Generating Iterated Function Systems for the Vicsek Snowflake and the Koch Curve

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Abstract. We determine all generating iterated function systems for certain self-similar sets such as the Vicsek snowflake and the Koch curve.

1. INTRODUCTION. Our work is motivated by a basic problem in fractal geometry: How does one find all generating iterated function systems (IFSs) for a self-similar set? Applications of IFS can be seen in reptiles [**2**] and image compression [**1, 3, 6**].

We call a nonempty compact set $F \subseteq \mathbb{R}^d$ a *self-similar set* if it is a finite union of its self-similar copies; that is, there exists a family of contractive similitudes $\mathcal{F} = \{\phi_i(\vec{x}) = \rho_i U_i \vec{x} + \vec{b_i}\}_{i=1}^N$ (N is an integer no smaller than 2) such that $F = \bigcup_{i=1}^N \phi_i(F)$, where $\rho_i \in (0, 1)$, U_i is an orthonormal $d \times d$ matrix, and $\vec{b_i}$ is a translation vector. The family $\mathcal{F}$ is called a generating IFS for F. It is well-known that $\mathcal{F}$ determines F uniquely, but the converse is not true.

Investigating all generating IFSs for a self-similar set was first done by Feng and Wang in $\mathbb{R}$ [**5**]. However, in the higher-dimensional case, the situation is somewhat different since the form of an orthonormal matrix is much more complicated. The discussion is limited either to homogeneous IFSs (all $\rho_i U_i$ are the same) with the strong separation condition [**4**] or to special kinds of planar self-similar sets [**8**].

In this note, we first give an easy-to-check theorem. We then use it to deal with all generating IFSs for some self-similar sets that cannot be covered by the above works.

We denote by $\mathcal{I}_E$ the collection of all isometries of a set $E \subseteq \mathbb{R}^d$. Readers can refer to [**7**] for more information about $\mathcal{I}_E$. By $f_{\mathbf{i}}$, we mean $f_{i_1} \circ \cdots \circ f_{i_\ell}$ if $\mathbf{i} = i_1 \dots i_\ell$ is a finite sequence in $\bigcup_{k=1}^\infty \{1, \dots, N\}^k$, which is the set of all finite words over $\{1, \dots, N\}$. Then we have the following.

Theorem. *Let $E \subseteq \mathbb{R}^d$ be the self-similar set generated by an IFS $\{\phi_i(x)\}_{i=1}^N$. Assume that for each contractive similitude $\phi(x)$ with $\phi(E) \subseteq E$, we have $\phi(E) \subseteq \phi_i(E)$ for some $i \in \{1, \dots, N\}$. Then every contractive similitude ψ satisfying $\psi(E) \subseteq E$ can be written as $\phi_{\mathbf{i}} \circ S$ for some $\mathbf{i} \in \bigcup_{k=1}^\infty \{1, \dots, N\}^k$ and $S \in \mathcal{I}_E$.*

Remark. As an application, we investigate all generating IFSs for the triadic Cantor set $\mathbf{C}$ generated by the IFS $\{\phi_1(x) = x/3, \phi_2(x) = (x + 2)/3\}$. Note that each contractive similitude ϕ with $\phi(\mathbf{C}) \subseteq \mathbf{C}$ satisfies $\phi(\mathbf{C}) \subseteq \phi_i(\mathbf{C})$ for some $i \in \{1, 2\}$, and $\mathcal{I}_{\mathbf{C}} = \{x, 1 - x\}$. Then by the **Theorem**, every $\psi_k(x)$ in a generating IFS $\{\psi_k\}_{k=1}^\ell$ for $\mathbf{C}$ must be of the form $\phi_{\mathbf{i}}(x)$ or $\phi_{\mathbf{i}}(1 - x)$.

