Nonlinearity 21 (2008) 1745–1758 doi:10.1088/0951-7715/21/8/004

A random version of McMullen–Bedford general Sierpinski carpets and its application

Yongxin Gui[1,2] **and Wenxia Li**[1]

[1] Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China
[2] Department of Mathematics, Xianning College, Xianning 437005, People's Republic of China

E-mail: wxli@math.ecnu.edu.cn

Received 8 September 2007, in final form 28 May 2008
Published 26 June 2008
Online at stacks.iop.org/Non/21/1745

Recommended by C P Dettmann

Abstract
We consider a random version of the McMullen–Bedford general Sierpinski carpet which is constructed by randomly choosing patterns in each step instead of a single pattern in its original form. Their Hausdorff, packing and box-counting dimensions are determined. A sufficient condition and a necessary condition for the Hausdorff measures in their dimensions to be positive are given. As an application, we discuss the issue on the intersection of the general Sierpinski carpet with its translations.

Mathematics Subject Classification: 28A80, 28A78

1. Introduction

Let T be an expanding endomorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by a matrix $\mathrm{diag}(n, m)$, where $2 \leqslant m < n$ are integers. The simplest invariant sets for T have the form

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} \mathrm{diag}(n^{-1}, m^{-1})^k d_k : \ d_k \in D \text{ for all } k \geqslant 1 \right\}, \tag{1}$$

where $D \subseteq I \times J$ is the set of digits with $I = \{0, 1, \cdots, n-1\}$ and $J = \{0, 1, \cdots, m-1\}$. The geometric description of this construction is the following: we divide the unit square into $n \times m$ congruent rectangles by drawing n vertical strips of equal width and m horizontal strips of equal height, choose some rectangles according to the pattern described by D, again divide each chosen rectangle into $n \times m$ congruent ones, choose the rectangles according to the same pattern D and repeat the procedure inductively. Then we obtain the limit set $K(T, D)$, the *general Sierpinski carpet* which was first studied independently by McMullen [31] and Bedford [2].

For $d \in \mathbb{Z}^2$ let

$$f_d(x) = T^{-1}(x + d) = \mathrm{diag}(n^{-1}, m^{-1})(x + d), \qquad x \in \mathbb{R}^2. \tag{2}$$

Then each f_d is contractive. For a finite subset Ω of $\mathbb{Z}^2$ let $\Pi : \Omega^{\mathbb{N}} \longrightarrow \mathbb{R}^2$ be defined by

$$\Pi(\boldsymbol{d}) := \sum_{k=1}^{\infty} \mathrm{diag}(n^{-1}, m^{-1})^k d_k \qquad \text{for } \boldsymbol{d} = (d_k)_{k\geqslant 1} \in \Omega^{\mathbb{N}}. \tag{3}$$

Clearly, we have $\Pi(D^{\mathbb{N}}) = K(T, D)$ and $\Pi((I \times J)^{\mathbb{N}}) = [0, 1]^2$. Some simple observations are as follows:

(I) $\Pi(\boldsymbol{d}) = \lim_{\ell\to\infty} \sum_{k=1}^{\ell} \mathrm{diag}(n^{-1}, m^{-1})^k d_k = \lim_{\ell\to\infty} f_{d_1} \circ f_{d_2} \circ \cdots \circ f_{d_\ell}(0)$;

(II) the set $\Pi(\Omega^{\mathbb{N}})$ is the unique nonempty compact invariant set of the family $\{f_d : d \in \Omega\}$ of contractive affine maps, i.e.

$$\Pi(\Omega^{\mathbb{N}}) = \bigcup_{d\in\Omega} f_d(\Pi(\Omega^{\mathbb{N}})).$$

A nonempty compact set satisfying the above equality is called a *self-affine set*. A family $\{f_d : d \in \Omega\}$ is said to satisfy the *open set condition* if there exists a nonempty bounded open set $O \subset \mathbb{R}^2$ such that $\bigcup_{d\in\Omega} f_d(O) \subset O$ with disjoint union on the left side. For example, when $\Omega \subset I \times J$ the family $\{f_d : d \in \Omega\}$ satisfies the open set condition with respect to the open set $O = (0, 1)^2$.

For $x \in \Pi(\Omega^{\mathbb{N}})$, a sequence $\boldsymbol{d} = (d_k)_{k\geqslant 1} \in \Omega^{\mathbb{N}}$ is called an Ω-*code* of x if $\Pi(\boldsymbol{d}) = x$. Thus each point in $\Pi(\Omega^{\mathbb{N}})$ has at least one Ω-code.

Let proj_y denote the projection of $\mathbb{R}^2$ onto its second coordinate. For each $b \in \mathrm{proj}_y D$ put $n_b = |D \cap (I \times \{b\})|$ where, and throughout this paper, $|A|$ denotes the cardinality of A. We denote by $\dim_{\mathrm{H}} E$, $\dim_{\mathrm{P}} E$ and $\dim_{\mathrm{B}} E$ the Hausdorff, packing and box-counting dimensions of the set E, respectively. One can refer to [8, 29] for their definitions. Let

$$\alpha := \log_n m.$$

Some known results are (cf [2, 31, 35])

$$\dim_{\mathrm{H}} K(T, D) = \log_m \sum_{b\in \mathrm{proj}_y D} n_b^{\alpha} = \log_m \sum_{d\in D} n_{\mathrm{proj}_y d}^{\alpha-1}, \tag{4}$$

and

$$\dim_{\mathrm{B}} K(T, D) = \dim_{\mathrm{P}} K(T, D) = \log_m(|\mathrm{proj}_y D|^{1-\alpha} |D|^{\alpha}). \tag{5}$$

In the recent years, some further problems related to the general Sierpinski carpet $K(T, D)$ and its various modifications have been proposed and considered by lots of authors. For instance, readers can refer to Peres [35, 36], Kenyon and Peres [17, 18], King [20], Olsen [34], Barański [1], Gatzouras and Lalley [11], Gui and Li [13], etc just to list a few.

Many random constructions related to the self-similar sets have been studied in [6, 9, 10, 28, 30] and elsewhere. In 1994, Gatzouras and Lalley [12] studied the randomization of the general Sierpinski carpet by means of branching processes. They gave exact expressions for the Hausdorff and box dimensions of the random general Sierpinski carpet. In this paper, we consider a more straightforward random version of the general Sierpinski carpet which, compared with the geometric construction of $K(T, D)$, is constructed by randomly choosing patterns in different levels. This is motivated by an issue on the intersection of the general Sierpinski carpet with its translations (we will discuss the details later in this section). We denote by $\mathcal{M}$ the set of all nonempty subsets of $I \times J$. We first divide the unit square into $n \times m$ congruent rectangles by drawing n vertical strips of equal width and m horizontal strips of equal height and choose some rectangles according to some randomly chosen pattern from $\mathcal{M}$, say Γ_1, again divide each of the chosen rectangles in the first level into $n \times m$ congruent ones and choose some rectangles according to some randomly chosen pattern from $\mathcal{M}$, say Γ_2, repeat the procedure inductively according to some randomly chosen pattern from $\mathcal{M}$, say Γ_k at the kth level. The pattern Γ_k at the kth level is chosen independent of all the previous patterns Γ_ℓ, $1 \leqslant \ell \leqslant k-1$, and is applied to all chosen $n^{k-1} \times m^{k-1}$ rectangles. We would like to point out that in [12] Gatzouras and Lalley used independently chosen patterns $\Gamma_{k,i}$ applying to each of the chosen $n^{k-1} \times m^{k-1}$ rectangles at the kth level. Let U_k be the union of all the rectangles chosen at the kth level. Then $(U_k)_{k\geqslant 1}$ is a sequence of decreasing compact sets of $[0, 1]^2$. Thus, the compact set constructed above is just the limit set of U_k, i.e. $\bigcap_{k\geqslant 1} U_k$. Alternatively, for such a (compact) set K, if $(\Gamma_k)_{k=1}^{\infty}$ are the patterns chosen in the above process then

$$K := K((\Gamma_k)_{k\geqslant 1}) = \Pi\left(\prod_{k=1}^{\infty} \Gamma_k\right) = \left\{\sum_{k=1}^{\infty} \operatorname{diag}(n^{-1}, m^{-1})^k d_k : \ d_k \in \Gamma_k \text{ for all } k \geqslant 1\right\}. \tag{6}$$

