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# Self-similar structure on intersection of homogeneous symmetric Cantor sets

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For a homogeneous symmetric Cantor set $C$, we consider all real numbers $t$ such that the intersection $C\cap(C+t)$ is a self-similar set and investigate the form of the corresponding iterated function systems.

## 1 Introduction

The classical *middle-third Cantor set*, denoted by $C$, may be simply generated by the IFS $\{f_1(x) = x/3, f_2(x) = x/3 + 2/3\}$, i.e., $C$ is the unique nonempty compact set in $\mathbf{R}$ satisfying

$$C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right),$$

where here, and throughout this paper, $A + c := \{x + c : x \in A\}$ and $cA := \{cx : x \in A\}$ for $c \in \mathbf{R}$, $A \subseteq \mathbf{R}$. In addition, $A - B := \{x - y : x \in A, y \in B\}$ and $A + B := \{x + y : x \in A, y \in B\}$ for $A, B \subset \mathbf{R}$. It is easy to check that

$$C \cap (C + t) \neq \emptyset \quad \text{if and only if} \quad t \in C - C = \{x - y : x, y \in C\} = [-1, 1].$$

However, $C \cap (C + t)$ presents complicated structure. In fact, in the past two decades, intersection of Cantor sets (not limited to the middle-third Cantor set) has been the subject of several studies (cf. [1], [6]–[17]). The context and motivations are numerous, but mainly come from the discipline of dynamical systems.

An algebraic description for $C \cap (C + t)$ is as follows. Note that each $t \in [-1, 1]$ can be represented as

$$t = \sum_{k=1}^{\infty} \frac{2t_k}{3^k} \quad \text{for some} \quad (t_k)_{k=1}^{\infty} \in \{-1, 0, 1\}^{\mathbf{N}},$$

where the sequence $(t_k)_{k=1}^{\infty}$, called the code of $t$, is uniquely determined by $t$ except a countable number of points in $[-1, 1]$, each of which has just two codes of the form $t_1 \dots t_{k-1} t_k 1 \dots 1 \dots$ and

$$t_1 \dots t_{k-1}(t_k + 1)(-1) \dots (-1) \dots \ \text{with}\ t_k = -1 \text{ or } 0.$$

When $t \in [-1, 1]$ has a unique code $(t_k)_{k=1}^{\infty} \in \{-1, 0, 1\}^{\mathbf{N}}$

$$C \cap (C + t) = \left\{ \sum_{k=1}^{\infty} \frac{2x_k}{3^k} : x_k = 1 \text{ if } t_k = 1;\ x_k = 0 \text{ if } t_k = -1;\ x_k = 0 \text{ or } 1 \text{ if } t_k = 0 \right\}.$$

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When $t \in [-1,1]$ has two codes, $C \cap (C+t)$ is a finite set with cardinality bigger than one. This representation for $C \cap (C+t)$ enable one to determine their Hausdorff and packing dimensions (cf. [12]).

A natural question is when and how the sets $\mathbf{C} \cap (\mathbf{C}+t)$ become self-similar sets. It is completely answered by Deng, He, and Wen in their recent paper [2]. Let $t \in [-1,1]$ has a unique code $(t_k)_{k=1}^{\infty} \in \{-1,0,1\}^{\mathbf{N}}$. It induces a sequence $(1-|t_k|)_{k=1}^{\infty} \in \{0,1\}^{\mathbf{N}}$. For a finite sequence $\mathbf{i} \in \{0,1\}^k$ ($k \in \mathbf{N}$), $\bar{\mathbf{i}} := \mathbf{i}\mathbf{i}\cdots \in \{0,1\}^{\mathbf{N}}$ denotes the infinite repeating of $\mathbf{i}$. To investigate the geometric structure of $C \cap (C+t)$ Deng, He, and Wen define a notion of "strong periodic" for a sequence from $\{0,1\}^{\mathbf{N}}$. A sequence $\mathbf{i} \in \{0,1\}^{\mathbf{N}}$ is said to be *strong p-periodic* (or simply, *strong periodic*) if there exist two finite sequences $\mathbf{u}, \mathbf{v} \in \{0,1\}^p$ for some $p \in \mathbf{N}$ such that $\mathbf{i} = \mathbf{u}\bar{\mathbf{v}}$ and $\mathbf{u} \preceq \mathbf{v}$, where $\mathbf{u} \preceq \mathbf{v}$ means $u_n \le v_n$, $1 \le n \le p$ for $\mathbf{u} = u_1 \dots u_p$ and $\mathbf{v} = v_1 \dots v_p$. Obviously, if $t \in [-1,1]$ has two codes, then any of its two induced codes is not strong periodic and $C \cap (C+t)$ is not a self-similar set. In addition, a sequence $\mathbf{i} = (i_k)_{k=1}^{\infty} \in \{0,1\}^{\mathbf{N}}$ is said to be eventually periodic if there exist two integers $d, m$ such that $i_{k+d} = i_k$ for all $k \ge m$, and the integer $d$ is called a period of $\mathbf{i}$. Thus a strong periodic sequence is eventually periodic. The following theorem gives the sufficient and necessary conditions for the set $C \cap (C+t)$ to be a self-similar set.

**Theorem A** ([2, Theorem 1.1]). *Let $(t_k)_{k=1}^{\infty} \in \{-1,0,1\}^{\mathbf{N}}$ be a code of $t \in [-1,1]$. Then the set $C \cap (C+t)$ is a self-similar set if and only if $(1-|t_k|)_{k=1}^{\infty}$ is strong periodic. Furthermore, if $C \cap (C+t)$ is a self-similar set with more than one point, then there exists an IFS which satisfies the strong separation condition.*

For a given self-similar set, an interesting (also very complicated) question is that what are all of its generating iterated function systems? One can refer to [4] for a detailed discussion about this topic. Note that the set $C \cap (C+t)$ is centrally symmetric, i.e., $C \cap (C+t) = c - C \cap (C+t)$ for some $c \in \mathbf{R}$. Thus, when $C \cap (C+t)$ is a self-similar set one can focus on its those generating iterated function systems for which all similarity ratios are positive (cf. Lemma 2.9(III)).

**Theorem B** ([2, Theorem 1.2]). *Let $(t_k)_{k=1}^{\infty} \in \{-1,0,1\}^{\mathbf{N}}$ be a code of $t \in [-1,1]$. If $(1-|t_k|)_{k=1}^{\infty}$ is strong p-periodic, then any IFS $\{f_i(x) = r_i x + b_i,\ \text{with } r_i > 0\}_{i=1}^{N}$ for $C \cap (C+t)$ satisfies that $r_i = 3^{-q_i}$ for some positive integer $q_i$ and*

$$b_i = \sum_{k=1}^{p+q_i} \frac{2b_{ik}}{3^k},\ i = 1, \dots, N,$$

*where all $b_{ik} = 0$ or $1$. Moreover, each $q_i$ is a period of $(1-|t_k|)_{k=1}^{\infty}$.*

In the present paper, we show that the Theorems A and B still hold for general homogeneous symmetric Cantor sets on $\mathbf{R}$ under some conditions. Let $\tau \ge 2$ be a positive integer and let $\beta \in (0,1)$. For each $k \in \mathbf{Z}$ (the set of integers) let

$$\phi_k(x) = \beta x + k(1-\beta)/(\tau-1),\ \ x \in \mathbf{R}. \tag{1.1}$$

For a finite set $\Omega \subset \mathbf{Z}$, by $C_{\beta,\tau,\Omega}$ we denote the self-similar set generated by the IFS $\{\phi_k(x) : k \in \Omega\}$, i.e., the set $C_{\beta,\tau,\Omega}$ is the unique nonempty compact satisfying

$$C_{\beta,\tau,\Omega} = \bigcup_{k \in \Omega} \phi_k(C_{\beta,\tau,\Omega}). \tag{1.2}$$

Thus

$$C_{\beta,\tau,\Omega} = \left\{ \sum_{k=1}^{\infty} j_k \beta^{k-1}(1-\beta)/(\tau-1) : (j_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}} \right\}.$$

The sequence $(j_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}}$ is call a *$\Omega$-code* of $x$ if $x = \sum_{k=1}^{\infty} j_k \beta^{k-1}(1-\beta)/(\tau-1)$. Obviously, the set $C_{1/3,2,\{0,1\}}$ is just the middle-third Cantor set.

**Definition 1.1** Let $\beta \in (0, \tau^{-1})$. The set $C_{\beta,\tau,\{0,1,\dots,\tau-1\}}$ is called a *homogeneous symmetric Cantor set*.

Thus a homogeneous symmetric Cantor set can be obtained as a limit via the following construction: place $\tau$ intervals of length $\beta$ equally spaced inside the unit interval (so that the leftmost point of the leftmost interval is 0, and the rightmost point of the rightmost interval is 1) and continue this construction inductively on each of the intervals. It is known that (cf. [13, 14])

$$C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t) \neq \emptyset \quad \text{if and only if} \quad t \in C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} - C_{\beta,\tau,\{0,1,\ldots,\tau-1\}},$$

and

$$C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} - C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} = C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}.$$

When $\beta \in (0, 1/(2\tau-1))$, the IFS $\{\phi_k : k \in \{0, \pm1, \ldots, \pm(\tau-1)\}\}$ satisfies the strong separation condition so that each $t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}$ has a unique $\{0, \pm1, \ldots, \pm(\tau-1)\}$-code.

When $\beta = 1/(2\tau-1)$, the IFS $\{\phi_k : k \in \{0, \pm1, \ldots, \pm(\tau-1)\}\}$ satisfies the open set condition so that each $t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}} = [-1, 1]$ has a unique $\{0, \pm1, \ldots, \pm(\tau-1)\}$-code except a countable number of points in $[-1, 1]$, each of which has just two codes of the form $t_1 \ldots t_{k-1} t_k \overline{(\tau-1)}$ and $t_1 \ldots t_{k-1}(t_k+1)\overline{(1-\tau)}$ with $t_k \neq \tau-1$. When $t \in [-1, 1]$ has two $\{0, \pm1, \ldots, \pm(\tau-1)\}$-codes the set

$$C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t)$$

is a finite set containing more than one point and so it is not a self-similar set.

Therefore, for $\beta \in (0, 1/(2\tau-1)]$ and $t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}$ if $t$ has a unique $\{0, \pm1, \ldots, \pm(\tau-1)\}$-code $(t_k)_{k=1}^\infty$, then

$$\begin{aligned} &C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t) \\ &\quad = \left\{ \sum_{k=1}^\infty x_k \beta^{k-1}(1-\beta)/(\tau-1) : x_k \in \{0, 1, \ldots, \tau-1\} \cap (\{0, 1, \ldots, \tau-1\} + t_k) \right\}. \end{aligned}$$

Like in [2], for a finite sequence $\mathbf{i} \in \{0, 1, \ldots, \tau-1\}^k$ ($k \in \mathbf{N}$), $\bar{\mathbf{i}} := \mathbf{i}\mathbf{i}\cdots \in \{0, 1, \ldots, \tau-1\}^{\mathbf{N}}$ denote the infinite repeating of $\mathbf{i}$. A sequence $\mathbf{i} \in \{0, 1, \ldots, \tau-1\}^{\mathbf{N}}$ is said to be *strong p-periodic* (or simply, *strong periodic*) if there exist two finite sequences $\mathbf{u}, \mathbf{v} \in \{0, 1, \ldots, \tau-1\}^p$ for some $p \in \mathbf{N}$ such that $\mathbf{i} = \mathbf{u}\bar{\mathbf{v}}$ and $\mathbf{u} \preceq \mathbf{v}$, where $\mathbf{u} \preceq \mathbf{v}$ means $u_n \leq v_n$, $1 \leq n \leq p$ for $\mathbf{u} = u_1 \ldots u_p$ and $\mathbf{v} = v_1 \ldots v_p$. In addition, a sequence $\mathbf{i} = (i_k)_{k=1}^\infty \in \{0, 1, \ldots, \tau-1\}^{\mathbf{N}}$ is said to be eventually periodic if there exist two integers $d, m$ such that $i_{k+d} = i_k$ for all $k \geq m$, and the integer $d$ is called a period of $\mathbf{i}$. Like in [2, Theorem 1.1], we first show a theorem which positively answers when $C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t)$ is a self-similar set.