2. PROOF OF THEOREM AND SOME EXAMPLES.

Proof of Theorem. Suppose that $\psi(E) \subseteq \phi_{i_1}(E)$ for some $i_1 \in \{1, \dots, N\}$. Then $\phi_{i_1}^{-1} \circ \psi(E) \subseteq E$, so either $\phi_{i_1}^{-1} \circ \psi(E) = E$ or, by repeating the above process, $\phi_{i_2}^{-1} \circ$

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$\phi_{i_1}^{-1} \circ \psi(E) \subseteq E$ for some $i_2 \in \{1, \ldots, N\}$. We can obtain by induction that $\phi_{\mathbf{i}}^{-1} \circ \psi(E) = E$ for some $\mathbf{i} \in \bigcup_{k=1}^{\infty}\{1, \ldots, N\}^k$, and hence, $\phi_{\mathbf{i}}^{-1} \circ \psi \in \mathcal{I}_E$. ■

Let E be the self-similar set generated by an IFS $\{\phi_i\}_{i=1}^N$. Assume that F is a compact set satisfying $\phi_i(F) \subseteq F$ for all $1 \leq i \leq N$. Then the following fact about self-similar sets will be used in all our examples:

$$E = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{i}\in\{1,\ldots,N\}^k} \phi_{\mathbf{i}}(F) \subseteq F. \tag{1}$$

The first example is *Vicsek snowflake*, a just-touching self-similar set.

Example 1. The Vicsek snowflake V (Figure 1) is generated by the IFS $\{f_i\}_{i=1}^5$, where

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix}, f_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}, f_3\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix},$$
$$f_4\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}, f_5\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Suppose that ψ is a contractive similitude in $\mathbb{R}^2$ with $\psi(V) \subseteq V$. Then $\psi = f_{\mathbf{i}} \circ S$ for some $\mathbf{i} \in \bigcup_{k=1}^{\infty}\{1, \ldots, 5\}^k$ and $S \in \mathcal{I}_V$.

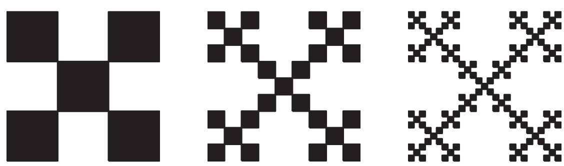

Figure 1. The first three iterations of Vicsek snowflake

Proof. By the **Theorem**, it suffices to prove that $\psi(V) \subseteq f_i(V)$ for some $i \in \{1, \ldots, 5\}$.

By (1), V is a subset of $[0, 1] \times [0, 1]$ and contains its two diagonals $\overline{AC}$ and $\overline{BD}$ (Figure 2). We claim that each line segment I in V is parallel to $\overline{AC}$ or $\overline{BD}$.

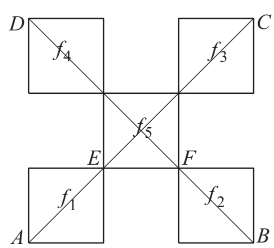

Figure 2. The diagonals of $[0, 1] \times [0, 1]$ are in the Vicsek snowflake.

Assume that $I \subseteq f_{\mathbf{i}}(V)$ for some $\mathbf{i} \in \cup_{k=0}^{\infty}\{1, \ldots, 5\}^k$ and $I \not\subseteq f_{\mathbf{i}i}(V)$ for all $i \in \{1, \ldots, 5\}$ with $f_{\{1,\ldots,5\}^0}$ being the identity. Then $f_{\mathbf{i}}^{-1}(I)$ is a line segment in V and it intersects at least two different $f_i([0, 1] \times [0, 1])$s.

Let M and N be the two endpoints of $f_{\mathbf{i}}^{-1}(I)$. We claim that $\overline{MN}$ lies on either $\overline{AC}$ or $\overline{BD}$. We can reduce its proof into the following two cases.

Case 1. $M \in f_1(V)$ and $N \in f_2(V)$. Then $\overline{EF} \subseteq \overline{MN}$, which contradicts the fact that $\overline{EF} \cap V$ is the Cantor set and $\overline{MN}$ is a line segment in V.