Now, take $\{D_1, \ldots, D_s\} \subset \mathcal{M}$ with $1 \leqslant s \leqslant 2^{mn-1}$. We endow $\{D_1, \ldots, D_s\}^{\mathbb{N}}$ with a probability measure $\mathbb{P}$. For a fixed probability vector $(p_i)_{i=1}^{s}$ (i.e. $p_i \in [0, 1]$ and $\sum_{i=1}^{s} p_i = 1$), let

$$\mathbb{P} := \prod_{\mathbb{N}} \left(\sum_{i=1}^{s} p_i \delta_i\right), \tag{7}$$

where δ_i denotes the Dirac measure concentrated at D_i. Thus, each $\omega = (D_{\omega(k)})_{k\geqslant 1} \in \{D_1, \ldots, D_s\}^{\mathbb{N}}$ corresponds to a compact set $K(\omega) = K((D_{\omega(k)})_{k\geqslant 1})$ defined by (6). In particular, when $s = 1$, $\{D_1\}^{\mathbb{N}}$ is a singleton and $K(\omega) = K(T, D_1)$—the general Sierpinski carpet. The main result presented in this paper is the following theorem.

Theorem 1.1. *Let $\mathbb{P}$ be defined as in (7). For $\mathbb{P}$-a.e. $\omega \in \{D_1, \ldots, D_s\}^{\mathbb{N}}$*

$$\dim_{\mathrm{H}} K(\omega) = \sum_{i=1}^{s} p_i \dim_{\mathrm{H}} K(T, D_i),$$

and

$$\dim_{\mathrm{B}} K(\omega) = \dim_{\mathrm{P}} K(\omega) = \sum_{i=1}^{s} p_i \dim_{\mathrm{B}} K(T, D_i) = \sum_{i=1}^{s} p_i \dim_{\mathrm{P}} K(T, D_i),$$

where $K(T, D_i)$ is defined as in (1).

Investigation of $K(\omega)$, in a sense, is motivated by the following issue related to the intersection of the general Sierpinski carpet with its translation. For the self-similar case, this has been the subject of several studies (cf [5, 7, 16, 19, 21–24, 26, 27, 32, 39]).

Let $K(T, D)$ be defined as in (1). For $t \in \mathbb{R}^2$, it is easy to see that

$$K(T, D) \cap (K(T, D) + t) \neq \emptyset \ \text{ if and only if } \ t \in K(T, D) - K(T, D),$$

where $K(T, D) - K(T, D) := \{x - y : x, y \in K(T, D)\}$. Let $D - D := \{x - y : x, y \in D\}$. Then, by (3)

$$K(T, D) - K(T, D) = \Pi(D^{\mathbb{N}}) - \Pi(D^{\mathbb{N}}) = \Pi((D - D)^{\mathbb{N}}).$$

Thus $K(T, D) - K(T, D)$ is a self-affine set generated by the family $\{f_d : d \in D - D\}$ of contractive affine maps, i.e.

$$K(T, D) - K(T, D) = \bigcup_{d\in D-D} f_d(K(T, D) - K(T, D)).$$

Each point in $K(T, D) - K(T, D)$ has at least one $(D - D)$-code and some points may have infinite number of $(D - D)$-codes.

For $t \in K(T, D) - K(T, D)$ we denote by $\mathcal{C}(t)$ the set of all $(D - D)$-codes of t, i.e. $\mathcal{C}(t) = \Pi^{-1}(t) = \{\boldsymbol{d} \in (D - D)^{\mathbb{N}} : \Pi(\boldsymbol{d}) = t\}$. For a $\boldsymbol{d} = (d_k)_{k\geqslant 1} \in \mathcal{C}(t)$ let $\Gamma_k = D \cap (D + d_k)$. Then each Γ_k is nonempty subset of D. Thus, each $\boldsymbol{d} = (d_k)_{k\geqslant 1} \in \mathcal{C}(t)$ uniquely determines such a sequence $(\Gamma_k)_{k\geqslant 1}$ of subsets of D, denoted by $\omega_{\boldsymbol{d}}$. We claim that

$$K(T, D) \cap (K(T, D) + t) = \bigcup_{\boldsymbol{d}\in\mathcal{C}(t)} K(\omega_{\boldsymbol{d}}), \tag{8}$$

where, according to the notation described above, $K(\omega_{\boldsymbol{d}}) = \Pi(\prod_{k=1}^{\infty} \Gamma_k)$ (so $K(T, D) \cap (K(T, D) + t)$ is then related to the sets $K(\omega_{\boldsymbol{d}})$ which are treated in theorem 1.1). In fact, each $K(\omega_{\boldsymbol{d}}) \subset K(T, D) \cap (K(T, D) + t)$ since

$$K(\omega_{\boldsymbol{d}}) = \left\{ \sum_{k=1}^{\infty} \operatorname{diag}(n^{-1}, m^{-1})^k u_k : u_k \in \Gamma_k = D \cap (D + d_k) \subset D \right\}.$$

On the other hand, for any $x \in K(T, D) \cap (K(T, D) + t)$ there exist $(u_k)_{k\geqslant 1}, (v_k)_{k\geqslant 1} \in D^{\mathbb{N}}$ such that

$$x = \sum_{k=1}^{\infty} \operatorname{diag}(n^{-1}, m^{-1})^k u_k,$$

and

$$t = \sum_{k=1}^{\infty} \operatorname{diag}(n^{-1}, m^{-1})^k u_k - \sum_{k=1}^{\infty} \operatorname{diag}(n^{-1}, m^{-1})^k v_k = \sum_{k=1}^{\infty} \operatorname{diag}(n^{-1}, m^{-1})^k (u_k - v_k).$$

Taking $d_k = u_k - v_k$, we have $\boldsymbol{d} = (d_k)_{k\geqslant 1} \in \mathcal{C}(t)$ and $u_k \in D \cap (D + d_k)$ which implies that $x \in K(\omega_{\boldsymbol{d}})$.

Therefore, by (8) one can determine the Hausdorff, packing and box-counting dimensions of $K(T, D) \cap (K(T, D) + t)$ if $\mathcal{C}(t)$ is at most countable. When the family $\{f_d : d \in D - D\}$ satisfies the open set condition, $\mathcal{C}(t)$ is a finite set for each $t \in K(T, D) - K(T, D)$. An easy condition for the family $\{f_d : d \in D - D\}$ to satisfy the open set condition is that $\max\{|\operatorname{proj}_x(d_1 - d_2)|, |\operatorname{proj}_y(d_1 - d_2)|\} \geqslant 2$ for any distinct $d_1, d_2 \in D - D$. This can be simply verified by taking $(-1, 1)^2$ as the open set.