**Theorem 1.2** *Let* $0 < \beta \leq 1/(2\tau-1)$ *and let* $t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}$. *Then* $C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t)$ *is a self-similar set if and only if t has a unique* $\{0, \pm1, \ldots, \pm(\tau-1)\}$*-code* $(t_k)_{k=1}^\infty \in \{0, \pm1, \ldots, \pm(\tau-1)\}^{\mathbf{N}}$ *and* $(\tau-1-|t_k|)_{k=1}^\infty \in \{0, 1, \ldots, \tau-1\}^{\mathbf{N}}$ *is strong periodic.*

To investigate the generating iterated function systems for $C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t)$ we consider its some translation instead. Let $\eta_k = \min\{x : x \in \{0, 1, \ldots, \tau-1\} \cap (\{0, 1, \ldots, \tau-1\} + t_k)\} = \max\{0, t_k\}$ where $(t_k)_{k=1}^\infty$ is the unique $\{0, \pm1, \ldots, \pm(\tau-1)\}$-code of $t$, and let

$$\begin{aligned} E &:= C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t) - \sum_{k=1}^\infty \eta_k \beta^{k-1}(1-\beta)/(\tau-1) \\ &= \left\{ \sum_{k=1}^\infty x_k \beta^{k-1}(1-\beta)/(\tau-1) : (x_k)_{k=1}^\infty \in \{0, \ldots, \tau-1\}^{\mathbf{N}}, x_k \leq \tau-1-|t_k| \right\}. \end{aligned} \tag{1.3}$$

When $(\tau-1-|t_k|)_{k=1}^\infty = \bar{0}$, $E = \{0\}$ is a self similar set. In this case each generating iterated function system of $E$ is of the form $\{f_i(x) = r_i x : 0 < |r_i| < 1\}_{i=1}^N$.

Like in [2, Theorem 1.2], the form of generating iterated function systems for $E$ is described as follows.

**Theorem 1.3** *Let* $0 < \beta \leq 1/(2\tau-1)$. *Let* $t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}$ *have a unique* $\{0, \pm1, \ldots, \pm(\tau-1)\}$*-code* $(t_k)_{k=1}^\infty \in \{0, \pm1, \ldots, \pm(\tau-1)\}^{\mathbf{N}}$. *Let E be defined as in* (1.3). *If the sequence* $(\tau-1-|t_k|)_{k=1}^\infty \in \{0, 1, \ldots, \tau-1\}^{\mathbf{N}}$ *is strong p-periodic and* $(\tau-1-|t_k|)_{k=1}^\infty \neq \bar{0}$, *then any IFS* $\{f_i(x) = r_i x + b_i\}_{i=1}^N$ *with* $r_i \in (0, 1)$ *for E satisfies* $r_i = \beta^{q_i}$ *for some positive integers* $q_i$ *and*

$$b_i = \sum_{k=1}^{p+q_i} b_{ik} \beta^{k-1}(1-\beta)/(\tau-1), \quad i = 1, 2, \ldots, N$$

*where* $b_{ik} \in \{0, \ldots, \tau-1-|t_k|\}$. *Moreover, each* $q_i$ *is a period of the sequence* $(\tau-1-|t_k|)_{k=1}^\infty$.

We remark that according to Lemma 2.9 (III) one only need to investigate those generating iterated function systems for $E$ for which the contractive ratios are all positive.

The main idea for the proofs of Theorems 1.2 and 1.3 comes from [2]. When $\tau = 2$ (correspondingly $\beta \in (0, 1/3]$), $C_{\beta,2,\{0,1\}}$ is called the middle-$(1 - 2\beta)$ Cantor set. For this case, the proofs of Theorems 1.2 and 1.3 can follow the approaches in [2]. However, when $\tau > 2$ substantial effort is required. For example, suppose $E$ (equivalently, $C_{\beta,\tau,\{0,1,...,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,...,\tau-1\}} + t)$) is a self-similar set. Let $\{f_i(x) = r_i x + b_i\}_{i=1}^N$ with $r_i \in (0, 1)$ be a generating IFS for $E$. One can show

$$r_1 = \sum_{i=1}^{\infty} \alpha_i \beta^i \quad \text{where} \quad \alpha_i \in \left\{0, \dots, \max_k(\tau - 1 - |t_k|)\right\}.$$

Thus for each $s \in \mathbf{N}$ (we have assumed $b_1 = 0$ and this is reasonable)

$$f_1\left(\frac{(\tau - 1 - |t_s|)\beta^{s-1}(1 - \beta)}{\tau - 1}\right) = \sum_{i=1}^{\infty} \frac{\alpha_i(\tau - 1 - |t_s|)\beta^{s+i-1}(1 - \beta)}{\tau - 1} \in E.$$

If $\tau = 2$, then all $\alpha_i \in \{0, 1\}$ so that the right side in the above equality gives a standard representation for a point of $E$ by Lemma 2.9 (IV). Let $m \in \mathbf{N}$ be such that $\alpha_m = 1$. This leads to $\tau - 1 - |t_{m+s}| \geq \tau - 1 - |t_s|$ for all $s \in \mathbf{N}$ immediately and so $(\tau - 1 - |t_k|)_{k=1}^{\infty}$ is strong periodic by Lemma 3.1. If $\tau > 2$ the above argument does not work any more.

This paper is arranged as follows. In the next section, we define some notations and give some lemmas to describe the geometric structure and properties of $C_{\beta,\tau,\Omega}$, which are used in the proofs of the main theorems. Sections 3 and 4 are devoted to the proofs of Theorems 1.2 and 1.3, respectively.

## 2 Preliminaries

Let $\{f_i(x) = r_i x + b_i, i = 1, \dots, N\}$ be a family of functions on $\mathbf{R}$ with $0 < |r_i| < 1$. It is well-known that there exists a unique nonempty compact set $T \subseteq \mathbf{R}$ such that $T = \bigcup_{i=1}^N f_i(T)$ (cf. [3, 5]). In this case, $T$ is called a *self-similar set* generated by the iterated function system (IFS) $\{f_i(x)\}_{i=1}^N$. The IFS $\{f_i(x)\}_{i=1}^N$ is said to satisfy the *strong separation condition* (SSC) (*open set condition* (OSC)) if there exits a nonempty compact set (nonempty open set, or equivalently nonempty bounded open set) $O$ such that $\bigcup_{i=1}^N f_i(O) \subseteq O$ with disjoint union on the left-hand side. Clearly, SSC implies OSC, but the converse is not true.

For a finite set $\Omega \subset \mathbf{Z}$, let $C_{\beta,\tau,\Omega}$ be defined as in (1.2). Clearly, $C_{\beta,\tau,\Omega_1} \subseteq C_{\beta,\tau,\Omega_2}$ if $\Omega_1 \subseteq \Omega_2$. In order to get more detailed information about $C_{\beta,\tau,\Omega}$ we need some notations. Let

$$\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k \quad \text{where} \quad \Omega^k = \{j_1 \dots j_k : j_n \in \Omega \ \text{ for all } \ n = 1, \dots, k\},$$

and

$$\Omega^{\mathbf{N}} = \{j_1 j_2 \cdots : j_n \in \Omega \quad \text{for all} \quad n \in \mathbf{N}\}.$$

Thus, $\Omega^*$ is the family of all finite sequences $j_1 \dots j_n$ with entries $j_i$ from $\Omega$ and $\Omega^{\mathbf{N}}$ denotes the family of all infinite sequences $j_1 j_2 \dots$ with entries $j_i$ from $\Omega$. We denote by $\mathbf{ij}$ the concatenation of $\mathbf{i} \in \Omega^*$ and $\mathbf{j} \in \Omega^* \cup \Omega^{\mathbf{N}}$, e.g., $\mathbf{ij} = i_1 \dots i_k j_1 \dots j_n$ for $\mathbf{i} = i_1 \dots i_k$ and $\mathbf{j} = j_1 \dots j_n$. For any $\mathbf{i} \in \Omega^*$, $|\mathbf{i}|$ denotes its length and $\bar{\mathbf{i}} := \mathbf{ii} \cdots \in \Omega^{\mathbf{N}}$, the infinite repeating of $\mathbf{i}$. A sequence $\mathbf{i} \in \Omega^{\mathbf{N}}$ is said to be *eventually periodic* if there exist $\mathbf{u}, \mathbf{v} \in \Omega^*$ such that $\mathbf{i} = \mathbf{u}\bar{\mathbf{v}}$, and $|\mathbf{v}|$ is called a *period* of $\mathbf{i}$. A sequence $\mathbf{i} \in \Omega^{\mathbf{N}}$ is said to be *strong $p$-periodic* (or simply, *strong periodic*) if there exist two words $\mathbf{u}, \mathbf{v} \in \Omega^p$ for some $p \in \mathbf{N}$ such that $\mathbf{i} = \mathbf{u}\bar{\mathbf{v}}$ and $\mathbf{u} \preceq \mathbf{v}$, where $\mathbf{u} \preceq \mathbf{v}$ means $u_n \leq v_n$, $1 \leq n \leq p$, for $\mathbf{u} = u_1 \dots u_p$ and $\mathbf{v} = v_1 \dots v_p$. Therefore, a strong $p$-periodic infinite sequence is eventually periodic with period $p$. For $\mathbf{j} = (j_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}}$ and a positive integer $n$, let $\mathbf{j}|n = j_1 \dots j_n$ denote the truncation of $\mathbf{j}$ to the $n$-th place. For $\mathbf{i}, \mathbf{j} \in \Omega^{\mathbf{N}}$ we say that $\mathbf{i} \preceq \mathbf{j}$ if $\mathbf{i}|n \preceq \mathbf{j}|n$ for all $n \in \mathbf{N}$.

Let $\phi_k, k \in \mathbf{Z}$ be defined as in (1.1). Denote $\phi_{\mathbf{j}} = \phi_{j_1} \circ \cdots \circ \phi_{j_n}$ for $\mathbf{j} = j_1 \dots j_n \in \Omega^*$. A so-called *coding map* $\Pi : \Omega^{\mathbf{N}} \to C_{\beta,\tau,\Omega}$ is defined by

$$\Pi(\mathbf{j}) = \lim_{n\to\infty} \phi_{\mathbf{j}|n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} j_k \beta^{k-1}(1 - \beta)/(\tau - 1) = \sum_{k=1}^{\infty} j_k \beta^{k-1}(1 - \beta)/(\tau - 1),$$

for $\mathbf{j} = (j_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}}$. The map $\Pi$ is surjective, i.e., $C_{\beta,\tau,\Omega} = \Pi(\Omega^{\mathbf{N}})$ (cf. [3, 15]). Therefore,

$$C_{\beta,\tau,\Omega} = \left\{ \sum_{k=1}^{\infty} j_k \beta^{k-1}(1-\beta)/(\tau-1) : (j_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}} \right\}. \tag{2.1}$$

**Definition 2.1** Let $\mathbf{j} = (j_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}}$ and let $x \in C_{\beta,\tau,\Omega}$. If $\Pi(\mathbf{j}) = x$ then $\mathbf{j}$ is called a $\Omega$-*code* of $x$ and $\sum_{k=1}^{\infty} j_k \beta^{k-1}(1-\beta)/(\tau-1)$ is called as a *standard* $\Omega$-*representation* of $x$.

Note that $x$ may have multiple $\Omega$-codes and standard $\Omega$-representations. We add a prefix $\Omega$ prior to the "code" (or "representation") to avoid confusion when multiple $C_{\beta,\tau,\Omega}$ are considered simultaneously. For example, a code of $x \in C_{\beta,\tau,\{0,1\}} \subseteq C_{\beta,\tau,\{0,1,2\}}$ consists of digits from $\{0,1\}$ when $x$ is considered as a member of $C_{\beta,\tau,\{0,1\}}$, from $\{0,1,2\}$ when $x$ is considered as a member of $C_{\beta,\tau,\{0,1,2\}}$. However, when no confusion occurs in the context, we often omit the prefix $\Omega$.

Let $I \subset \mathbf{R}$ be a compact set such that $\phi_k(I) \subseteq I, k \in \Omega$ (this $I$ always exists, cf. [3, 15]). Then, for a sequence $\mathbf{j} \in \Omega^{\mathbf{N}}$, $(\phi_{\mathbf{j}|n}(I))_{n\geq 1}$ is a decreasing sequence of compact sets and the diameters of $\phi_{\mathbf{j}|n}(I)$ tend to 0. This leads to an alternative way to define $\Pi$:

$$\{\Pi(\mathbf{j})\} = \bigcap_{n=1}^{\infty} \phi_{\mathbf{j}|n}(I), \ \ \mathbf{j} \in \Omega^{\mathbf{N}}.$$

Therefore, each point of $C_{\beta,\tau,\Omega}$ has a unique $\Omega$-code (standard $\Omega$-representation) if the IFS $\{\phi_k(x) : k \in \Omega\}$ satisfies SSC.

Let $C_{\beta,\tau,\{0,1,\ldots,\tau-1\}}$ be a homogeneous symmetric Cantor set (then $\beta \in (0, 1/\tau)$ by Definition 1.1). Obviously, each $x \in C_{\beta,\tau,\{0,1,\ldots,\tau-1\}}$ has a unique $\{0,1,\ldots,\tau-1\}$-code (standard $\Omega$-representation), and the closed intervals $\phi_0([0,1]), \phi_1([0,1]), \ldots, \phi_{\tau-1}([0,1])$ have the same length $\beta$ and are evenly laid in $[0,1]$ in this order, i.e., the left endpoint of $\phi_0([0,1])$ coincides with the left endpoint of $[0,1]$, the right endpoint of $\phi_{\tau-1}([0,1])$ coincides with the right endpoint of $[0,1]$ and the gaps between them have the same length $(1-\tau\beta)/(\tau-1)$. For an $\mathbf{i} = (i_n)_{n=1}^{k} \in \{0,1,\ldots,\tau-1\}^k$, the left and right endpoints of $\phi_{\mathbf{i}}([0,1])$ are, respectively

$$\phi_{\mathbf{i}}(0) = \sum_{n=1}^{k} i_n \beta^{n-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\{0,1,\ldots,\tau-1\}},$$

and

$$\phi_{\mathbf{i}}(1) = \phi_{\mathbf{i}}(0) + \beta^k = \beta^k + \sum_{n=1}^{k} i_n \beta^{n-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\{0,1,\ldots,\tau-1\}}.$$

The following observations, Lemmas 2.2–2.5 are direct and useful.