Case 2. $M \in f_1(V)$ and $N \in f_5(V)$. In this case, $E \in \overline{MN}$. By $f_5^{-1}(E) = A$, we can assume $\overline{EN} \subseteq f_{51^{*m}}(V)$ and $\overline{EN} \not\subseteq f_{51^{*(m+1)}}(V)$ for some nonnegative integer m, where 1^{*m} is the word by repeating 1 a total of m times. Then $f_{51^{*m}}^{-1}(\overline{EN})$ is a subset of V and it intersects both $f_1(V)$ and some $f_i(V)$ with $i \neq 1$, so it lies on $\overline{AC}$, giving $\overline{EN} \subseteq f_{51^{*m}}(\overline{AC}) \subseteq \overline{AC}$. Therefore, $\overline{MN} \subseteq \overline{AC}$, proving our claim.

The two diagonals of $\psi([0,1] \times [0,1])$ are in V; hence, they are parallel to $\overline{AC}$ and $\overline{BD}$, respectively. Thus, all sides of $\psi([0,1] \times [0,1])$ are parallel to either $\overline{AB}$ or $\overline{CD}$. Suppose that $\psi(V) \not\subseteq f_i(V)$ for all $i \in \{1, \ldots, 5\}$. Then the four vertices of $\psi([0,1] \times [0,1])$ are in four different regions of the form $f_i([0,1] \times [0,1])$ with $i \neq 5$. Therefore, $\psi(\overline{AB} \cap V)$ is a scaled copy of the Cantor set and contains a gap of length at least $\frac{1}{3}$, which is impossible as ψ is a contractive similitude. ■

The second example is a self-similar set for which substantial overlaps occur.

Example 2. The self-similar set W is generated by the IFS $\{g_1, g_2, g_3\}$ where

$$g_1\begin{pmatrix} x \\ y \end{pmatrix} = \rho\begin{pmatrix} x \\ y \end{pmatrix}, g_2\begin{pmatrix} x \\ y \end{pmatrix} = \rho\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \rho^2 \\ 0 \end{pmatrix}, g_3\begin{pmatrix} x \\ y \end{pmatrix} = \rho^2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

with $\rho = \frac{\sqrt{5}-1}{2}$. Suppose that ψ is a contractive similitude in $\mathbb{R}^2$ with $\psi(W) \subseteq W$. Then $\psi = g_{\mathbf{i}}$ for some $\mathbf{i} \in \bigcup_{k=1}^{\infty}\{1, 2, 3\}^k$.

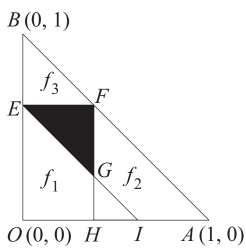

Figure 3a.

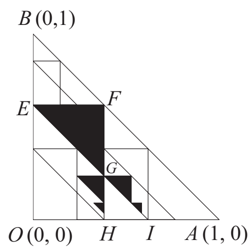

Figure 3b.

Proof. We point out that $g_{122} = g_{211}$, which is a complete overlap. Note that $\mathcal{I}_W = \{\text{identity}\}$. By the **Theorem**, we only need to show that $\psi(W) \subseteq g_i(W)$ for some $i \in \{1, 2, 3\}$. Figure 3 may help in following the proof.

Using (1), we can check that W is a subset of ΔOAB and contains all its three sides but no points of $(\Delta EFG)^\circ$, where $(\Delta EFG)^\circ$ is the interior of ΔEFG (Figure 3a).

We first prove that $\psi(\Delta OAB) \subseteq g_i(\Delta OAB)$ for some $i \in \{1, 2, 3\}$. Suppose otherwise. Let $\overline{P_1P_2} \subseteq W$ be one side of $\psi(\Delta OAB)$; there are two cases to consider.

Case 1. $P_1 \in \Delta BEF \setminus \{E, F\}$, $P_2 \in (\Delta EOI \setminus \{E\}) \cup (\Delta FHA \setminus \{F\})$.

Then $\overline{P_1P_2}$ lies on either $\overline{BO}$ or $\overline{BA}$, say $\overline{BO}$ since it cannot intersect $(\Delta EFG)^\circ$. Reasoning similarly, the third vertex of $\psi(\Delta OAB)$ lies on $\overline{BA}$. So P_1 equals B and $\Delta EFG \subseteq \psi(\Delta OAB)$, which contradicts the fact that ψ is a contractive similitude.