The rest of this paper is organized as follows. In section 2, some basic facts and known results needed in the proof of our theorems are described. The proof of theorem 1.1 is arranged in section 3. We focus on those $K(\omega)$ with $\omega \in \{D_1, \ldots, D_s\}^{\mathbb{N}}$ for which each D_k occurs in ω with frequency p_k, $1 \leqslant k \leqslant s$. As an application of theorem 1.1, we consider the dimensions of the intersection of $K(T, D)$ with its translations. In section 4, a necessary condition and a sufficient condition are obtained, respectively, for the Hausdorff measure of $K(\omega)$ in its dimension to be positive (see theorem 4.1).

2. Preliminaries

Following [11, 31, 33, 35, 36], we use approximate squares to calculate dimension. For each $\boldsymbol{x} = (x_j)_{j=1}^{\infty} \in (I \times J)^{\mathbb{N}}$ and each positive integer k, let

$$Q_k(\boldsymbol{x}) = \{\Pi(\boldsymbol{y}) : \boldsymbol{y} = (y_j)_{j=1}^{\infty} \in (I \times J)^{\mathbb{N}},\ y_j = x_j \ \text{ for } 1 \leqslant j \leqslant [\alpha k] \ \text{ and } \operatorname{proj}_y y_j = \operatorname{proj}_y x_j \text{ for } [\alpha k] + 1 \leqslant j \leqslant k\},$$

where here, and throughout this paper, $[x]$ with $x \in \mathbb{R}$ denotes the greatest integer function. The rectangles $Q_k(\boldsymbol{x})$ are approximate squares in $[0, 1]^2$, whose sides have length $n^{-[\alpha k]}$ and m^{-k}. Note that the ratio of the sides of $Q_k(\boldsymbol{x})$ is at most n, and their diameters $\operatorname{diam} Q_k(\boldsymbol{x})$ satisfy

$$\sqrt{2} m^{-k} \leqslant \operatorname{diam} Q_k(\boldsymbol{x}) \leqslant \sqrt{2} n m^{-k}.$$

So in the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares $Q_k(\boldsymbol{x})$. The following lemma appears in [33] in which the approximate square $Q_k(\boldsymbol{x})$ behaves as an analogue as the ball does in the classical density theorems. It is just a reformulation of the Rogers–Taylor density theorem as stated by Peres in [36, section 2].

Lemma 2.1 [33, lemma 4]. Suppose that μ is a finite Borel measure on $[0, 1]^2$, and that E is a subset of $(I \times J)^{\mathbb{N}}$ such that $\Pi(E)$ is a Borel subset of $[0, 1]^2$, and $\mu(\Pi(E)) > 0$. Let δ be a positive number. For each point $\boldsymbol{x} \in E$, put

$$A(\boldsymbol{x}) = \limsup_{k\to\infty} (k\delta + \log_m \mu(Q_k(\boldsymbol{x}))).$$

(1) If $A(\boldsymbol{x}) = -\infty$ for all $x \in E$, then $\mathcal{H}^\delta(\Pi(E)) = +\infty$;
(2) If $A(\boldsymbol{x}) = +\infty$ for all $x \in E$, then $\mathcal{H}^\delta(\Pi(E)) = 0$;
(3) If there are real numbers a and b such that $a \leqslant A(\boldsymbol{x}) \leqslant b$ for all $x \in E$, then $0 < \mathcal{H}^\delta(\Pi(E)) < +\infty$.

For a fixed $\omega = (D_{\omega(k)})_{k\geqslant 1} \in \{D_1, \ldots, D_s\}^{\mathbb{N}}$, in order to apply lemma 2.1 to $K(\omega) = \Pi(\prod_{k=1}^{\infty} D_{\omega(k)})$ we first construct a finite Borel measure on $\prod_{k=1}^{\infty} D_{\omega(k)} \subset (I \times J)^{\mathbb{N}}$. For each $1 \leqslant i \leqslant s$, let $\boldsymbol{p}(i) = (p_{d,\,i})_{d\in D_i}$ be a probability vector on D_i, i.e. $\sum_{d\in D_i} p_{d,\,i} = 1$ with each $p_{d,\,i} \in (0, 1)$ (note that we require each $p_{d,\,i}$ is positive!). This, for each $1 \leqslant i \leqslant s$, induces a probability vector $(q_{b,\,i})_{b\in \mathrm{proj}_y D_i}$ on $\mathrm{proj}_y D_i$ by letting

$$q_{b,\,i} = \sum_{d\in D_i,\ \mathrm{proj}_y d=b} p_{d,\,i}.$$

Denote $\boldsymbol{p} = (\boldsymbol{p}(1), \boldsymbol{p}(2), \ldots, \boldsymbol{p}(s))$. A Borel probability measure $\mu_{\boldsymbol{p}}$ on $\prod_{k=1}^{\infty} D_{\omega(k)} \subset (I\times J)^{\mathbb{N}}$ is then defined in terms of $\boldsymbol{p} = (\boldsymbol{p}(1), \boldsymbol{p}(2), \ldots, \boldsymbol{p}(s))$ as follows. For any finite sequence $(x_1, x_2, \ldots, x_k) \in \prod_{j=1}^{k} D_{\omega(j)}$

$$\mu_{\boldsymbol{p}}([x_1, x_2, \ldots, x_k]) = \prod_{j=1}^{k} p_{x_j,\,\omega(j)}, \tag{9}$$

where $[x_1, x_2, \ldots, x_k] := \{\boldsymbol{d} = (d_j)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} D_{\omega(j)} : d_j = x_j \text{ for } 1 \leqslant j \leqslant k\}$ is a cylinder set of $\prod_{j=1}^{\infty} D_{\omega(j)}$ with base $(x_1, x_2, \ldots, x_k)$. Let $\widetilde{\mu}_{\boldsymbol{p}}$ be the image measure of $\mu_{\boldsymbol{p}}$ induced by Π, i.e., for Borel set $A \subseteq \mathbb{R}^2$,

$$\widetilde{\mu}_{\boldsymbol{p}}(A) = \mu_{\boldsymbol{p}}(\Pi^{-1}A), \tag{10}$$

where the map Π is defined as in (3) and is restricted on the set $\prod_{j=1}^{k} D_{\omega(j)}$. Then

$$\widetilde{\mu}_{\boldsymbol{p}}(K(\omega)) = \widetilde{\mu}_{\boldsymbol{p}}\left(\Pi\left(\prod_{k=1}^{\infty} D_{\omega(k)}\right)\right) = \mu_{\boldsymbol{p}}\left(\prod_{k=1}^{\infty} D_{\omega(k)}\right) = 1.$$

From the construction of $Q_k(\boldsymbol{x})$ it follows that for any $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ (cf [33, formula (4)], also [11, formula (4.4)])

$$\widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) = \prod_{j=1}^{[\alpha k]} p_{x_j,\,\omega(j)} \cdot \prod_{j=[\alpha k]+1}^{k} q_{\mathrm{proj}_y x_j,\,\omega(j)}. \tag{11}$$

Let us recall the definition of the Hausdorff dimension of a measure μ. It is defined as the infimum of Hausdorff dimensions of sets of full μ-measure. The following lemma appears in [36] as a version of the well-known Billingsley lemma [3] for which the the ball is replaced by the approximate square.

Lemma 2.2 [36, Corollary]. Let $\widetilde{\mu}_{\boldsymbol{p}}$ be defined as above. If

$$\liminf_{k\to\infty} \frac{\log \widetilde{\mu}_{\boldsymbol{P}}(Q_k(\boldsymbol{x}))}{\log m^{-k}} = \beta,$$

for $\mu_{\boldsymbol{p}}$-almost every $\boldsymbol{x} \in \prod_{j=1}^{\infty} D_{\omega(j)}$, then $\dim_{\rm H} \widetilde{\mu}_{\boldsymbol{p}} = \beta$.

