$. Note that $V_{\mathbf{t}} = \{0, 1, \dots, N\}^\tau \backslash \mathcal{W}_{\mathbf{t}}^{\tau}$ in the directed graph $G_{\mathbf{t}} = (V_{\mathbf{t}}, E_{\mathbf{t}})$. The following cycle

$$0^{\tau-1}N \to 0^{\tau-2}N^2 \to \cdots \to 0N^{\tau-1} \to N^{\tau-1}0 \to N^{\tau-2}0^2 \to \cdots \to N0^{\tau-1} \to 0^{\tau-1}N$$

is in $G_{\mathbf{t}}$. By Proposition 1.3, $\mathbf{t}$ is not an admissible translation vector.

Therefore, we conclude that $\Gamma \cup (\Gamma + t)$ with $t > 0$ is a self-similar set if and only if

$$t = \frac{j(1-\beta)}{N} \beta^{-k},$$

where $j \in \left\{1, 2, \cdots, \lfloor \frac{N+1}{2} \rfloor\right\}$ and $k \in \mathbb{Z}_+$.

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Data availability All data included in this study are available upon request by contact with the corresponding author.

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