**Lemma 2.2** *Let $\Omega_1, \Omega_2 \subset \mathbf{Z}$ be finite sets. Then $C_{\beta,\tau,\Omega_1} - C_{\beta,\tau,\Omega_2} = C_{\beta,\tau,\Omega_1-\Omega_2}$ and $C_{\beta,\tau,\Omega_1} + C_{\beta,\tau,\Omega_2} = C_{\beta,\tau,\Omega_1+\Omega_2}$. In particular, $C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} - C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} = C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}$ and $C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} = C_{\beta,\tau,\{0,1,\ldots,2(\tau-1)\}}$.*

Proof. We only prove $C_{\beta,\tau,\Omega_1} - C_{\beta,\tau,\Omega_2} = C_{\beta,\tau,\Omega_1-\Omega_2}$. Let $x = \sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\Omega_1}$ with $(x_k)_{k=1}^{\infty} \in \Omega_1^{\mathbf{N}}$ and let $y = \sum_{k=1}^{\infty} y_k \beta^{k-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\Omega_2}$ with $(y_k)_{k=1}^{\infty} \in \Omega_2^{\mathbf{N}}$. Then

$$x - y = \sum_{k=1}^{\infty} (x_k - y_k)\beta^{k-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\Omega_1-\Omega_2},$$

by (2.1). Conversely, let $z = \sum_{k=1}^{\infty} z_k \beta^{k-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\Omega_1-\Omega_2}$ with $(z_k)_{k=1}^{\infty} \in (\Omega_1 - \Omega_2)^{\mathbf{N}}$. Let $z_k = x_k - y_k$ with $x_k \in \Omega_1$ and $y_k \in \Omega_2$. Then

$$z = \sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta)/(\tau-1) - \sum_{k=1}^{\infty} y_k \beta^{k-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\Omega_1} - C_{\beta,\tau,\Omega_2},$$

by (2.1). □

For $\mathbf{i} = (i_k)_{k=1}^p \in \{0, \pm 1, \dots, \pm(\tau-1)\}^p, p \in \mathbf{N}$, we denote $\mathbf{i}' := i_1 \dots i_{p-1}(i_p+1)$ when $i_p < \tau - 1$. Thus $\mathbf{i}' \in \{0, \pm 1, \dots, \pm(\tau-1)\}^p$.

**Lemma 2.3**

(I) *When* $\beta \in (1/(2\tau-1), 1)$, $C_{\beta,\tau,\{0,\pm1,\dots,\pm(\tau-1)\}} = [-1,1]$ *and an* $x \in [-1,1]$ *may have multiple* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-codes.*

(II) *When* $\beta = 1/(2\tau-1)$, $C_{\beta,\tau,\{0,\pm1,\dots,\pm(\tau-1)\}} = [-1,1]$. *The IFS* $\{\phi_k(x) : k \in \{0, \pm 1, \dots, \pm(\tau-1)\}\}$ *satisfies the OSC and each* $x \in [-1,1]$ *has a unique* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-code except a countable number of points of* $[-1,1]$*, each of which has just two* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-codes of the form* $\mathbf{i}\overline{(\tau-1)}$ *and* $\mathbf{i}'\overline{(1-\tau)}$ *where* $\mathbf{i} = (i_n)_{n=1}^k \in \{0, \pm 1, \dots, \pm(\tau-1)\}^k$ *for some* $k \in \mathbf{N}$. *One can check that only the endpoints, except* $-1$ *and* $1$*, of intervals* $\phi_{\mathbf{i}}([-1,1])$, $\mathbf{i} \in \{0, \pm 1, \dots, \pm(\tau-1)\}^*$ *have two* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-codes. In particular,* $-1$ *has a unique* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-code* $\overline{(1-\tau)}$, $1$ *has a unique* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-code* $\overline{(\tau-1)}$.

(III) *When* $\beta \in (0, 1/(2\tau-1))$, $C_{\beta,\tau,\{0,\pm1,\dots,\pm(\tau-1)\}}$ *is nowhere dense and each* $x \in C_{\beta,\tau,\{0,\pm1,\dots,\pm(\tau-1)\}}$ *has a unique* $\{0, \pm 1, \dots, \pm(\tau-1)\}$*-code.*

(IV) *When* $\beta \in (1/(2\tau-1), 1)$, $C_{\beta,\tau,\{0,1,\dots,2(\tau-1)\}} = [0,2]$ *and an* $x \in [0,2]$ *may have multiple* $\{0, 1, \dots, 2(\tau-1)\}$*-codes. But for* $\beta = 1/(2\tau-1)$*, the IFS* $\{\phi_k(x) : k \in \{0, 1, \dots, 2(\tau-1)\}\}$ *satisfies the OSC and each* $x \in [0,2]$ *has a unique* $\{0, 1, \dots, 2(\tau-1)\}$*-code except a countable number of points of* $[0,2]$*, each of which has only two* $\{0, 1, \dots, 2(\tau-1)\}$*-codes of the form* $\mathbf{i}\overline{(2\tau-2)}$ *and* $\mathbf{i}'\overline{0}$ *with* $\mathbf{i} \in \{0, 1, \dots, 2(\tau-1)\}^k$ *for some* $k \in \mathbf{N}$. *When* $\beta \in (0, 1/(2\tau-1))$, $C_{\beta,\tau,\{0,1,\dots,2(\tau-1)\}}$ *is nowhere dense and each* $x \in C_{\beta,\tau,\{0,1,\dots,2(\tau-1)\}}$ *has a unique* $\{0, 1, \dots, 2(\tau-1)\}$*-code.*

**Lemma 2.4** *If each point of* $C_{\beta,\tau,\Omega}$ *has a unique* $\Omega$*-code and* $\Omega_1 \subseteq \Omega$*, then each point of* $C_{\beta,\tau,\Omega_1}$ *has a unique* $\Omega_1$*-code which is identical to its* $\Omega$*-code.*

Proof. Let $(x_k)_{k=1}^\infty \in \Omega_1^{\mathbf{N}}$ be a $\Omega_1$-code of $x \in C_{\beta,\tau,\Omega_1}$. Then $(x_k)_{k=1}^\infty \in \Omega_1^{\mathbf{N}}$ is also a $\Omega$-code of $x$. The desired result is then obtained by the uniqueness of $\Omega$-code. □

The following lemma is important in the proofs of the main theorems. It shows that each point

$$x \in C_{\beta,\tau,\{0,\dots,m\}} \subset C_{\beta,\tau,\{-m,\dots,2m\}}$$

has identical $\{0, \dots, m\}$ and $\{-m, \dots, 2m\}$-codes.

**Lemma 2.5** *Let* $0 < \beta < 1/(2\tau-1)$*. Let* $m \le \tau - 1$ *be a positive integer. Suppose* $x \in C_{\beta,\tau,\{-m,\dots,2m\}}$ *with a code* $(x_k)_{k=1}^\infty \in \{-m, \dots, 2m\}^{\mathbf{N}}$*. Then* $x \notin C_{\beta,\tau,\{0,\dots,m\}}$ *if there exists some* $x_k \notin \{0, \dots, m\}$.

Proof. At first, let us notice the fact that each point in $C_{\beta,\tau,\{-m,\dots,m\}}$ has a unique $\{-m, \dots, m\}$-code by Lemmas 2.3 (III) and 2.4. Suppose $x \in C_{\beta,\tau,\{0,\dots,m\}}$. By Lemmas 2.3 (III) and 2.4, $x$ has a unique $\{0, \dots, m\}$-code, say $(y_k)_{k=1}^\infty \in \{0, \dots, m\}^{\mathbf{N}}$. Then

$$x = \sum_{k=1}^\infty y_k \beta^{k-1}(1-\beta)/(\tau-1) = \sum_{k=1}^\infty x_k \beta^{k-1}(1-\beta)/(\tau-1).$$

Let $\mathcal{I} = \{k \in \mathbf{N} : m+1 \le x_k \le 2m\}$. Then $\mathcal{I} \neq \emptyset$. Otherwise, $x \in C_{\beta,\tau,\{-m,\dots,m\}}$ and so $x$ has two distinct $\{-m, \dots, m\}$-codes $(x_k)_{k=1}^\infty$ and $(y_k)_{k=1}^\infty$. Now we have

$$\begin{aligned} z &:= \sum_{k \in \mathcal{I}^c} y_k \beta^{k-1}(1-\beta)/(\tau-1) + \sum_{k \in \mathcal{I}} (y_k - m)\beta^{k-1}(1-\beta)/(\tau-1) \\ &= \sum_{k \in \mathcal{I}^c} x_k \beta^{k-1}(1-\beta)/(\tau-1) + \sum_{k \in \mathcal{I}} (x_k - m)\beta^{k-1}(1-\beta)/(\tau-1). \end{aligned} \tag{2.2}$$

It turns out that $z \in C_{\beta,\tau,\{-m,\dots,m\}}$ has two distinct $\{-m, \dots, m\}$-codes since, for $k \in \mathcal{I}$, $x_k - m \in \{1, \dots, m\}$ and $y_k - m \in \{-m, \dots, 0\}$. □

The following lemma is an analog of the previous lemma for the case $\beta = 1/(2\tau-1)$. It shows that each non-endpoint $x \in C_{\beta,\tau,\{0,\ldots,m\}} \subset C_{\beta,\tau,\{-m,\ldots,2m\}}$ has identical $\{0,\ldots,m\}$ and $\{-m,\ldots,2m\}$-codes.

**Lemma 2.6** *Let* $\beta = 1/(2\tau-1)$. *Let* $m \le \tau - 1$ *be a positive integer. Suppose* $x \in C_{\beta,\tau,\{-m,\ldots,2m\}}$ *with a code* $(x_k)_{k=1}^{\infty} \in \{-m,\ldots,2m\}^{\mathbf{N}}$. *We have*

(I) *if there exists some* $x_k \notin \{0,\ldots,m\}$ *and* $m < \tau - 1$, *then* $x \notin C_{\beta,\tau,\{0,\ldots,m\}}$;

(II) *if* $m = \tau - 1$ *and there exists some* $x_k \notin \{0,\ldots,\tau-1\}$, *then either* $x \notin C_{\beta,\tau,\{0,\ldots,\tau-1\}}$ *or* $x \in C_{\beta,\tau,\{0,\ldots,\tau-1\}}$ *is an endpoint of interval* $\phi_{\mathbf{i}}([0,1])$ *for some* $\mathbf{i} \in \{0,1,\ldots,\tau-1\}^*$.

Proof. Part (I) can be proved by the same argument as in Lemma 2.5 since each point in $C_{\beta,\tau,\{-m,\ldots,m\}}$ has a unique $\{-m,\ldots,m\}$-code when $m < \tau - 1$.

In the following we prove part (II). Suppose that $x \in C_{\beta,\tau,\{0,\ldots,\tau-1\}}$ and $(y_k)_{k=1}^{\infty} \in \{0,\ldots,\tau-1\}^{\mathbf{N}}$ is the unique $\{0,\ldots,\tau-1\}$-code of $x$. It suffices to show that $(y_k)_{k=1}^{\infty} = \mathbf{i}\overline{(\tau-1)}$ or $(y_k)_{k=1}^{\infty} = \mathbf{i}\,\overline{0}$ for some $\mathbf{i} \in \{0,\ldots,\tau-1\}^*$. Let $\mathcal{I} = \{k \in \mathbf{N} : \tau \le x_k \le 2(\tau-1)\}$.