3. Proof of theorem 1.1 and intersection of $K(T, D)$ with its translation

In this section, we first give the proof of theorem 1.1. Then we apply theorem 1.1 to determine the dimensions of the intersection of $K(T, D)$ with its translation. Recall that $\mathbb{P}$ defined as in (7) is a probability on $\{D_1, D_2, \ldots, D_s\}^{\mathbb{N}}$. From the Birkhoff ergodic theorem (cf [37, theorem 1.14]) it follows that for $\mathbb{P}$-a.e. $\omega = (D_{\omega(j)})_{j\geqslant 1} \in \{D_1, D_2, \ldots, D_s\}^{\mathbb{N}}$

$$\lim_{k\to\infty} \frac{N_k(D_i)}{k} = p_i, \quad 1 \leqslant i \leqslant s, \tag{12}$$

where $N_k(D_i) = |\{1 \leqslant j \leqslant k : D_{\omega(j)} = D_i\}|$ for $1 \leqslant i \leqslant s$. Therefore, theorem 1.1 follows directly from the following proposition, (4) and (5).

Proposition 3.1. *Let* $\omega = (D_{\omega(j)})_{j\geqslant 1} \in \{D_1, D_2, \ldots, D_s\}^{\mathbb{N}}$ *satisfy* (12). *For each* $b \in \mathrm{proj}_y D_i$ *let* $n_{b,\,i} := |D_i \cap (I \times \{b\})|$, $1 \leqslant i \leqslant s$. *Then*

$$\dim_{\rm H} K(\omega) = \sum_{i=1}^{s} p_i \log_m \sum_{b\in \mathrm{proj}_y D_i} n_{b,\,i}^{\alpha},$$

and

$$\dim_{\rm B} K(\omega) = \dim_{\rm P} K(\omega) = \sum_{i=1}^{s} p_i \log_m(|\mathrm{proj}_y D_i|^{1-\alpha}|D_i|^{\alpha}).$$

Proof.

Step 1. $\dim_{\rm H} K(\omega) \geqslant \sum_{i=1}^{s} p_i \log_m \sum_{b\in \mathrm{proj}_y D_i} n_{b,\,i}^{\alpha}$.

We first show that

$$\dim_{\rm H} \widetilde{\mu}_{\boldsymbol{p}} = \sum_{i=1}^{s}\sum_{d\in D_i} p_i \left(-\alpha p_{d,\,i} \log_m p_{d,\,i} - (1-\alpha) p_{d,\,i} \log_m q_{\mathrm{proj}_y d,\,i}\right), \tag{13}$$

where $\widetilde{\mu}_{\boldsymbol{p}}$ is defined as in (9) and (10).

For $n \in \mathbb{N}$ let X_n be the random variable on $(\prod_{j=1}^{\infty} D_{\omega(j)}, \mathcal{B}, \mu_{\boldsymbol{p}})$ such that for $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$

$$X_n(\boldsymbol{x}) = \log_m p_{x_n,\,\omega(n)} - \sum_{d\in D_{\omega(n)}} p_{d,\,\omega(n)} \log_m p_{d,\,\omega(n)}.$$

Then, $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent random variables with

$$\mathcal{E}(X_n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mathcal{E}(X_n^2)}{n^2} < \infty,$$

where $\mathcal{E}(X_n)$ denotes the expectation of the random variable X_n. From the strong law of large numbers (cf [4, theorem 1 in section 5.2]) it follows that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} X_j(\boldsymbol{x}) = 0 \quad \text{for} \quad \mu_{\boldsymbol{p}}\text{-}a.e.\ \boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}.$$

Similarly, by letting $Y_n(\boldsymbol{x}) = \log_m q_{\mathrm{proj}_y x_n,\,\omega(n)} - \sum_{d\in D_{\omega(n)}} p_{d,\,\omega(n)} \log_m q_{\mathrm{proj}_y x_n,\,\omega(n)}$ we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} Y_j(\boldsymbol{x}) = 0 \quad \text{for} \quad \mu_{\boldsymbol{p}}\text{-}a.e.\ \boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}.$$

For $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ and integer $k \in \mathbb{N}$, by taking logarithm in (11) we have

$$\log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) = \sum_{j=1}^{[\alpha k]} \log_m p_{x_j,\,\omega(j)} + \sum_{j=[\alpha k]+1}^{k} \log_m q_{\mathrm{proj}_y x_j,\,\omega(j)}.$$

Then

$$\begin{aligned}\frac{\log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x}))}{-k} &= -\frac{1}{k}\left(\sum_{j=1}^{[\alpha k]} \log_m p_{x_j,\,\omega(j)} + \sum_{j=[\alpha k]+1}^{k} \log_m q_{\mathrm{proj}_y x_j,\,\omega(j)}\right)\\ &= -\frac{1}{k}\left(\sum_{j=1}^{[\alpha k]} X_j(x) + \sum_{j=[\alpha k]+1}^{k} Y_j(x)\right) - \frac{1}{k}\left(\sum_{j=1}^{[\alpha k]} \sum_{d\in D_{\omega(j)}} p_{d,\,\omega(j)} \log_m p_{d,\,\omega(j)}\right.\\ &\quad \left. + \sum_{j=[\alpha k]+1}^{k} \sum_{d\in D_{\omega(j)}} p_{d,\,\omega(j)} \log_m q_{\mathrm{proj}_y x_j,\,\omega(j)}\right),\end{aligned}$$

which implies

$$\lim_{k\to\infty} \frac{\log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x}))}{-k} = \sum_{i=1}^{s}\sum_{d\in D_i} p_i \left(-\alpha p_{d,\,i} \log_m p_{d,\,i} - (1-\alpha) p_{d,\,i} \log_m q_{\mathrm{proj}_y d,\,i}\right),$$

for $\mu_{\boldsymbol{p}}$-$a.e.$ $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$. So (13) is then obtained by lemma 2.2.

In particular, we take, for each $1 \leqslant i \leqslant s$, the probability vector $\boldsymbol{p}(i) = (p_{d,i})_{d\in D_i}$ on D_i by letting

$$p_{d,\,i} = \frac{n^{\alpha-1}_{\mathrm{proj}_y d,\,i}}{\sum_{d\in D_i} n^{\alpha-1}_{\mathrm{proj}_y d,\,i}} \qquad \text{for } d \in D_i. \tag{14}$$

Therefore, from (13) it follows

$$\dim_{\mathrm{H}} \widetilde{\mu}_{\boldsymbol{p}} = \sum_{i=1}^{s} p_i \log_m \sum_{b\in \mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}.$$

This gives that $\dim_{\mathrm{H}} K(\omega) \geqslant \dim_{\mathrm{H}} \widetilde{\mu}_{\boldsymbol{p}} = \sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}$.

Step 2. $\dim_{\mathrm{H}} K(\omega) \leqslant \sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}$.

To do this, we take $\boldsymbol{p} = (\boldsymbol{p}(1), \boldsymbol{p}(2), \ldots, \boldsymbol{p}(s))$ where each $\boldsymbol{p}(i)$ is given by (14). We use lemma 2.1 and show that for any given $\delta > \sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}$, $A(\boldsymbol{x}) = +\infty$ for all $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$.