Case 2. $P_1 \in \Delta EOI \setminus \Delta GHI$, $P_2 \in \Delta FHA \setminus \Delta GHI$.

Recall that $(\Delta EFG)^\circ$ contains no points of W. So does $\left(g^k(\Delta EFG)\right)^\circ$ by similarity, where $g = g_{12}$ or g_{21}, and g^k is the kth iteration of g.

Notice that $g^k(\Delta EFG)$ connect each other one by one. They are located along the line FH (or EI) and approach to the point H (or I) as $k \to \infty$. Thus, either $\overline{P_1P_2}$ or another side of $\psi(\Delta OAB)$ passes through Y_n for some n (Figure 3b gives the case Y_2), where

$$Y_n = \Big(\bigcup_{k=0}^{n} g_{12}^k(\Delta EFG)\Big)^\circ \cup \Big(\bigcup_{k=0}^{n} g_{21}^k(\Delta EFG)\Big)^\circ.$$

This is impossible since all three sides of $\psi(\Delta OAB)$ are in W.

Having known that $\psi(\Delta OAB) \subseteq g_i(\Delta OAB)$ for some $i \in \{1, 2, 3\}$, we will prove that $\psi(W) \subseteq g_i(W)$ by verifying $g_1(W) \cap \Delta GHI = g_2(W) \cap \Delta GHI$.

It follows from $g_{122} = g_{211}$ that $g_{21}^n \circ g_1 = g_1 \circ g_{22}^n$. Moreover, for each $n \in \mathbb{N}$, we can check that $g_{21}^n \circ g_3(O)$ lies in $\overline{EI}$, $g_{21}^n \circ g_{22}(O)$ equals I, and $g_{21}^n \circ g_{23}(O)$ lies over $\overline{EI}$. So

$$\begin{aligned}
&\Delta GHI \cap g_{21}^n(W) = \Delta GHI \cap \big(g_1 \circ g_{22}^n(W) \cup g_{21}^n \circ g_2(W) \cup g_{21}^n \circ g_3(W)\big) \\
&\subseteq g_1(W) \cup \big(\Delta GHI \cap g_{21}^n \circ g_2(W)\big) \\
&= g_1(W) \cup \Big(\Delta GHI \cap \big(g_{21}^{n+1}(W) \cup g_{21}^n \circ g_{22}(W) \cup g_{21}^n \circ g_{23}(W)\big)\Big) \\
&\subseteq g_1(W) \cup \big(\Delta GHI \cap g_{21}^{n+1}(W)\big).
\end{aligned}$$

Then

$\Delta GHI \cap g_2(W) = \Delta GHI \cap g_{21}(W) \subseteq g_1(W) \cup \big(\Delta GHI \cap g_{21}^n(W)\big)$.

Therefore, $\Delta GHI \cap g_2(W) \subseteq g_1(W)$ as $\bigcap_{n=1}^\infty g_{21}^n(W) = \{I\} \subseteq g_1(W)$. Using a similar argument yields $\Delta GHI \cap g_1(W) \subseteq g_2(W)$, which completes the proof. ■

All generating IFSs for the above examples can be iterated from their defining IFS. However, the famous *Koch curve*, which is a fractal invented by the Swedish mathematician Helge von Koch in 1904, is one of the exceptions.

Start with $E_0 = [0, 1]$. Let E_1 be the set consists of the four line segments obtained by replacing the middle third of E_0 by the other two sides of the equilateral triangle based on the removed segment. Inductively, we construct E_k by applying the same procedure to each line segments in E_{k-1} (Figure 4). Finally, we arrive at the Koch curve.