Case 1. $\mathcal{I} \ne \emptyset$. Then $z \in C_{\beta,\tau,\{0,\pm1,,\ldots,\pm(\tau-1)\}}$ defined by (2.2) has two distinct $\{0,\pm1,,\ldots,\pm(\tau-1)\}$-codes $(s_k)_{k=1}^{\infty}, (t_k)_{k=1}^{\infty} \in \{0,\pm1,,\ldots,\pm(\tau-1)\}^{\mathbf{N}}$ where

$$\begin{cases} s_k = y_k - (\tau-1) \in \{-(\tau-1),\ldots,0\} \quad \text{and} \quad t_k = x_k - (\tau-1) \in \{1,\ldots,\tau-1\} \text{ for } k \in \mathcal{I}, \\ s_k = y_k \in \{0,\ldots,\tau-1\} \quad \text{and} \quad t_k = x_k \in \{0,\pm1,,\ldots,\pm(\tau-1)\} \quad \text{for} \quad k \in \mathcal{I}^c. \end{cases}$$

By Lemma 2.3 (II), we have that $(s_k)_{k=1}^{\infty} = \mathbf{i}\overline{(\tau-1)}$ and $(t_k)_{k=1}^{\infty} = \mathbf{i}'\overline{(1-\tau)}$, or $(s_k)_{k=1}^{\infty} = \mathbf{i}'\overline{(1-\tau)}$ and $(t_k)_{k=1}^{\infty} = \mathbf{i}\,\overline{(\tau-1)}$ for some $\mathbf{i} \in \{0,\pm1,\cdots\pm(\tau-1)\}^*$. For a set $A$ we denote by $|A|$ its cardinality.

When $(s_k)_{k=1}^{\infty} = \mathbf{i}\,\overline{(\tau-1)}$ and $(t_k)_{k=1}^{\infty} = \mathbf{i}'\overline{(1-\tau)}$, it must be $|\mathcal{I}^c| = \infty$ and $|\mathcal{I}| < \infty$. More precisely,

$$(s_k)_{k=1}^{\infty} = \mathbf{j}0\overline{(\tau-1)} \quad \text{and} \quad (t_k)_{k=1}^{\infty} = \mathbf{j}1\overline{(1-\tau)}, \quad \mathbf{j}0 \in \{0,1,\ldots,\tau-1\}^*,$$

and so

$$(y_k)_{k=1}^{\infty} = \mathbf{j}(\tau-1)\overline{(\tau-1)} \quad \text{and} \quad (x_k)_{k=1}^{\infty} = \mathbf{j}\tau\overline{(1-\tau)}, \quad \mathbf{j}0 \in \{0,1,\ldots,\tau-1\}^*.$$

When $(s_k)_{k=1}^{\infty} = \mathbf{i}'\overline{(1-\tau)}$ and $(t_k)_{k=1}^{\infty} = \mathbf{i}\overline{(\tau-1)}$, it must be $|\mathcal{I}^c| < \infty$ and $|\mathcal{I}| = \infty$. More precisely,

$$(s_k)_{k=1}^{\infty} = \mathbf{j}(i+1)\overline{(1-\tau)} \quad \text{and} \quad (t_k)_{k=1}^{\infty} = \mathbf{j}i\overline{(\tau-1)}, \quad \mathbf{j}(i+1) \in \{0,1,\ldots,\tau-1\}^*,$$

and so

$$(y_k)_{k=1}^{\infty} = \mathbf{j}(i+1)\overline{0} \quad \text{and} \quad (x_k)_{k=1}^{\infty} = \mathbf{j}i\overline{(2\tau-2)}, \quad \mathbf{j}(i+1) \in \{0,1,\ldots,\tau-1\}^*.$$

Case 2. $\mathcal{I} = \emptyset$. In this case, $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ are two distinct $\{0,\pm1,\ldots,\pm(\tau-1)\}$-codes of $x$. So

$$(y_k)_{k=1}^{\infty} = \mathbf{i}\overline{(\tau-1)} \quad \text{and} \quad (x_k)_{k=1}^{\infty} = \mathbf{i}'\overline{(1-\tau)}$$

for some $\mathbf{i} \in \{0,\ldots,\tau-1\}^*$. □

From Lemma 2.2 it follows that

$$C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap \big(C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t\big) \ne \emptyset \quad \text{if and only if} \quad t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}.$$

**Lemma 2.7** (Cf. [13, Lemma 3.3]) *Let* $\tau \ge 2$ *be a positive integer and let* $\beta \in (0,1)$. *Then for each* $t \in C_{\beta,\tau,\{0,\pm1,\ldots,\pm(\tau-1)\}}$

$$C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} \cap \big(C_{\beta,\tau,\{0,1,\ldots,\tau-1\}} + t\big) = \bigcup_{\tilde{t}} C_{\beta,\tau,\{0,1,\ldots,\tau-1\}}|_{\tilde{t}},$$

*where the union is taken over all* $\{0,\pm1,\ldots,\pm(\tau-1)\}$-*codes* $\tilde{t}$ *of* $t$, *and for an appointed* $\{0,\pm1,\ldots,\pm(\tau-1)\}$-*code* $\tilde{t} = (t_k)_{k=1}^{\infty}$ *of* $t$

$$C_{\beta,\tau,\{0,1,\ldots,\tau-1\}}|_{\tilde{t}} := \left\{ \sum_{k=1}^{\infty} \frac{x_k \beta^{k-1}(1-\beta)}{\tau-1} : x_k \in \{0,1,\ldots,\tau-1\} \cap (\{0,1,\ldots,\tau-1\} + t_k) \right\}.$$

Proof. Let $(t_k)_{k=1}^{\infty}$ be a $\{0, \pm 1, \dots, \pm(\tau-1)\}$-code of $t$. Let $(x_k)_{k=1}^{\infty}$ be such that

$$x_k \in \{0,1,\dots,\tau-1\} \cap (\{0,1,\dots,\tau-1\} + t_k).$$

Then $\sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta)/(\tau-1) \in C_{\beta,\tau,\{0,1,\dots,\tau-1\}}$. Note that $x_k \in (\{0,1,\dots,\tau-1\} + t_k)$ implies $x_k - t_k \in \{0,1,\dots,\tau-1\}$. Thus

$$\sum_{k=1}^{\infty} \frac{x_k \beta^{k-1}(1-\beta)}{\tau-1} = \sum_{k=1}^{\infty} \frac{(x_k - t_k)\beta^{k-1}(1-\beta)}{\tau-1} + \sum_{k=1}^{\infty} \frac{t_k \beta^{k-1}(1-\beta)}{\tau-1} \in C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t.$$

On the other hand, for any $x \in C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t)$ there exist $(x_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty} \in \{0,1,\dots,\tau-1\}^{\mathbf{N}}$ such that $x = \sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta)/(\tau-1)$ and

$$t = \sum_{k=1}^{\infty} \frac{x_k \beta^{k-1}(1-\beta)}{\tau-1} - \sum_{k=1}^{\infty} \frac{y_k \beta^{k-1}(1-\beta)}{\tau-1} = \sum_{k=1}^{\infty} \frac{(x_k - y_k)\beta^{k-1}(1-\beta)}{\tau-1}.$$

Then $(x_k - y_k)_{k=1}^{\infty} \in \{0, \pm 1, \dots, \pm(\tau-1)\}^{\mathbf{N}}$ is a $\{0, \pm 1, \dots, \pm(\tau-1)\}$- code of $t$. Therefore,

$$x_k = y_k + (x_k - y_k) \in \{0,1,\dots,\tau-1\} \cap (\{0,1,\dots,\tau-1\} + x_k - y_k),$$

leading to $x \in C_{\beta,\tau,\{0,1,\dots,\tau-1\}}|_{\tilde{t}}$ with $\tilde{t} = (x_k - y_k)_{k=1}^{\infty}$. □

From Lemmas 2.3 (I) and 2.7 it follows that for $\beta \in (1/(2\tau-1), 1)$ the set $C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t)$ is very complicated. However, when $\beta \in (0, 1/(2\tau-1)]$ we have the following corollary by Lemmas 2.3 (II)(III) and 2.7.

**Corollary 2.8** *Let $\tau \geq 2$ be a positive integer. If $\beta \in (0, 1/(2\tau-1))$, then for each $t \in C_{\beta,\tau,\{0,\pm 1,\dots,\pm(\tau-1)\}}$*

$$\begin{aligned} &C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t) \\ &\quad = \left\{ \sum_{k=1}^{\infty} \frac{x_k \beta^{k-1}(1-\beta)}{\tau-1} : x_k \in \{0,1,\dots,\tau-1\} \cap (\{0,1,\dots,\tau-1\} + t_k) \right\}, \end{aligned} \tag{2.3}$$

*where $(t_k)_{k=1}^{\infty} \in \{0, \pm 1, \dots, \pm(\tau-1)\}^{\mathbf{N}}$ is the unique $\{0, \pm 1, \dots, \pm(\tau-1)\}$-code of $t$.*

*Let $\beta = 1/(2\tau-1)$. If $t \in C_{\beta,\tau,\{0,\pm 1,\dots,\pm(\tau-1)\}}$ has a unique $\{0, \pm 1, \dots, \pm(\tau-1)\}$-code, then the set $C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t)$ can be represented as* (2.3). *If $t \in C_{\beta,\tau,\{0,\pm 1,\dots,\pm(\tau-1)\}}$ has two $\{0, \pm 1, \dots, \pm(\tau-1)\}$-codes, then the set $C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t)$ is a finite set containing more than one point.*

Let $\beta \in (0, 1/(2\tau-1)]$ and let $t \in C_{\beta,\tau,\{0,\pm 1,\dots,\pm(\tau-1)\}}$ have a unique $\{0, \pm 1, \dots, \pm(\tau-1)\}$-code $(t_k)_{k=1}^{\infty}$. Let $\eta_k = \min\{x : x \in \{0,1,\dots,\tau-1\} \cap (\{0,1,\dots,\tau-1\} + t_k)\} = \max\{0, t_k\}$ and let

$$\begin{aligned} E &:= C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t) - \sum_{k=1}^{\infty} \eta_k \beta^{k-1}(1-\beta)/(\tau-1) \\ &= \left\{ \sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta)/(\tau-1) : \{0,\dots \tau-1\}^{\mathbf{N}} \ni (x_k)_{k=1}^{\infty} \preceq (\tau-1-|t_k|)_{k=1}^{\infty} \right\}, \end{aligned} \tag{2.4}$$

where the second identity is obtained by (2.3). The set $E$ presents better geometric and algebraic properties and so is dealt with easier than the set $C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap (C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t)$ does.

**Lemma 2.9** *Let $\tau \geq 2$ be an integer and let $\beta \in (0, 1/(2\tau-1)]$. Let $t \in C_{\beta,\tau,\{0,\pm 1,\dots,\pm(\tau-1)\}}$ have a unique $\{0, \pm 1, \dots, \pm(\tau-1)\}$-code $(t_k)_{k=1}^{\infty}$. Let $E$ be defined by* (2.4). *Then*

(I) *$0 \in E$ and $E \subset C_{\beta,\tau,\{0,1,\dots,\tau-1\}}$;*

(II) *If $\sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta)/(\tau-1) \in E$ then $\sum_{k=1}^{\infty} y_k \beta^{k-1}(1-\beta)/(\tau-1) \in E$ for all $(y_k)_{k=1}^{\infty} \preceq (x_k)_{k=1}^{\infty}$ (i.e., $y_k \leq x_k$ for all $k$). In particular, $h\beta^{k-1}(1-\beta)/(\tau-1) \in E$ for $h \in \{0,\dots,\tau-1-|t_k|\}$;*

(III) *When* $E$ *is generated by an IFS, say* $\{f_i(x) = r_i x + b_i\}_{i=1}^N$*, one can require that all* $r_i > 0, 1 \le i \le N$*, and* $0 = b_1 \le b_2 \le \cdots \le b_N$*;*

(IV) *If* $(x_k)_{k=1}^\infty \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$ *is such that* $\sum_{k=1}^\infty x_k \beta^{k-1}(1-\beta)/(\tau-1) \in E$*, then* $(x_k)_{k=1}^\infty \preceq (\tau - 1 - |t_k|)_{k=1}^\infty$.

Proof. (I) and (II) follows just from (2.4). (III) is just Proposition 2.3 in [2] (also cf. [4]). In fact, $E$ is centrally symmetric, i.e.,

$$E = \sum_{k=1}^\infty (\tau - 1 - |t_k|)\beta^{k-1}(1-\beta)/(\tau-1) - E.$$

Thus, if $r_i < 0$, one can replace $f_i(x)$ by $f_i^*(x) := -r_i x + r_i \sum_{k=1}^\infty (\tau - 1 - |t_k|)\beta^{k-1}(1-\beta)/(\tau-1) + b_i$ and

$$\begin{aligned} f_i^*(E) &= -r_i E + r_i \sum_{k=1}^\infty (\tau - 1 - |t_k|)\beta^{k-1}(1-\beta)/(\tau-1) + b_i \\ &= r_i \left( \sum_{k=1}^\infty (\tau - 1 - |t_k|)\beta^{k-1}(1-\beta)/(\tau-1) - E \right) + b_i \\ &= r_i E + b_i = f_i(E). \end{aligned}$$

By (I) we have $b_i = f_i(0) \in E \subset [0,1]$. Finally, if all $b_i > 0$ then $0 \notin f_i(E)$, implying $0 \notin E$.

(IV) is just a version of Lemma 2.4. □

## 3 Proof of Theorem 1.2

We begin with a lemma which gives the description of a strong periodic infinite sequence. It still works when $\{0, 1, \ldots, \tau - 1\}$ is replaced with any $\Omega$.