For a fixed $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ any $k \in \mathbb{N}$, let

$$S_k(i, \boldsymbol{x}) = \sum_{\substack{x_j\ \in D_{\omega(j)} = D_i \\ 1\leqslant j\leqslant k}} \log_m n_{\mathrm{proj}_y x_j,\,i} \qquad \text{for } \ i = 1, 2, \ldots, s.$$

Note that

$$\begin{aligned}\log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) &= \sum_{j=1}^{[\alpha k]} \log_m p_{x_j,\,\omega(j)} + \sum_{j=[\alpha k]+1}^{k} \log_m q_{\mathrm{proj}_y x_j,\,\omega(j)} \\ &= \sum_{i=1}^{s} \sum_{\substack{x_j\in D_{\omega(j)}=D_i \\ 1\leqslant j\leqslant[\alpha k]}} \log_m \frac{n^{\alpha-1}_{\mathrm{proj}_y x_j,\,i}}{\sum_{d\in D_i} n^{\alpha-1}_{\mathrm{proj}_y d,\,i}} + \sum_{i=1}^{s} \sum_{\substack{x_j\in D_{\omega(j)}=D_i \\ [\alpha k]+1\leqslant j\leqslant k}} \log_m \frac{n^{\alpha}_{\mathrm{proj}_y x_j,\,i}}{\sum_{d\in D_i} n^{\alpha-1}_{\mathrm{proj}_y d,\,i}} \\ &= \sum_{i=1}^{s} \big(\alpha S_k(i,\boldsymbol{x}) - S_{[\alpha k]}(i,\boldsymbol{x})\big) - \sum_{i=1}^{s} N_k(D_i)\log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{k\to\infty} \frac{1}{k}\log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) \\ &\quad= \limsup_{k\to\infty} \frac{1}{k}\left(\sum_{i=1}^{s} \big(\alpha S_k(i,\boldsymbol{x}) - S_{[\alpha k]}(i,\boldsymbol{x})\big) - \sum_{i=1}^{s} N_k(D_i)\log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}\right) \\ &\quad= \limsup_{k\to\infty} \frac{1}{k}\sum_{i=1}^{s} \big(\alpha S_k(i,\boldsymbol{x}) - S_{[\alpha k]}(i,\boldsymbol{x})\big) - \sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i} \\ &\quad= -\sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i} + \alpha \limsup_{k\to\infty} \sum_{i=1}^{s}\left(\frac{S_k(i,\boldsymbol{x})}{k} - \frac{S_{[\alpha k]}(i,\boldsymbol{x})}{\alpha k}\right).\end{aligned}$$

Since $\sup_k |S_{k+1}(i,\boldsymbol{x}) - S_k(i,\boldsymbol{x})| < +\infty$ for each $1 \leqslant i \leqslant s$, it follows from [17, lemma 4.1]

$$\limsup_{k\to\infty} \sum_{i=1}^{s}\left(\frac{S_k(i,\boldsymbol{x})}{k} - \frac{S_{[\alpha k]}(i,\boldsymbol{x})}{\alpha k}\right) \geqslant 0.$$

Hence, we have

$$A(\boldsymbol{x}) = \limsup_{k\to\infty} \big(k\delta + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x}))\big) = \limsup_{k\to\infty} k\left(\delta + \frac{1}{k}\log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x}))\right) = +\infty,$$

for any $\delta > \sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}$. This leads to $\dim_{\mathrm{H}} K(\omega) \leqslant \sum_{i=1}^{s} p_i \log_m \sum_{b\in\mathrm{proj}_y D_i} n^{\alpha}_{b,\,i}$ by lemma 2.1.

Step 3. $\dim_{\mathrm{B}} K(\omega) = \sum_{i=1}^{s} p_i \log_m(|\mathrm{proj}_y D_i|^{1-\alpha}|D_i|^{\alpha})$.

Let $M_k := \{Q_k(\boldsymbol{x}) : \boldsymbol{x} \in (I \times J)^{\mathbb{N}}\}$ with $k \in \mathbb{N}$. Then M_k consists of pairwise nonoverlapping approximate squares which cover $[0, 1] \times [0, 1]$. From the definition of box-counting dimension it is not difficult to prove

$$\dim_{\mathrm{B}} K(\omega) = \lim_{k\to\infty} \frac{\log|\{Q_k(\boldsymbol{x}) \in M_k : Q_k(\boldsymbol{x}) \cap K(\omega) \neq \emptyset\}|}{\log m^k},$$

if the limit exists. Now we calculate $|\{Q_k(\boldsymbol{x}) \in M_k : Q_k(\boldsymbol{x}) \cap K(\omega) \neq \emptyset\}|$. This is done by classifying points of the set $\{(x_j)_{j=1}^{k} \in \prod_{j=1}^{k} D_{\omega(j)}\}$ of finite sequences. Denote $x_j = (u_j, v_j)$. Then, $|\{Q_k(\boldsymbol{x}) \in M_k : Q_k(\boldsymbol{x}) \cap K(\omega) \neq \emptyset\}|$ is precisely the number of ways to choose $\{u_j\}_{j=1}^{[\alpha k]}$ and $\{v_j\}_{j=1}^{k}$ such that

(I) $(u_j, v_j) \in D_{\omega(j)}$ for $1 \leqslant j \leqslant [\alpha k]$;
(II) $(u, v_j) \in D_{\omega(j)}$ for $[\alpha k] + 1 \leqslant j \leqslant k$ and some choice of $u \in I$.

Hence, we have $|\{Q_k(\boldsymbol{x}) \in M_k : Q_k(\boldsymbol{x}) \cap K(\omega) \neq \emptyset\}| = \prod_{j=1}^{[\alpha k]} |D_{\omega(j)}| \prod_{j=[\alpha k]+1}^{k} |\mathrm{proj}_y D_{\omega(j)}|$. Therefore,

$$\begin{aligned}\dim_{\mathrm{B}} K(\omega) &= \lim_{k\to\infty} \frac{\log |\{Q_k(\boldsymbol{x}) \in M_k : Q_k(\boldsymbol{x}) \cap K(\omega) \neq \emptyset\}|}{\log m^k} \\ &= \lim_{k\to\infty} \frac{\sum_{i=1}^{s} \left(|\{1 \leqslant j \leqslant [\alpha k] : D_{\omega(j)} = D_i\}| \log |D_i| + |\{[\alpha k]+1 \leqslant j \leqslant k : D_{\omega(j)} = D_i\}| \log |\mathrm{proj}_y D_i|\right)}{k \log m} \\ &= \sum_{i=1}^{s} p_i \log_m (|\mathrm{proj}_y D_i|^{1-\alpha} |D_i|^{\alpha}).\end{aligned}$$

Step 4. $\dim_{\mathrm{P}} K(\omega) = \sum_{i=1}^{s} p_i \log_m (|\mathrm{proj}_y D_i|^{1-\alpha} |D_i|^{\alpha})$.

This is done directly from [8, corollary 3.9]. In fact, for any open set $V \subset \mathbb{R}^2$ with $V \cap K(\omega) \neq \emptyset$ there exist a $k \in \mathbb{N}$ and $(x_1, \ldots, x_k) \in \prod_{j=1}^{k} D_{\omega(j)}$ such that

$$f_{x_1} \circ f_{x_2} \circ \cdots \circ f_{x_k}([0,1]^2) \cap K(\omega) \subset V \cap K(\omega).$$

On the other hand, we have

$$K(\omega) = \bigcup_{(y_j)_{j=1}^{k} \in \prod_{j=1}^{k} D_{\omega(j)}} f_{y_1} \circ f_{y_2} \circ \cdots \circ f_{y_k}([0,1]^2) \cap K(\omega),$$

where the sets on the right-hand side are translations of each other. This implies that $\dim_{\mathrm{B}}(V \cap K(\omega)) = \dim_{\mathrm{B}} K(\omega)$, leading to $\dim_{\mathrm{P}} K(\omega) = \dim_{\mathrm{B}} K(\omega)$ by [8, Corollary 3.9]. □

A pattern $D \subset I \times J$ is said to have *uniformly horizontal fibres* if $n_b := |D \cap (I \times \{b\})| =$ constant for all $b \in \mathrm{proj}_y D$. The following corollary follows directly from proposition 3.1.