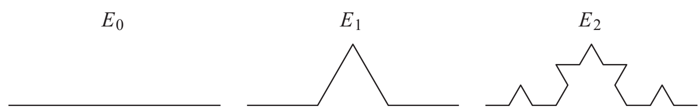

Figure 4. E_k with $k = 0, 1, 2$

Example 3. The Koch curve K is generated by the IFS $\{h_1, h_2\}$ where

$$h_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{\sqrt{3}}{6}y \\ \frac{\sqrt{3}}{6}x - \frac{y}{2} \end{pmatrix} \text{ and } h_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{x}{2} - \frac{\sqrt{3}}{6}y + 1 \\ \frac{\sqrt{3}}{6}x - \frac{y}{2} \end{pmatrix}.$$

Suppose that ψ is a contractive similitude in $\mathbb{R}^2$ with $\psi(K) \subseteq K$. Then $\psi = h_{\mathbf{i}}$ or $h_{\mathbf{i}} \circ T$ for some $\mathbf{i} \in \bigcup_{k=1}^\infty \{1, 2, 3\}^k$, where

$$h_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{3}+\frac{1}{3} \\ \frac{y}{3}+\frac{\sqrt{3}}{9} \end{pmatrix} \text{ and } T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-x \\ y \end{pmatrix}.$$

Proof. Note that h_1 and h_2 map $\triangle ABC$ to $\triangle DBA$ and $\triangle ECA$, respectively (Figure 5). Owing to (1), we have $K \subseteq \triangle ABC$. By the **Theorem**, it remains to prove that $\psi(K) \subseteq h_i(K)$ for some $i \in \{1, 2, 3\}$ as $\mathcal{I}_K = \{\text{identity}, T\}$.

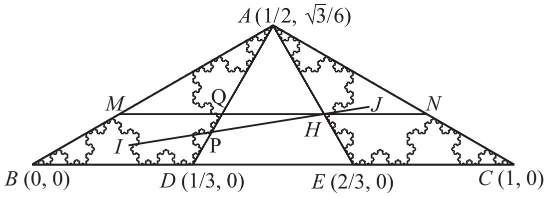

Figure 5. Possible positions of vertices of $\psi(\Delta ABC)$

Suppose that $\psi(K) \not\subseteq h_i(K)$ for $i \in \{1, 2\}$. Denote the three vertices of $\psi(\Delta ABC)$ by I, J and L. Without loss of generality, we can assume $I \in \triangle ABD \setminus \{A\}$ and $J \in \triangle AEC \setminus \{A\}$. Let H be the intersection point of $\overline{IJ}$ and $\overline{AE}$. Let $\overline{MN}$ be the line segment passing through H and parallel to $\overline{BC}$ with $M \in \overline{AB}$ and $N \in \overline{AC}$.

Note that $K \cap \overline{AB}$ (also $K \cap \overline{AC}$ and $K \cap \overline{BC}$) is a similar copy of the triadic Cantor set $\mathbf{C}$. So is $\psi(K) \cap \overline{IJ}$. Therefore, $|\overline{PH}| \leq \min\{|\overline{HJ}|, |\overline{PI}|\}$, which implies $I = M$ and $J = N$. Otherwise, as we can see in Figure 5, $|\overline{HJ}| < |\overline{HN}| = |\overline{QH}| < |\overline{PH}|$, a contradiction!

Finally, we claim that $L = A$. Otherwise, either $\overline{LM}$ or $\overline{LN}$ is parallel to $\overline{BC}$ by applying the same argument as above, which is impossible.

Now we get that $\psi(K \cap \overline{AB})$ is a subset of $K \cap \overline{AM}$ or $K \cap \overline{AN}$. We only consider the former case. A similar proof works for the latter case.

Notice that $K \cap \overline{AB} = \mathbf{C}/\sqrt{3}$. Letting $|\overline{AM}|/|\overline{AB}| = \rho$ yields $\rho \cdot \left(\mathbf{C}/\sqrt{3}\right) \subseteq \mathbf{C}/\sqrt{3}$ or, equivalently, $\rho\mathbf{C} \subseteq \mathbf{C}$. By the **Remark** in the introduction, we have $\rho = 3^{-n}$ for some positive integer n. Thus, $\psi(K) \subseteq h_3(K)$, which finishes our proof. ■

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