**Lemma 3.1** *Let* $\tilde{x} = (x_k)_{k=1}^\infty \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$*. If there exists a positive integer* $q$ *such that* $x_{k+q} \ge x_k$ *for all* $k \in \mathbf{N}$*, then* $\tilde{x}$ *is strong periodic and* $q$ *is a period of* $\tilde{x}$*.*

Proof. Put $\tau_i = \max\{x_{i+kq} : k \in \mathbf{N} \cup \{0\}\}, 1 \le i \le q$. Let

$$m = \max_{1 \le i \le q} k_i \quad \text{where} \quad k_i = \min\{k \in \mathbf{N} \cup \{0\} : x_{i+kq} = \tau_i\}.$$

Then $\tilde{x} = \overline{x_1 \ldots x_q}$ if $m = 0$, or $\tilde{x} = x_1 \ldots x_{mq} \overline{x_{1+mq} \ldots x_{q+mq}}$ if $m \ge 1$. Thus, the desired results follow. □

Proof of Theorem 1.2. It suffices to prove that $E$, given by (1.3) (or (2.4)), is a self-similar set if and only if $(\tau - 1 - |t_k|)_{k=1}^\infty \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$ is strong periodic.

To prove sufficiency, we restate an alternative representation of a self-similar set. Let $D = \{d_1, d_2, \ldots, d_N\}$ be a finite set of real numbers. Let $f_i(x) = r(x + d_i)$, $1 \le i \le N$, and $|r| < 1$. Then the self-similar set $T(r, D)$ generated by IFS $(f_i)_{i=1}^N$ can be represented as

$$T(r, D) = \left\{ \sum_{k=1}^\infty d_k r^k : d_k \in D \right\}.$$

Since $(\tau - 1 - |t_k|)_{k=1}^\infty \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$ is strong periodic, it can be written as $(\tau - 1 - |t_k|)_{k=1}^\infty = \mathbf{i}\overline{\mathbf{i} + \mathbf{j}} \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$ where $\mathbf{i} = i_1 i_2 \ldots i_p, \mathbf{j} = j_1 j_2 \ldots j_p \in \{0, 1, \ldots, \tau - 1\}^p$ for some $p \in \mathbf{N}$ and $\mathbf{i} + \mathbf{j} := (i_1 + j_1)(i_2 + j_2) \ldots (i_p + j_p) \in \{0, 1, \ldots, \tau - 1\}^p$. Take a finite set of real numbers (recall that $\mathbf{ij}$ is the concatenation of $\mathbf{i}$ and $\mathbf{j}$)

$$D = \left\{ \beta^{-p} \sum_{k=1}^{2p} \sigma_k \beta^{k-1}(1-\beta)/(\tau-1) : \{0, 1, \ldots, \tau - 1\}^{2p} \ni (\sigma_k)_{k=1}^{2p} \preceq \mathbf{ij} \right\}. \tag{3.1}$$

We shall show $E = T(\beta^p, D)$. Now arbitrarily fix an $x \in E$ with $\{0, 1, \ldots, \tau - 1\}$-code $(x_k)_{k=1}^{\infty}$. Then for each $k \in \mathbf{N}$, $x_{kp+1}x_{kp+2}\ldots x_{kp+p}(\preceq \mathbf{i} + \mathbf{j})$ can be represented as

$$x_{kp+1}x_{kp+2}\ldots x_{kp+p} = (y_{kp+1} + z_{kp+1})(y_{kp+2} + z_{kp+2})\ldots(y_{kp+p} + z_{kp+p})$$

where $y_{kp+1}y_{kp+2}\ldots y_{kp+p} \preceq \mathbf{i}$ and $z_{kp+1}z_{kp+2}\ldots z_{kp+p} \preceq \mathbf{j}$. Hence (note that $x_1x_2\ldots x_p \preceq \mathbf{i}$)

$$\begin{aligned} x &= \beta^p \sum_{k=1}^{p} \frac{x_k\beta^{k-1-p}(1-\beta)}{\tau - 1} + \beta^{2p} \sum_{k=p+1}^{2p} \frac{(y_k + z_k)\beta^{k-1-2p}(1-\beta)}{\tau - 1} \\ &\quad + \beta^{3p} \sum_{k=2p+1}^{3p} \frac{(y_k + z_k)\beta^{k-1-3p}(1-\beta)}{\tau - 1} + \cdots \\ &= \beta^p \left( \sum_{k=1}^{p} \frac{x_k\beta^{k-1-p}(1-\beta)}{\tau - 1} + \sum_{k=p+1}^{2p} \frac{z_k\beta^{k-1-p}(1-\beta)}{\tau - 1} \right) \\ &\quad + \beta^{2p} \left( \sum_{k=p+1}^{2p} \frac{y_k\beta^{k-1-2p}(1-\beta)}{\tau - 1} + \sum_{k=2p+1}^{3p} \frac{z_k\beta^{k-1-2p}(1-\beta)}{\tau - 1} \right) + \cdots \in T(\beta^p, D). \end{aligned}$$

Thus, $E \subseteq T(\beta^p, D)$. The converse inclusion is left for the readers.

In the following, we prove the necessity. By Lemma 2.9 (III) one can assume that $E$ is generated by an IFS $\{f_i(x) = r_i x + b_i\}_{i=1}^{N}$ with $r_i \in (0, 1)$ and $0 = b_1 \le b_2 \le \cdots \le b_N$. We assume

$$(\tau - 1 - |t_k|)_{k=1}^{\infty} \neq \bar{0} \quad \text{and} \quad (\tau - 1 - |t_k|)_{k=1}^{\infty} \neq \mathbf{i}\overline{(\tau - 1)} \text{ with } \mathbf{i} \in \{0, 1, \ldots, \tau - 1\}^*, \tag{3.2}$$

since $(\tau - 1 - |t_k|)_{k=1}^{\infty}$ is strong periodic in this two cases and so the result holds. Take $q \in \mathbf{N} \cup \{0\}$ such that $\beta^q(1 - \beta)/(\tau - 1) \in E$ (this $q$ exists since $(\tau - 1 - |t_k|)_{k=1}^{\infty} \neq \bar{0}$). So we have

$$E \ni f_1(\beta^q(1-\beta)/(\tau-1)) = \beta^q r_1(1-\beta)/(\tau-1) = \sum_{k=1}^{\infty} x_k\beta^{k-1}(1-\beta)/(\tau-1),$$

for some $(x_k)_{k=1}^{\infty} \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$ with $(x_k)_{k=1}^{\infty} \preceq (\tau - 1 - |t_k|)_{k=1}^{\infty}$. So $r_1 = \sum_{k=1}^{\infty} x_k\beta^{k-q-1}$, $x_k \le \tau - 1 - |t_k|$. Since $0 < r_1 < 1$, we have $x_k = 0$ for $k \le q + 1$. Thus $r_1$ is of the form

$$r_1 = \sum_{i=1}^{\infty} x_{i+q+1}\beta^i := \sum_{i=1}^{\infty} \alpha_i\beta^i \text{ where } \alpha_i = x_{i+q+1} \in \{0, \ldots, \tau - 1 - |t_{i+q+1}|\}. \tag{3.3}$$

Let

$$\ell := \max_k\{\tau - 1 - |t_k|\}.$$

So $1 \le \ell \le \tau - 1$. We like to point out that the sequence $(\alpha_k)_{k=1}^{\infty}$ is not of the form $\mathbf{i}\,\bar{\ell}$. Otherwise, we have $(\tau - 1 - |t_k|)_{k=1}^{\infty} = \mathbf{j}\bar{\ell}$. Now for $s \in \mathbf{N}$ with $s > |\mathbf{j}|$ we have

$$f_1\left(\frac{\beta^{s-1}(1-\beta)}{\tau - 1}\right) = \sum_{i=1}^{\infty} \frac{\alpha_i\beta^{i+s-1}(1-\beta)}{\tau - 1} \in E.$$

Note that there exists a positive number $\epsilon$ such that

$$(f_1(\beta^{s-1}(1-\beta)/(\tau-1)), f_1(\beta^{s-1}(1-\beta)/(\tau-1)) + \epsilon) \cap E = \emptyset$$

since $(\alpha_k)_{k=1}^{\infty} = \mathbf{i}\,\bar{\ell}$. Thus, when $k > |\mathbf{j}|, k \neq s$ is big enough we have

$$\beta^{s-1}(1-\beta)/(\tau-1) + \beta^{k-1}(1-\beta)/(\tau-1) \in E$$

and

$$f_1\left(\frac{\beta^{s-1}(1-\beta)}{\tau-1}+\frac{\beta^{k-1}(1-\beta)}{\tau-1}\right)=f_1\left(\frac{\beta^{s-1}(1-\beta)}{\tau-1}\right)+\frac{r_1\beta^{k-1}(1-\beta)}{\tau-1}\notin E.$$

Therefore, the sequence $(\alpha_k)_{k=1}^{\infty}$ is not of the form $\mathbf{i}\,\bar{\ell}$.

Let $\alpha_{\max}=\max_k \alpha_k$. Then $\alpha_{\max}\geq 1$ since $r_1>0$. Suppose $\alpha_{\max}=1$. Fix $m$ such that $\alpha_m=1$. Then for each $s\in\mathbf{N}$

$$f_1((\tau-1-|t_s|)\beta^{s-1}(1-\beta)/(\tau-1))=\sum_{i=1}^{\infty}\alpha_i(\tau-1-|t_s|)\beta^{s+i-1}(1-\beta)/(\tau-1)\in E,$$

implying $\tau-1-|t_{m+s}|\geq\tau-1-|t_s|$ by Lemma 2.9 (IV). Thus $(\tau-1-|t_k|)_{k=1}^{\infty}$ is strong periodic by Lemma 3.1.

We like to point out that if $\tau=2$, then $\alpha_{\max}=1$ (at this moment $\ell=1$) and so the proof is finished. In the following, we show that $\alpha_{\max}=1$ must hold even if $\tau>2$. Otherwise, $1<\alpha_{\max}\leq\ell$. We consider an auxiliary set

$$E_\ell:=\left\{\sum_{k=1}^{\infty}x_k\beta^{k-1}(1-\beta)/(\tau-1):(x_k)_{k=1}^{\infty}\in\{0,1,\ldots,\ell\}^{\mathbf{N}}\right\}=\Pi\left(\{0,1,\ldots,\ell\}^{\mathbf{N}}\right).$$

Then $E\subseteq E_\ell$.

For any $p\in\mathbf{N}$, take $(x_k)_{k=1}^{p}\in\{0,1,\ldots,\ell\}^p$ with $x_p=\ell$. We claim that

$$\sum_{i=1}^{p}x_i\beta^{i-1}(1-\beta)/(\tau-1)+\delta\notin E_\ell\quad\text{if}\quad\frac{\ell\beta^p}{\tau-1}<\delta<\frac{\beta^{p-2}(1-\beta)}{\tau-1}-\frac{\ell\beta^{p-1}(1-\beta)}{\tau-1}.\tag{3.4}$$

We remark that $\ell\beta^p/(\tau-1)<\beta^{p-2}(1-\beta)/(\tau-1)-\ell\beta^{p-1}(1-\beta)/(\tau-1)$ holds when $0<\beta\leq 1/(2\tau-1)$ (in fact, only need $\beta\tau<1$). Now we verify (3.4), which is done in two cases.

Case 1. $x_1=\cdots=x_p=\ell$.

In this case, $\sum_{i=1}^{p}x_i\beta^{i-1}(1-\beta)/(\tau-1)+\delta>\Pi(\bar{\ell})$ since $\ell\beta^p/(\tau-1)=\sum_{k=p+1}^{\infty}\ell\beta^{k-1}(1-\beta)/(\tau-1)$. So (3.4) holds.

Case 2. $x_{p-w}<\ell$ and $x_{p-w+1}=\cdots=x_p=\ell$ for some $1\leq w\leq p-1$.