Corollary 3.2. *Let* $\omega = (D_{\omega(j)})_{j\geqslant 1} \in \{D_1, D_2, \ldots, D_s\}^{\mathbb{N}}$ *satisfy* (12)*. Then* $\dim_{\mathrm{H}} K(\omega) = \dim_{\mathrm{B}} K(\omega)$ *if and only if each* D_i *with* $p_i \neq 0$, $i = 1, 2, \cdots, s$ *has uniformly horizontal fibres.*

Proof. Note that $x \to x^{\alpha}$ is a strictly concave function because of $0 < \alpha = \log_n m < 1$. Thus, for each $1 \leqslant i \leqslant s$ with $p_i \neq 0$

$$\begin{aligned}|\mathrm{proj}_y D_i|^{1-\alpha} |D_i|^{\alpha} &= |\mathrm{proj}_y D_i| \left(\sum_{b\in \mathrm{proj}_y D_i} \frac{n_{b,i}}{|\mathrm{proj}_y D_i|} \right)^{\alpha} \\ &\geqslant |\mathrm{proj}_y D_i| \sum_{b\in \mathrm{proj}_y D_i} \frac{n_{b,i}^{\alpha}}{|\mathrm{proj}_y D_i|} = \sum_{b\in \mathrm{proj}_y D_i} n_{b,i}^{\alpha},\end{aligned}$$

where the equality holds if and only if all $n_{b,i}$ are same. Thus, the desired result follows from proposition 3.1. □

Now we apply theorem 1.1 (proposition 3.1) to the subject on the intersection of $K(T, D)$ with its translation. As discussed in section 1, when $\max\{|\mathrm{proj}_x(d_1-d_2)|, |\mathrm{proj}_y(d_1-d_2)|\} \geqslant 2$ for any distinct $d_1, d_2 \in D - D$, the family $\{f_d : d \in D - D\}$ of contractive affine maps satisfies the open set condition with respect to the open set $(-1, 1)^2$. In this case, $|\mathcal{C}(t)| \leqslant 4$ for each $t \in K(T, D) - K(T, D)$ (recall $\mathcal{C}(t) = \Pi^{-1}(t)$ where the map Π is defined by (3) with $\Omega = D - D$).

Corollary 3.3. *Suppose that* $\max\{|\mathrm{proj}_x(d_1 - d_2)|, |\mathrm{proj}_y(d_1 - d_2)|\} \geqslant 2$ *for any distinct* $d_1, d_2 \in D - D$*. If* $t \in K(T, D) - K(T, D)$ *has a unique* $(D - D)$*-code* $(t_k)_{k=1}^{\infty} \in (D - D)^{\mathbb{N}}$ *(i.e.* $\mathcal{C}(t) = \{(t_k)_{k=1}^{\infty}\}$ *is a singleton) and*

$$\lim_{\ell\to\infty} \frac{|\{1 \leqslant j \leqslant \ell : t_j = d_i\}|}{\ell} = p_i \qquad \text{for } 1 \leqslant i \leqslant s,$$

where we assume $D - D = \{d_1, \ldots, d_s\}$. *Then*

$$\dim_{\mathrm{H}} K(T, D) \cap (K(T, D) + t) = \sum_{i=1}^{s} p_i \log_m \sum_{b \in \mathrm{proj}_y D_i} n_{b,\,i}^{\alpha},$$

and

$$\begin{aligned}\dim_{\mathrm{B}} K(T, D) \cap (K(T, D) + t) &= \dim_{\mathrm{P}} K(T, D) \cap (K(T, D) + t)\\ &= \sum_{i=1}^{s} p_i \log_m (|\mathrm{proj}_y D_i|^{1-\alpha} |D_i|^{\alpha}),\end{aligned}$$

where $D_i = D \cap (D + d_i)$, $1 \leqslant i \leqslant s$.

Proof. By (8) we have that $K(T, D) \cap (K(T, D) + t) = \Pi(\prod_{i=1}^{\infty}(D \cap (D + t_i)))$. Thus, the desired result follows directly from proposition 3.1. We remark that it may happen that some D_is are identical, i.e. $D \cap (D + d_i) = D \cap (D + d_j)$ may happen for distinct d_i and d_j. But it is easy to see that it does not affect the results. □

Remarks.

(I) For each probability vector $(p_i)_{i=1}^{s}$ the set of t satisfying the conditions in corollary 3.3 is dense in $K(T, D) - K(T, D)$. The assumption that $\max\{|\mathrm{proj}_x(d_1 - d_2)|, |\mathrm{proj}_y(d_1 - d_2)|\} \geqslant 2$ for any distinct $d_1, d_2 \in D - D$ can be replaced by the family $\{f_d : d \in D - D\}$ satisfying the open set condition.

(II) If $|\mathcal{C}(t)| > 1$ (note that $|\mathcal{C}(t)| \leqslant 4$ under the condition on $D - D$), then the set $K(T, D) \cap (K(T, D) + t)$ is reduced to a finite union of the generalized Moran sets. In this case, the Hausdorff, packing, lower and upper box-counting dimensions of $K(T, D) \cap (K(T, D) + t)$ can be determined without the assumption on the digit frequencies of the $(D - D)$-codes of t. But it may have different lower and upper box-counting dimensions. However, the packing and upper box-counting dimensions are always identical. Readers can refer to, for instance, [14, 15, 25, 38] for detailed information on the generalized Moran sets.

(III) When the family $\{f_d : d \in D - D\}$ fails to satisfy the open set condition, the approach described in this paper does not work any more. The situation becomes much more complicated. However, when t is a rational pair the set $K(T, D) \cap (K(T, D) + t)$ represents a graph-directed structure to which the results in [18] can be applied.

4. Further discussion

In this section, a necessary condition and a sufficient condition are obtained, respectively, such that the Hausdorff measure of $K(\omega)$ in its dimension is positive. If the pattern D_i has uniformly horizontal fibres, we denote

$$\ell_i = |\mathrm{proj}_y D_i|, \qquad n_{b,\,i} = t_i \quad \text{for all } \ b \in \mathrm{proj}_y D_i.$$

In addition, patterns D_i and D_j are said to have *uniformly horizontal fibres of the same type* if both of them have uniformly horizontal fibres and $\ell_i = \ell_j$, $t_i = t_j$.

Theorem 4.1. *Let* $\omega = (D_{\omega(j)})_{j \geqslant 1} \in \{D_1, D_2, \ldots, D_s\}^{\mathbb{N}}$ *satisfy* (12). *Let* γ *be the Hausdorff dimension of* $K(\omega)$.

(I) *If* $0 < \mathcal{H}^{\gamma}(K(\omega)) < \infty$, *then each* D_i *with* $p_i \neq 0$, $i = 1, 2, \cdots, s$ *has uniformly horizontal fibres;*

(II) *If all patterns* D_i, $1 \leqslant i \leqslant s$ *have uniformly horizontal fibres of the same type, then* $0 < \mathcal{H}^{\gamma}(K(\omega)) < \infty$.