From the geometric structure of $E_\ell$ we have

$$(\Pi(x_1\ldots x_{p-w}\bar{\ell}),\Pi(x_1\ldots x_{p-w-1}(x_{p-w}+1)\bar{0}tobe))\cap E_\ell=\emptyset.$$

However, we have

$$\begin{aligned}&\sum_{i=1}^{p}x_i\beta^{i-1}(1-\beta)/(\tau-1)+\delta\\&\quad>\sum_{i=1}^{p}x_i\beta^{i-1}(1-\beta)/(\tau-1)+\ell\beta^p/(\tau-1)=\Pi(x_1\ldots x_{p-w}\bar{\ell})\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{p}\frac{x_i\beta^{i-1}(1-\beta)}{\tau-1}+\delta &< \sum_{i=1}^{p}\frac{x_i\beta^{i-1}(1-\beta)}{\tau-1}+\frac{\beta^{p-2}(1-\beta)}{\tau-1}-\frac{\ell\beta^{p-1}(1-\beta)}{\tau-1}\\
&= \sum_{i=1}^{p-w}\frac{x_i\beta^{i-1}(1-\beta)}{\tau-1}+\sum_{i=p-w+1}^{p}\frac{\ell\beta^{i-1}(1-\beta)}{\tau-1}+\frac{\beta^{p-2}(1-\beta)}{\tau-1}-\frac{\ell\beta^{p-1}(1-\beta)}{\tau-1}\\
&= \sum_{i=1}^{p-w}\frac{x_i\beta^{i-1}(1-\beta)}{\tau-1}+\frac{\ell(\beta^{p-w}-\beta^{p})}{\tau-1}+\frac{\beta^{p-2}(1-\beta)}{\tau-1}-\frac{\ell\beta^{p-1}(1-\beta)}{\tau-1}\\
&\le \sum_{i=1}^{p-w}\frac{x_i\beta^{i-1}(1-\beta)}{\tau-1}+\frac{\beta^{p-w-1}(1-\beta)}{\tau-1}\\
&= \Pi\big(x_1\ldots x_{p-w-1}(x_{p-w}+1)\overline{0}\big).
\end{aligned}$$

For the above inequality " $\le$ ", the condition $\beta\le 1/(2\tau-1)$ (in fact, only need $\beta\tau<1$) is being used, and more precisely, the inequality " $\le$ " is strict when $w>1$ and the inequality " $\le$ " becomes " $=$ " when $w=1$.

Let $\gamma=\min\{i:\alpha_i\ge 2\}$. Let $s$ be such that $\tau-1-|t_s|=\ell$. Then $h\beta^{s-1}(1-\beta)/(\tau-1)\in E$ for $h\in\{0,\ldots,\ell\}$ by Lemma 2.9 (II). Thus by (3.3)

$$\begin{aligned}
f_1(h\beta^{s-1}(1-\beta)/(\tau-1)) &= r_1h\beta^{s-1}(1-\beta)/(\tau-1)\\
&= \sum_{i=1}^{\gamma-1}\alpha_ih\beta^{s+i-1}(1-\beta)/(\tau-1)\\
&\quad+\sum_{i=\gamma}^{\infty}\alpha_ih\beta^{s+i-1}(1-\beta)/(\tau-1)\in E.
\end{aligned}\tag{3.5}$$

Let $h^*\in\{1,\ldots,\ell\}$ be such that

$$\begin{cases}
h^*\displaystyle\sum_{i=\gamma}^{\infty}\frac{\alpha_i\beta^{s+i-1}(1-\beta)}{\tau-1}>\frac{\ell\beta^{s+\gamma-1}(1-\beta)}{\tau-1}+\frac{\ell\beta^{s+\gamma}}{\tau-1}\\
(h^*-1)\displaystyle\sum_{i=\gamma}^{\infty}\frac{\alpha_i\beta^{s+i-1}(1-\beta)}{\tau-1}\le\frac{\ell\beta^{s+\gamma-1}(1-\beta)}{\tau-1}+\frac{\ell\beta^{s+\gamma}}{\tau-1}.
\end{cases}\tag{3.6}$$

We point out that $h^*$ is well-determined. In fact, when $h^*=\ell$ we have

$$\begin{aligned}
h^*\sum_{i=\gamma}^{\infty}\frac{\alpha_i\beta^{s+i-1}(1-\beta)}{\tau-1} &= \ell\sum_{i=\gamma}^{\infty}\frac{\alpha_i\beta^{s+i-1}(1-\beta)}{\tau-1}\\
&\ge \frac{\ell\alpha_\gamma\beta^{s+\gamma-1}(1-\beta)}{\tau-1}\\
&> \frac{\ell\beta^{s+\gamma-1}(1-\beta)}{\tau-1}+\frac{\ell\beta^{s+\gamma}}{\tau-1},
\end{aligned}$$

where the assumption $\alpha_\gamma\ge 2$ is being used. Taking into account that the second inequality in (3.6) is true when $h^*=1$, we can get a unique $h^*\in\{1,\ldots,\ell\}$ to satisfy (3.6). Therefore,

$$\begin{aligned}
f_1(h^*\beta^{s-1}(1-\beta)/(\tau-1)) &= \sum_{i=1}^{\gamma-1}\alpha_ih^*\beta^{s+i-1}(1-\beta)/(\tau-1)+\ell\beta^{s+\gamma-1}(1-\beta)/(\tau-1)\\
&\quad+h^*\sum_{i=\gamma}^{\infty}\alpha_i\beta^{s+i-1}(1-\beta)/(\tau-1)-\ell\beta^{s+\gamma-1}(1-\beta)/(\tau-1).
\end{aligned}$$

Let $\delta = h^* \sum_{i=\gamma}^{\infty} \alpha_i \beta^{s+i-1}(1-\beta)/(\tau-1) - \ell\beta^{s+\gamma-1}(1-\beta)/(\tau-1)$. Then $\delta > \ell\beta^{s+\gamma}/(\tau-1)$ by the first inequality in (3.6). On the other hand, by the second inequality in (3.6) we have

$$\begin{aligned}
\delta &\le \sum_{i=\gamma}^{\infty} \alpha_i \beta^{s+i-1}(1-\beta)/(\tau-1) + \ell\beta^{s+\gamma}/(\tau-1) \\
&< \ell\beta^{s+\gamma-1}/(\tau-1) + \ell\beta^{s+\gamma}/(\tau-1) \\
&\le \beta^{s+\gamma-2}(1-\beta)/(\tau-1) - \ell\beta^{s+\gamma-1}(1-\beta)/(\tau-1),
\end{aligned}$$

where the condition $0 < \beta \le 1/(2\tau-1)$ is being used for the last inequality, and the second inequality " $<$ " is obtained since $(\alpha_k)_{k=1}^{\infty}$ is not of the form $\mathbf{i}\,\bar{\ell}$. Thus (recall $\alpha_i = 0$ or 1 for $i < \gamma$)

$$f_1\left(\frac{h^*\beta^{s-1}(1-\beta)}{\tau-1}\right) = \sum_{i=1}^{\gamma-1} \frac{\alpha_i h^* \beta^{s+i-1}(1-\beta)}{\tau-1} + \frac{\ell\beta^{s+\gamma-1}(1-\beta)}{\tau-1} + \delta \notin E_\ell(\supseteq E),$$

by (3.4), contradicting (3.5). □

From the "if part" proof of Theorem 1.2 it follows that if $(\tau-1-|t_k|)_{k=1}^{\infty} = \mathbf{i}\overline{\mathbf{i}+\mathbf{j}} \in \{0,1,\dots,\tau-1\}^{\mathbf{N}}$ where $\mathbf{i} = i_1 i_2 \dots i_p, \mathbf{j} = j_1 j_2 \dots j_p \in \{0,1,\dots,\tau-1\}^p$ for some $p \in \mathbf{N}$ and $\mathbf{i}+\mathbf{j} := (i_1+j_1)(i_2+j_2)\dots(i_p+j_p) \in \{0,1,\dots,\tau-1\}^p$, then $E$ can be generated by the IFS $\{f_i(x) = \beta^p(x+d_i) : d \in D\}$ where $D$ is determined by (3.1). More precisely,

$$E = \bigcup_{\{0,\dots,\tau-1\}^{2p} \ni \mathbf{k} \preceq \mathbf{ij}} (\beta^p E + \Pi(\mathbf{k}\bar{0})). \tag{3.7}$$

When $\tau = 2$, $(\tau-1-|t_k|)_{k=1}^{\infty} = (1-|t_k|)_{k=1}^{\infty} = \mathbf{i}\overline{\mathbf{i}+\mathbf{j}} \in \{0,1\}^{\mathbf{N}}$ which implies that either $i_k$ or $j_k$ is 0 for each $1 \le k \le p$. Therefore, from the following theorem it follows that the right side of (3.7) is a disjoint union, and so the resulting IFS satisfies the SSC (this can be also verified directly). Thus

$$\dim_H C_{\beta,2,\{0,1\}} \cap (C_{\beta,2,\{0,1\}} + t) = \dim_H E = \frac{\log \# D}{-p\log\beta} = \frac{\sum_{k=1}^{p}(i_k+j_k)\log 2}{-p\log\beta}.$$

However, it is not this situation for $\tau \ge 3$.

**Theorem 3.2** *Let $\tau \ge 2$ be an integer and let $\beta \in (0, 1/(2\tau-1)]$. Let $t$ has a unique code $(t_k)_{k=1}^{\infty} \in \{0, \pm 1, \dots, \pm(\tau-1)\}$. Suppose that $(\tau-1-|t_k|)_{k=1}^{\infty} = \mathbf{i}\overline{\mathbf{i}+\mathbf{j}} \in \{0,1,\dots,\tau-1\}^{\mathbf{N}}$ where $\mathbf{i} = i_1 i_2 \dots i_p$, $\mathbf{j} = j_1 j_2 \dots j_p \in \{0,1,\dots,\tau-1\}^p$ for some $p \in \mathbf{N}$. Then*

$$\dim_H C_{\beta,\tau,\{0,1,\dots,\tau-1\}} \cap \big(C_{\beta,\tau,\{0,1,\dots,\tau-1\}} + t\big) = \dim_H E = \frac{\log \prod_{k=1}^{p}(i_k+j_k+1)}{-p\log\beta}. \tag{3.8}$$

*In addition, the right side of* (3.7) *is a disjoint union if and only if either $i_k$ or $j_k$ is 0 for each $1 \le k \le p$.*

Proof. For a given $\mathbf{k} = k_1 \dots k_p k_{p+1} \dots k_{2p} \preceq \mathbf{ij}$ we have

$$\begin{aligned}
\beta^p E + \Pi(\mathbf{k}\bar{0}) &= \left\{ \beta^p x + \sum_{j=1}^{2p} k_j \beta^{j-1}(1-\beta)/(\tau-1) : x \in E \right\} \\
&= \left\{ \sum_{j=1}^{p} k_j \beta^{j-1}(1-\beta)/(\tau-1) + \sum_{j=p+1}^{2p} (k_j + x_{j-p})\beta^{j-1}(1-\beta)/(\tau-1) \right. \\
&\qquad \left. + \sum_{j>2p} x_{j-p}\beta^{j-1}(1-\beta)/(\tau-1) : (x_k)_{k=1}^{\infty} \preceq (\tau-1-|t_k|)_{k=1}^{\infty} = \mathbf{i}\overline{\mathbf{i}+\mathbf{j}} \right\}.
\end{aligned}$$

Thus each point of $\beta^p E + \Pi\big(\mathbf{k}\bar{0}\big)$ has a unique $\{0,1,\dots,\tau-1\}$-code of the form

$$k_1 \dots k_p (k_{p+1}+x_1)\dots(k_{2p}+x_p) x_{p+1} x_{p+2} \dots \quad \text{for some} \quad (x_k)_{k=1}^{\infty} \preceq (\tau-1-|t_k|)_{k=1}^{\infty} = \mathbf{i}\overline{\mathbf{i}+\mathbf{j}}.$$

Take distinct $\mathbf{k} = k_1 \dots k_p k_{p+1} \dots k_{2p} \preceq \mathbf{ij}$ and $\mathbf{k}^* = k_1^* \dots k_p^* k_{p+1}^* \dots k_{2p}^* \preceq \mathbf{ij}$. If $k_1 \dots k_p \neq k_1^* \dots k_p^*$, then $\big(\beta^p E + \Pi(\mathbf{k}\bar{0})\big) \cap \big(\beta^p E + \Pi(\mathbf{k}^*\bar{0})\big) = \emptyset$ follows directly from above arguments. If $k_1 \dots k_p = k_1^* \dots k_p^*$, we have $k_{p+u} \neq k_{p+u}^*$ for some $1 \leq u \leq p$. This forces $j_u \geq 1$.

When either $i_k$ or $j_k$ is 0 for each $1 \leq k \leq p$, we have $i_u = 0$ (i.e., $\tau - 1 - |t_u| = 0$) and so $k_{p+u} + x_u \neq k_{p+u}^* + x_u^*$ for any $(x_k)_{k=1}^\infty, \big(x_k^*\big)_{k=1}^\infty \preceq \big(\tau - 1 - |t_k|\big)_{k=1}^\infty = \mathbf{i}\overline{\mathbf{i} + \mathbf{j}}$. Then $\big(\beta^p E + \Pi\big(\mathbf{k}\bar{0}\big)\big) \cap \big(\beta^p E + \Pi\big(\mathbf{k}^*\bar{0}\big)\big) = \emptyset$ follows from above arguments. This proves the sufficiency.

On the other hand, suppose both $i_u$ and $j_u$ are not zero for some $1 \leq u \leq p$. Take $\mathbf{k} = k_1 \dots k_{p+u-1} j_u k_{p+u+1} \dots k_{2p} \preceq \mathbf{ij}$, $\mathbf{k}^* = k_1 \dots k_{p+u-1}(j_u - 1)k_{p+u+1} \dots k_{2p} \preceq \mathbf{ij}$, $x = x_1 \dots x_{u-1} 0 x_{u+1} \cdots \preceq \mathbf{i}\overline{\mathbf{i} + \mathbf{j}}$ and $x^* = x_1 \dots x_{u-1} 1 x_{u+1} \cdots \preceq \mathbf{i}\overline{\mathbf{i} + \mathbf{j}}$. Then $\beta^p \Pi(x) + \Pi\big(\mathbf{k}\bar{0}\big) = \beta^p \Pi(x^*) + \Pi\big(\mathbf{k}^*\bar{0}\big)$, leading to $\big(\beta^p E + \Pi\big(\mathbf{k}\bar{0}\big)\big) \cap \big(\beta^p E + \Pi\big(\mathbf{k}^*\bar{0}\big)\big) \neq \emptyset$.

Finally, if either $i_k$ or $j_k$ is 0 for each $1 \leq k \leq p$ we have

$$\#\{\mathbf{k} \in \{0,1\}^{2p} : \mathbf{k} \preceq \mathbf{ij}\} = \prod_{k=1}^{p} (i_k + j_k + 1),$$

and so (3.8) holds by (3.7). Otherwise, denote $\mathbf{0} := 0 \dots 0 \in \{0, \dots, \tau - 1\}^p$ and set

$$F = \left\{ \sum_{k=1}^{\infty} x_k \beta^{k-1} (1 - \beta)/(\tau - 1) : \{0, \dots \tau - 1\}^{\mathbf{N}} \ni (x_k)_{k=1}^\infty \preceq \mathbf{0}\overline{\mathbf{i} + \mathbf{j}} \right\}.$$

Like in (3.7), we have

$$F = \bigcup_{\{0,\dots,\tau-1\}^{2p} \ni \mathbf{k} \preceq \mathbf{0}(\mathbf{i}+\mathbf{j})} \big(\beta^p F + \Pi\big(\mathbf{k}\bar{0}\big)\big).$$

By the same argument as above,

$$\dim_H F = \frac{\log \prod_{k=1}^{p} (i_k + j_k + 1)}{-p \log \beta}.$$

Thus, (3.8) holds since $E$ is a finite union of translations of $F$. □

In fact, we have $\dim_H E = \dim_B E$ and the formula (3.8) can be obtained directly from a dimension formula for the generalized Moran sets (cf. [12, 13]).

## 4 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. For simplicity, we denote $(\ell_k)_{k=1}^\infty := (\tau - 1 - |t_k|)_{k=1}^\infty$. We assume $(\ell_k)_{k=1}^\infty$ is strong $p$-periodic and $(\ell_k)_{k=1}^\infty \neq \bar{0}$. Then it can be written as

$$(\ell_k)_{k=1}^\infty = \mathbf{i}\overline{\mathbf{i} + \mathbf{j}} \quad \text{where} \quad \mathbf{i} = (\ell_k)_{k=1}^p \quad \text{and} \quad \mathbf{j} = (\ell_{p+k} - \ell_k)_{k=1}^p.$$

Let $\ell := \max_k \ell_k = \max_k \{\tau - 1 - |t_k|\}$. For an $x \in E$ with code $(x_k)_{k=1}^\infty$, let $\mathcal{L}(x) = \sup\{k : x_k \geq 1\}$ with the convention $\mathcal{L}(\bar{0}) = 0$. $\mathcal{L}(x)$ may be infinite.

Let $E$ be generated by an IFS $\{f_i(x) = r_i x + b_i\}_{i=1}^N$ with $r_i \in (0,1)$. We first show that $r_i = \beta^{q_i}$ with $q_i \in \mathbf{N}$ and $\mathcal{L}(b_i) < \infty$ for all $i$, i.e., each $b_i$ is finitely represented. These are done in the following Lemmas 4.1–4.5. In the following, we assume that all conditions in Theorem 1.3 are satisfied.

**Lemma 4.1** *If $E$ is generated by an IFS $\{f_i(x) = r_i x + b_i\}_{i=1}^N$ with $r_i \in (0,1)$, then there exists a $q_i \in \mathbf{N}$ such that*

$$r_i = \xi_{q_i} \beta^{q_i} + \sum_{k > q_i} \xi_k \beta^k,$$

*where $\xi_{q_i} \in \{1, \dots, \ell\}$ and $\xi_k \in \{0, \pm 1, \dots, \pm \ell\}$ for $k > q_i$.*

Proof. Since $b_i = f_i(0) \in E$, we use $(b_{ik})_{k=1}^{\infty} \preceq (\ell_k)_{k=1}^{\infty}$ to denote the unique $\{0, 1, \dots \tau - 1\}$-code of $b_i$. Note that for each $x \in E$, $f_i(x) = r_i x + b_i = r_i x + \sum_{k=1}^{\infty} b_{ik}\beta^{k-1}(1-\beta)/(\tau-1) \in E$. Take $t$ such that $\ell_t > 0$. Then

$$f_i\left(\frac{\beta^{t-1}(1-\beta)}{\tau-1}\right) = \frac{r_i\beta^{t-1}(1-\beta)}{\tau-1} + \sum_{k=1}^{\infty}\frac{b_{ik}\beta^{k-1}(1-\beta)}{\tau-1} := \sum_{k=1}^{\infty}\frac{y_k\beta^{k-1}(1-\beta)}{\tau-1}, \tag{4.1}$$

where $(y_k)_{k=1}^{\infty} \in \{0, 1, \dots, \tau-1\}^{\mathbf{N}}$. Taking into account $r_i \in (0, 1)$, this gives

$$r_i = \sum_{k=t+1}^{\infty}(y_k - b_{ik})\beta^{k-t} := \sum_{k=1}^{\infty}\xi_k\beta^k \quad \text{where} \quad \xi_k = y_{t+k} - b_{i(t+k)} \in \{0, \pm 1, \dots, \pm\ell_{t+k}\}.$$

More precisely, there exists a $q_i \in \mathbf{N}$ such that

$$r_i = \xi_{q_i}\beta^{q_i} + \sum_{k>q_i}\xi_k\beta^k \quad \text{where} \quad \xi_{q_i} \in \{1, \dots, \ell_{t+q_i}\},\ \xi_k \in \{0, \pm 1, \dots, \pm\ell_{t+k}\} \text{ for } k > q_i.$$

The $\xi_{q_i} > 0$ is derived from the facts that $r_i > 0$ and

$$\sum_{k>q_i}^{\infty}\xi_k\beta^k \le \frac{\ell\beta^{q_i+1}}{1-\beta} \le \beta^{q_i}.$$

Thus the desired result is proved. ☐

**Corollary 4.2** *The sequence* $(\xi_k)_{k\geq q_i}$ *defined above cannot be of the form* $\mathbf{u}\overline{(\tau-1)}$. *The sequence* $(\xi_k)_{k\geq q_i}$ *defined above cannot be of the form* $\mathbf{u}\overline{(1-\tau)}$, *equivalently,* $(b_{ik})_{k=1}^{\infty} \in \{0, 1, \dots, \tau-1\}^{\mathbf{N}}$ *cannot be of the form* $\mathbf{v}\overline{(\tau-1)}$

Proof. Note that $(\xi_k)_{k\geq q_i} = \mathbf{u}\overline{(\tau-1)}$ implies that $(y_k)_{k=1}^{\infty} = \mathbf{v}\overline{(\tau-1)} \in \{0, 1, \dots, \tau-1\}^{\mathbf{N}}$. Then

$$f_i\left(\frac{\beta^{t-1}(1-\beta)}{\tau-1}\right) = \Pi(\mathbf{v}\overline{(\tau-1)})$$

is the right endpoint of the interval $\phi_{\mathbf{v}}([0, 1])$. Take a positive integer $n > t$ such that $\beta^{n-1}(1-\beta)/(\tau-1) \in E$. Then $\beta^{t-1}(1-\beta)/(\tau-1) + \beta^{n-1}(1-\beta)/(\tau-1) \in E$, but

$$f_i\left(\frac{\beta^{t-1}(1-\beta)}{\tau-1} + \frac{\beta^{n-1}(1-\beta)}{\tau-1}\right) = \Pi(\mathbf{v}\overline{(\tau-1)}) + \frac{r_i\beta^{n-1}(1-\beta)}{\tau-1} \notin E$$

when $n$ is big enough.

If $(b_{ik})_{k=1}^{\infty} = \mathbf{v}\overline{(\tau-1)}$, then $b_i$ is the right endpoint of the interval $\phi_{\mathbf{v}}([0, 1])$. By (4.1) this leads to $f_i(\beta^{t-1}(1-\beta)/\tau-1)) \notin E$ when $t$ with $\ell_t > 0$ is big enough. ☐

**Lemma 4.3** *Using the notation of Lemma* 4.1 *we have* $\xi_{q_i} = 1$ *and* $\xi_k \in \{-1, 0, 1\}$ *for* $k > q_i$.

Proof. When $\tau = 2$, we have $\ell = 1$ so that the result is true by Lemma 4.1. However, we need lots of effort when $\tau \geq 3$.

Recall that $b_i$ is represented as

$$b_i = \sum_{k=1}^{\infty} b_{ik}\beta^{k-1}(1-\beta)/(\tau-1), \quad 0 \le b_{ik} \le \ell_k.$$

Take $p \le s \le 2p$ such that $\ell_s = \ell$. Thus $u\beta^{s-1}(1-\beta)/(\tau-1) \in E$ for each $1 \le u \le \ell$. Note that

$$\begin{aligned}\frac{r_i\beta^{s-1}(1-\beta)}{\tau-1} &= \left(\xi_{q_i}\beta^{q_i} + \sum_{k>q_i}^{\infty}\xi_k\beta^k\right)\frac{\beta^{s-1}(1-\beta)}{\tau-1}\\ &= \frac{\xi_{q_i}\beta^{q_i+s-1}(1-\beta)}{\tau-1} + \sum_{k>q_i}^{\infty}\frac{\xi_k\beta^{k+s-1}(1-\beta)}{\tau-1}.\end{aligned}$$

Thus

$$\begin{aligned}f_i\left(\frac{\beta^{s-1}(1-\beta)}{\tau-1}\right) &= \sum_{k=q_i}^{\infty}\frac{\xi_k\beta^{k+s-1}(1-\beta)}{\tau-1} + \sum_{k=1}^{\infty}\frac{b_{ik}\beta^{k-1}(1-\beta)}{\tau-1}\\ &= \sum_{k<q_i+s}\frac{b_{ik}\beta^{k-1}(1-\beta)}{\tau-1} + \sum_{k=q_i}^{\infty}\frac{(\xi_k+b_{i(k+s)})\beta^{k+s-1}(1-\beta)}{\tau-1} \in E.\end{aligned}$$

Note $\xi_k + b_{i(k+s)} \in \{0, \pm 1, , \dots, \pm\ell, \ell+1, \dots, 2\ell\}$.

Case 1. $\beta < 1/(2\tau-1)$. In this case, from Lemma 2.5 it follows $\xi_k + b_{i(k+s)} \in \{0, 1, \dots, \ell\}$ and more precisely, $\xi_k + b_{i(k+s)} \in \{0, 1, \dots, \ell_{k+s}\}$.

Case 2. $\beta = 1/(2\tau-1)$. By Lemma 2.6 if one can show that $(\xi_k + b_{i(k+s)})_{k\ge q_i}$ is not of the form $\mathbf{j}\overline{(1-\tau)}$ or $\mathbf{j}\overline{(2\tau-2)}$, then $\xi_k + b_{i(k+s)} \in \{0, 1, \dots, \ell\}$.

Suppose that $(\xi_k + b_{i(k+s)})_{k\ge q_i} = \mathbf{j}\overline{(1-\tau)}$. Then $(\xi_k)_{k\ge q_i} = \mathbf{u}\overline{(1-\tau)}$, contradicting Corollary 4.2. Suppose that $(\xi_k + b_{i(k+s)})_{k\ge q_i} = \mathbf{j}\overline{(2\tau-2)}$. Then $(\xi_k)_{k\ge q_i} = \mathbf{u}\overline{(\tau-1)}$, contradicting Corollary 4.2.