Proof. (I) Without loss of generality, we assume that $1 \leqslant t \leqslant s$ is such that $\sum_{i=1}^{t} p_i = 1$ with $p_i > 0$ for $1 \leqslant i \leqslant t$ (this means other p_is are zeros if $t < s$). By the structure of $K(\omega)$ we have

$$K(\omega) = \bigcup_{d \in D_{\omega(1)}} K(\omega) \cap f_d([0, 1]^2),$$

where the sets are translations of each other on the right-hand side. Thus,

$$\mathcal{H}^{\gamma}(K(\omega) \cap f_d([0, 1]^2)) = \frac{1}{|D_{\omega(1)}|} \mathcal{H}^{\gamma}(K(\omega)),$$

for all $d \in D_{\omega(1)}$. For any $\boldsymbol{d} = (d_i)_{i=1}^{k} \in \prod_{i=1}^{k} D_{\omega(i)}$ let $f_{\boldsymbol{d}} := f_{d_1} \circ f_{d_2} \circ \cdots \circ f_{d_k}$. The same argument yields that

$$\mathcal{H}^{\gamma}(K(\omega) \cap f_{\boldsymbol{d}}([0, 1]^2)) = \frac{1}{\prod_{i=1}^{k} |D_{\omega(i)}|} \mathcal{H}^{\gamma}(K(\omega)),$$

for any $\boldsymbol{d} = (d_i)_{i=1}^{k} \in \prod_{i=1}^{k} D_{\omega(i)}$. Now we take $\overline{\boldsymbol{p}} = (\overline{\boldsymbol{p}}(1), \overline{\boldsymbol{p}}(2), \ldots, \overline{\boldsymbol{p}}(s))$ where $\overline{\boldsymbol{p}}(i) = (|D_i|^{-1}, \ldots, |D_i|^{-1})$ is a probability vector on D_i, $1 \leqslant i \leqslant s$. Let $\tilde{\mu}_{\overline{\boldsymbol{p}}}$ be the probability measure on $K(\omega)$ which is defined in the way shown by (9)–(11). Then

$$\tilde{\mu}_{\overline{\boldsymbol{p}}}(K(\omega) \cap f_{\boldsymbol{d}}([0, 1]^2)) = \frac{1}{\prod_{i=1}^{k} |D_{\omega(i)}|},$$

for any $\boldsymbol{d} = (d_i)_{i=1}^{k} \in \prod_{i=1}^{k} D_{\omega(i)}$. Therefore, for any Borel set $A \subset [0, 1]^2$

$$\tilde{\mu}_{\overline{\boldsymbol{p}}}(K(\omega) \cap A) = \frac{1}{\mathcal{H}^{\gamma}(K(\omega))} \mathcal{H}^{\gamma}(K(\omega) \cap A).$$

Note that

$$\dim_{\mathrm{H}} \tilde{\mu}_{\overline{\boldsymbol{p}}} = \inf_{\substack{A \subset K(\omega) \\ \tilde{\mu}_{\overline{\boldsymbol{p}}}(A)=1}} \dim_{\mathrm{H}} A = \inf_{\substack{A \subset K(\omega) \\ \mathcal{H}^{\gamma}(A)=\mathcal{H}^{\gamma}(K(\omega))}} \dim_{\mathrm{H}} A = \gamma.$$

On the other hand, from (13) it follows

$$\begin{aligned} \dim_{\mathrm{H}} \tilde{\mu}_{\overline{\boldsymbol{p}}} &= \sum_{i=1}^{t} \sum_{d \in D_i} p_i \left(-\alpha \frac{1}{|D_i|} \log_m \frac{1}{|D_i|} - (1-\alpha) \frac{1}{|D_i|} \log_m \frac{n_{\mathrm{proj}_y d, i}}{|D_i|} \right) \\ &= \sum_{i=1}^{t} p_i \sum_{d \in D_i} \frac{1}{|D_i|} \log_m \left(|D_i| n_{\mathrm{proj}_y d, i}^{\alpha-1} \right). \end{aligned}$$

Recall that $\gamma = \sum_{i=1}^{t} p_i \log_m \sum_{b \in \mathrm{proj}_y D_i} n_{b, i}^{\alpha} = \sum_{i=1}^{t} p_i \log_m \sum_{d \in D_i} n_{\mathrm{proj} y d, i}^{\alpha-1}$ by proposition 3.1. Since $\log_m x$ is a strictly concave function in x we obtain that for each $1 \leqslant i \leqslant t$ all $n_{\mathrm{proj}_y d, i}$, $d \in D_i$ are same, i.e. each D_i, $i = 1, 2, \cdots, t$ has uniformly horizontal fibres.

(II) We first consider the case that each pattern D_i has uniformly horizontal fibres. Then by proposition 3.1 we have

$$\gamma = \dim_{\mathrm{H}} K(\omega) = \dim_{\mathrm{B}} K(\omega) = \alpha \sum_{i=1}^{s} p_i \log_m t_i + \sum_{i=1}^{s} p_i \log_m \ell_i.$$

We take $\boldsymbol{p} = (\boldsymbol{p}(1), \boldsymbol{p}(2), \ldots, \boldsymbol{p}(s))$ where $\boldsymbol{p}(i) = (p_{d, i})_{d \in D_i} = (|D_i|^{-1}, |D_i|^{-1}, \ldots, |D_i|^{-1})$ is a probability vector on D_i, $1 \leqslant i \leqslant s$. The probability measure $\tilde{\mu}_{\boldsymbol{p}}$ on $K(\omega)$ is constructed in the way shown by (9)–(11). Thus, for any $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ and any $k \in \mathbb{N}$

$$\begin{aligned} k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) &= \sum_{i=1}^{s}(p_i\alpha k - N_{[\alpha k]}(D_i))\log_m t_i + \sum_{i=1}^{s}(p_i k - N_k(D_i))\log_m \ell_i, \\ &= \sum_{i=1}^{s}(p_i[\alpha k] - N_{[\alpha k]}(D_i))\log_m t_i + \sum_{i=1}^{s}(p_i k - N_k(D_i))\log_m \ell_i \\ &\quad + \sum_{i=1}^{s}(p_i\alpha k - p_i[\alpha k])\log_m t_i. \end{aligned} \tag{15}$$

Thus, when all patterns D_i, $1 \leqslant i \leqslant s$ have uniformly horizontal fibres of the same type, (15) reduces to

$$k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) = \sum_{i=1}^{s}(p_i\alpha k - p_i[\alpha k])\log_m t_1,$$

which gives $0 < \mathcal{H}^{\gamma}(K(\omega)) < \infty$ by lemma 2.1. □

Combining theorem 4.1 (I) with corollary 3.2 we have that $\dim_{\mathrm{H}} K(\omega) = \dim_{\mathrm{B}} K(\omega)$ if $K(\omega)$ has positive Hausdorff measure in its dimension. Let us recall that in the case of general Sierpinski carpet we have $\dim_{\mathrm{H}} K(T, D) = \dim_{\mathrm{B}} K(T, D)$ if and only if $K(T, D)$ has positive Hausdorff measure in its dimension. However, this is not true for the case discussed in this paper. The following examples show that the conditions shown in theorem 4.1 are not necessary (examples 1 and 2 for case (I), example 3 for case (II)). For simplicity, we only consider the case that $s = 2$ although it becomes more complicated for $s > 2$.