Summarizing above cases 1 and 2, we have $\xi_k + b_{i(k+s)} \in \{0, 1, \dots, \ell_{k+s}\}$. By induction, we have for $1 \le u \le \ell$

$$\begin{aligned}f_i\left(\frac{u\beta^{s-1}(1-\beta)}{\tau-1}\right) &= \sum_{k<q_i+s}\frac{b_{ik}\beta^{k-1}(1-\beta)}{\tau-1}\\ &\quad + \sum_{k=q_i}^{\infty}\frac{\big((u-1)\xi_k + b_{i(k+s)} + \xi_k\big)\beta^{k+s-1}(1-\beta)}{\tau-1} \in E.\end{aligned}$$

The same argument as above shows that

$$u\xi_k + b_{i(k+s)} \in \{0, 1, \dots, \ell_{k+s}\} \quad \text{for} \quad u \in \{1, \dots, \ell\}.$$

Taking into account $\xi_{q_i} \ge 1$, we have $\xi_{q_i} = 1$. And $\xi_k \in \{-1, 0, 1\}$ for $k > q_i$. In addition, for $k \ge q_i$ if $\xi_k = \pm 1$ then $\ell_{k+s} = \ell$. In particular, $\ell_{q_i+s} = \ell$. ☐

**Lemma 4.4** *Using the notation of Lemma* 4.1 *we have* $\xi_k = 0$ *for* $k > q_i$.

Proof. Suppose that the statement it is not true. Let

$$r_i = \beta^{q_i} + \xi_q\beta^q + \sum_{k>q}\xi_k\beta^k$$

where $q > q_i$ and $\xi_q \in \{-1, 1\}$. Let $s$ be the positive integer taken in the proof of previous lemma. Then by the assertion at the end of the proof of Lemma 4.3 we have $\ell_{q_i+s} = \ell_{q+s} = \ell$. This shows that

$$\frac{u\beta^{q_i+s-1}(1-\beta)}{\tau-1} + \frac{v\beta^{q+s-1}(1-\beta)}{\tau-1} \in E \quad \text{for} \quad u, v \in \{0, 1, \dots, \ell\}.$$

By the argument shown in the previous lemma, we have for each $u \in \{0, 1, \ldots, \ell\}$

$$\begin{aligned} f_i\left(\frac{u\beta^{q_i+s-1}(1-\beta)}{\tau-1}\right) &= \frac{u\beta^{q_i+s-1}(1-\beta)}{\tau-1}\left(\beta^{q_i} + \xi_q\beta^q + \sum_{k>q}\xi_k\beta^k\right) + b_i \\ &:= \frac{(u\xi_q + b_{i(q_i+q+s)})\beta^{q_i+q+s-1}(1-\beta)}{\tau-1} + \sum_{k\neq q_i+q+s}\frac{x_k\beta^{k-1}(1-\beta)}{\tau-1} \in E, \end{aligned}$$

and

$$u\xi_q + b_{i(q_i+q+s)}, x_k \in \{0, 1, \ldots, \ell\}.$$

More precisely,

$$b_{i(q_i+q+s)} = \begin{cases} \ell & \text{if } \ \xi_{\mathrm{q}} = -1, \\ 0 & \text{if } \ \xi_{\mathrm{q}} = 1. \end{cases} \tag{4.2}$$

Note that for $v \in \{0, 1, \ldots, \ell\}$

$$\begin{aligned} &\frac{v\beta^{q+s-1}(1-\beta)}{\tau-1}\left(\beta^{q_i} + \xi_q\beta^q + \sum_{k>q}\xi_k\beta^k\right) \\ &\quad = \frac{v\beta^{q_i+q+s-1}(1-\beta)}{\tau-1} + \sum_{k\neq q_i+q+s}\frac{y_k\beta^{k-1}(1-\beta)}{\tau-1}, \end{aligned}$$

where $y_k \in \{0, \pm 1, \ldots, \pm(\tau-1)\}$. Therefore, for $u, v \in \{0, 1, \ldots, \ell\}$

$$\begin{aligned} &f_i\left(\frac{u\beta^{q_i+s-1}(1-\beta)}{\tau-1} + \frac{v\beta^{q+s-1}(1-\beta)}{\tau-1}\right) \\ &\quad = \frac{\big(u\xi_q + v + b_{i(q_i+q+s)}\big)\beta^{q_i+q+s-1}(1-\beta)}{\tau-1} + \sum_{k\neq q_i+q+s}\frac{(x_k+y_k)\beta^{k-1}(1-\beta)}{\tau-1}. \end{aligned}$$

The same argument shown in Lemma 4.3 gives $u\xi_q + v + b_{i(q_i+q+s)} \in \{0, 1, \ldots, \ell\}$ for $u, v \in \{0, 1, \ldots, \ell\}$. However, this is impossible by (4.2). ☐

**Lemma 4.5** *Using the notation of Lemma* 4.1 *we have* $\mathcal{L}(b_i) < \infty$.

Proof. As before, we write $b_i = \sum_{k=1}^{\infty} b_{ik}\beta^{k-1}(1-\beta)/(\tau-1)$ where $(b_{ik})_{k=1}^{\infty} \in \{0, 1, \ldots, \tau-1\}^{\mathbf{N}}$. Then $(b_{ik})_{k=1}^{\infty}$ can not be of the form $\mathbf{v}\overline{(\tau-1)}$ by Corollary 4.2.

Let $\{d_1, d_2, \ldots, d_m\} := \{1 \le k \le p : \ell_{k+p} \ge 1\} \subseteq \{1, \ldots, p\}$ where we assume that $d_1 < d_2 < \cdots < d_m$. Then (recall we have assumed $(\ell_k)_{k=1}^{\infty}$ is a strong $p$-periodic)

$$\ell_{d_u+p} = \ell_{d_u+np} \quad \text{for all} \quad u \in \{1, 2, \ldots, m\}, n \in \mathbf{N}.$$

Thus

$$f_i\left(\frac{\beta^{d_u+p-1}(1-\beta)}{\tau-1}\right) = \frac{(1+b_{q_i+d_u+p})\beta^{q_i+d_u+p-1}(1-\beta)}{\tau-1} + \sum_{k\neq q_i+d_u+p}\frac{b_{ik}\beta^{k-1}(1-\beta)}{\tau-1} \in E.$$

By Lemmas 2.5, 2.6 and Lemma 2.4 we know $1 + b_{q_i+d_u+p} \in \{1, \ldots, \ell_{q_i+d_u+p}\}$ and so

$$q_i + d_u + p \in \{d_v + np : v \in \{1, \ldots, m\} \quad \text{and} \quad n \in \mathbf{N}\} \quad \text{for all} \quad u \in \{1, \ldots, m\}.$$

Therefore,

$$q_i \in \bigcap_{u=1}^{m} \{d_v - d_u + np : v \in \{1, \ldots, m\} \quad \text{and} \quad n \in \mathbf{N} \cup \{0\}\}. \tag{4.3}$$

Let

$$\Re = \{\vartheta \in \{0, \ldots, p-1\} : \text{ for any } \ d_i, \text{ there exists } d_j \text{ such that } \vartheta + d_i \equiv d_j \pmod p\ \}. \tag{4.4}$$

Clearly $0 \in \Re$ and $\Re$ may contain other members other than 0, e.g., $\Re = \{0, 2\}$ when $\{d_1, d_2\} = \{2, 4\}$ and $p = 4$. Fix an $\vartheta \in \Re$. It is easy to see that for each $d_i$, there exists a unique $d_j$ satisfying $\vartheta + d_i \equiv d_j \pmod p$. This implies that

$$\{\vartheta + d_i : 1 \le i \le m\} (\operatorname{mod} p) = \{d_i : 1 \le i \le m\}. \tag{4.5}$$

By (4.3) and (4.4) we can write $q_i = \vartheta^* + k^* p$ for some $\vartheta^* \in \Re, k^* \in \mathbf{N} \cup \{0\}$. Let $\mathcal{T} : \{d_1, \ldots, d_m\} \longrightarrow \{d_1, \ldots, d_m\}$ by letting $\mathcal{T}(d_{j^*}) = d_j$ if $d_{j^*} + \vartheta^* \equiv d_j \pmod p$. $\mathcal{T}$ is a bijection by (4.5). We claim that $b_{is} = 0$ when $s > (k^* + 2)p$.

Let $s > (k^* + 2)p$ be such that $\ell_s > 0$. Then $s$ can be written as $s = d_j + \eta p$ with $\eta \ge k^* + 2$. In the following, we prove $b_{is} = 0$.

Let $d_{j^*} \in \{d_1, \ldots, d_m\}$ be such that $\mathcal{T}(d_{j^*}) = d_j$. Then $d_{j^*} + \vartheta^* = d_j$ or $d_j + p$. We take $x = \ell_{d_{j^*}+p} \beta^{d_{j^*} + (\eta - k^*)p - 1}(1 - \beta)/(\tau - 1)$ if $d_{j^*} + \vartheta^* = d_j$, or $x = \ell_{d_{j^*}+p} \beta^{d_{j^*} + (\eta - k^* - 1)p - 1}(1 - \beta)/(\tau - 1)$ if $d_{j^*} + \vartheta^* = d_j + p$. Then $x \in E$ and

$$\begin{aligned} f_i(x) = r_i x + b_i \ &= \beta^{\vartheta^* + k^* p} x + \sum_{h=1}^{\infty} \frac{b_{ih}\beta^{h-1}(1-\beta)}{\tau - 1} \\ &= \sum_{h \ne d_j + \eta p} \frac{b_{ih}\beta^{h-1}(1-\beta)}{\tau - 1} + \frac{\big(\ell_{d_{j^*}+p} + b_{i(d_j + \eta p)}\big)\beta^{d_j + \eta p - 1}(1-\beta)}{\tau - 1} \in E, \end{aligned}$$

implying $\ell_{d_{j^*}+p} + b_{i(d_j+\eta p)} \in \big\{0, 1, \ldots, \ell_{d_j + \eta p}\big\}$ by Lemmas 2.4, 2.5 and 2.6 i.e.

$$\ell_{d_{j^*}+p} + b_{is} = \ell_{d_{j^*}+p} + b_{i(\mathcal{T}(d_{j^*}) + \eta p)} \in \big\{0, 1, \ldots, \ell_{\mathcal{T}(d_{j^*}) + \eta p}\big\}.$$

So $\ell_{d_{j^*}+p} \le \ell_{\mathcal{T}(d_{j^*}) + \eta p} = \ell_{d_j + p}$. Equivalently, $\ell_{d_j + p} \ge \ell_{\mathcal{T}^{-1}(d_j) + p}$. Thus, $\ell_{d_j + p} \ge \ell_{\mathcal{T}^{-1}(d_j) + p} \ge \cdots \ge \ell_{\mathcal{T}^{-k}(d_j) + p}$ for any $k \in \mathbf{N}$. However, there exists a $1 \le k \le p$ such that $\mathcal{T}^{-k}(d_j) = d_j$. Therefore, $\ell_{d_{j^*}+p} = \ell_{\mathcal{T}(d_{j^*}) + \eta p}$ and so $b_{is} = b_{i(\mathcal{T}(d_{j^*}) + \eta p)} = 0$. ☐

Proof of Theorem 1.3. As above, by $(\ell_k)_{k=1}^{\infty}$ we denote the sequence $(\tau - 1 - |t_k|)_{k=1}^{\infty}$. Let $(\ell_k)_{k=1}^{\infty} = \mathbf{i}\overline{\mathbf{i} + \mathbf{j}} \in \{0, 1, \ldots, \tau - 1\}^{\mathbf{N}}$ with $\mathbf{i}, \mathbf{j} \in \{0, 1, \ldots, \tau - 1\}^p$ for some $p \in \mathbf{N}$.

Fix an $i \in \{1, \ldots, N\}$. Then $\mathcal{L}(b_i) < \infty$ by Lemma 4.5. Let $n \in \mathbf{N}$ be such that $np \ge \mathcal{L}(b_i)$. Thus, $\beta^{np} E \subset E$. Note that

$$f_i(\beta^{np} E) = b_i + r_i(\beta^{np} E) \subseteq f_i(E) \subseteq E.$$

We have $r_i(\beta^{np} E) \subseteq E$ since $y < \beta^{np} \le \beta^{\mathcal{L}(b_i)} < \beta^{\mathcal{L}(b_i) - 1}(1 - \beta)/(\tau - 1)$ for each $y \in r_i(\beta^{np} E)$.

In addition, $r_i(\beta^{np} E) \subset E$ means that for each $m \in \mathbf{N}$, we have $\beta^{np} r_i \ell_m \beta^{m-1}(1 - \beta)/(\tau - 1) = \ell_m \beta^{m + np + q_i - 1}(1 - \beta)/(\tau - 1) \in E$ which implies $\ell_{m + np + q_i} \ge \ell_m$. Thus $np + q_i$ is a period of $\overline{\mathbf{i} + \mathbf{j}}$ by Lemma 3.1. Let $p_0$ be the smallest period of $\overline{\mathbf{i} + \mathbf{j}}$. Then $p_0 | p$ and so $p_0 | (np + q_i)$, leading to $p_0 | q_i$, i.e., $q_i$ is a period of $(\ell_k)_{k=1}^{\infty}$.

Let $m > p + q_i$. Taking $x = \ell_{m - q_i} \beta^{m - q_i - 1}(1 - \beta)/(\tau - 1)$, we have

$$f_i(x) = \ell_{m - q_i} \beta^{m-1}(1 - \beta)/(\tau - 1) + b_i \in E.$$

As before, the sequence $(b_{ik})_{k=1}^{\infty} \preceq (\ell_k)_{k=1}^{\infty}$ denotes the unique $\{0, 1, \ldots \tau - 1\}$-code of $b_i$. By Lemmas 2.5, 2.6 and 2.9 (IV), $\ell_{m - q_i} + b_{im} = \ell_m + b_{im} \le \ell_m$ which implies $b_{im} = 0$ for all $m > p + q_i$. ☐

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