Example 1. Let $\omega = (D_{\omega(j)})_{j\geqslant 1} \in \{D_1, D_2\}^{\mathbb{N}}$ satisfy (12) with $p_1 = 1$ and $p_2 = 0$, where D_1 has uniformly horizontal fibres. Then

$$\dim_{\mathrm{H}} K(\omega) = \dim_{\mathrm{B}} K(\omega) = \log_m(|\mathrm{proj}_y D_1|^{1-\alpha}|D_1|^{\alpha}) = \log_m(\ell_1 t_1^{\alpha}) := \gamma.$$

Take the probability vectors $\boldsymbol{p}(i)$ on D_i as $\boldsymbol{p}(i) = (p_{d,\,i})_{d\in D_i} := (|D_i|^{-1}, \ldots, |D_i|^{-1})$, $i = 1, 2$. The probability measure $\tilde{\mu}_{\boldsymbol{p}}$ on $K(\omega)$ is constructed in the way shown by (9)–(11). Then for any $\boldsymbol{x} = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ and any $k \in \mathbb{N}$

$$\begin{aligned} k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) &= k\log_m(\ell_1 t_1^{\alpha}) + \sum_{j=1}^{[\alpha k]}\log_m p_{x_j,\,\omega(j)} + \sum_{j=[\alpha k]+1}^{k}\log_m q_{\mathrm{proj}_y x_j,\,\omega(j)} \\ &= k\log_m(\ell_1 t_1^{\alpha}) + N_{[\alpha k]}(D_1)\log_m\frac{1}{|D_1|} + \big(N_k(D_1) - N_{[\alpha k]}(D_1)\big)\log_m\frac{t_1}{|D_1|} \\ &\quad + N_{[\alpha k]}(D_2)\log_m\frac{1}{|D_2|} + \sum_{\substack{[\alpha k]+1\leqslant j\leqslant k \\ \omega(j)=2}}\log_m q_{\mathrm{proj}_y x_j,\,2} \\ &= (k\alpha - [\alpha k])\log_m t_1 + N_k(D_2)\log_m \ell_1 + N_{[\alpha k]}(D_2)\log_m t_1 \\ &\quad - N_{[\alpha k]}(D_2)\log_m|D_2| + \sum_{\substack{[\alpha k]+1\leqslant j\leqslant k \\ \omega(j)=2}}\log_m q_{\mathrm{proj}_y x_j,\,2} \\ &= (k\alpha - [\alpha k])\log_m t_1 + (N_k(D_2) - N_{[\alpha k]}(D_2))\log_m \ell_1 + N_{[\alpha k]}(D_2)\log_m\frac{|D_1|}{|D_2|} \\ &\quad + \sum_{\substack{[\alpha k]+1\leqslant j\leqslant k \\ \omega(j)=2}}\log_m q_{\mathrm{proj}_y x_j,\,2}. \end{aligned}$$

Denote $q^* = \max_{b\in\mathrm{proj}_y D_2} q_{b,2}$ and $q_* = \min_{b\in\mathrm{proj}_y D_2} q_{b,2}$. Then

$$\begin{aligned} k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) &\leqslant (k\alpha - [\alpha k])\log_m t_1 + (N_k(D_2) - N_{[\alpha k]}(D_2))\log_m(\ell_1 q^*) \\ &\quad + N_{[\alpha k]}(D_2)\log_m\frac{|D_1|}{|D_2|}, \end{aligned}$$

and

$$k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) \geqslant (k\alpha - [\alpha k]) \log_m t_1 + (N_k(D_2) - N_{[\alpha k]}(D_2)) \log_m(\ell_1 q_*) + N_{[\alpha k]}(D_2) \log_m \frac{|D_1|}{|D_2|},$$

for any $\boldsymbol{x} = (x_j)_{j \geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ and any $k \in \mathbb{N}$.

Now we take $\omega = (D_{\omega(j)})_{j \geqslant 1} \in \{D_1, D_2\}^{\mathbb{N}}$ such that $\omega(k^2) = 2$ and $= 1$ otherwise. Thus $N_k(D_1) = |\{1 \leqslant j \leqslant k : D_{\omega(j)} = D_1\}| = k - \left[\sqrt{k}\right]$ and $N_k(D_2) = |\{1 \leqslant j \leqslant k : D_{\omega(j)} = D_2\}| = \left[\sqrt{k}\right]$ (so ω satisfies (12) with $p_1 = 1$ and $p_2 = 0$). Hence, for any $\boldsymbol{x} = (x_j)_{j \geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$

$$A(\boldsymbol{x}) = \limsup_{k\to\infty} (k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) = \begin{cases} +\infty & \text{if } \ell_1 q_* \geqslant 1 \text{ and } |D_1| > |D_2| \\ -\infty & \text{if } \ell_1 q^* \leqslant 1 \text{ and } |D_1| < |D_2|. \end{cases}$$

By lemma 2.1, this means $\mathcal{H}^{\gamma}(K(\omega))$ takes, respectively, 0 and $+\infty$. □

Example 2. Let $\omega = (D_{\omega(j)})_{j \geqslant 1} \in \{D_1, D_2\}^{\mathbb{N}}$ satisfy (12) with $0 < p_1 < 1$ (so $p_2 = 1 - p_1 > 0$). Let both D_1 and D_2 have uniformly horizontal fibres. Then

$$\dim_{\mathrm{H}} K(\omega) = \dim_{\mathrm{B}} K(\omega) = p_1 \log_m(\ell_1 t_1^{\alpha}) + (1 - p_1) \log_m(\ell_2 t_2^{\alpha}) := \gamma.$$

With the same probability measure $\tilde{\mu}_{\boldsymbol{p}}$ as that in Example 1, it follows from (15) that

$$k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) = \left(p_1[\alpha k] - N_{[\alpha k]}(D_1)\right) \log_m \frac{t_1}{t_2} + (p_1 k - N_k(D_1)) \log_m \frac{\ell_1}{\ell_2} + \sum_{i=1}^{2} (p_i \alpha k - p_i[\alpha k]) \log_m t_i,$$

for any $x = (x_j)_{j \geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ and any $k \in \mathbb{N}$. Suppose that $t_1 = t_2$ and $\ell_1 > \ell_2$. Then, for any $x = (x_j)_{j \geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ we have $A(x) = \limsup_{k\to\infty} (k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})) = +\infty$ if we take $\omega = (D_{\omega(j)})_{j \geqslant 1} \in \{D_1, D_2\}^{\mathbb{N}}$ such that $\limsup_{k\to\infty}(p_1 k - N_k(D_1)) = +\infty$. In fact, the set of the ω possessing this property is of full $\mathbb{P}$ measure by the law of the iterated logarithm (cf [4, theorem 1 in section 10.2]). □

Example 3. In example 2, suppose that $(\ell_1, t_1) \neq (\ell_2, t_2)$. Then, for any $x = (x_j)_{j \geqslant 1} \in \prod_{j=1}^{\infty} D_{\omega(j)}$ we have that $A(x) = \limsup_{k\to\infty} (k\gamma + \log_m \widetilde{\mu}_{\boldsymbol{p}}(Q_k(\boldsymbol{x})))$ is a (finite) real number if we take $\omega = (D_{\omega(j)})_{j \geqslant 1} \in \{D_1, D_2\}^{\mathbb{N}}$ such that $-\xi \leqslant p_1 k - N_k(D_1) \leqslant \xi$ for some positive ξ, e.g. we let $\omega(2k - 1) = 1$ and $\omega(2k) = 2$ when $p_1 = p_2$. □

Acknowledgments

The authors would like to thank the anonymous referees for their careful reading, valuable comments and suggestions that led to the improvement of the manuscript. The first author is supported by the Science and Technology Commission of Shanghai Municipality #06ZR14029 and the Educational Office of Hubei province #Q20082802. The second author is supported by the National Science Foundations of China #10571058, #10771071 and Shanghai Leading Academic Discipline Project #B407.

References

[1] Barański K 2007 Hausdorff dimension of the limit sets of some planar geometric constructions *Adv. Math.* **210** 215–45

[2] Bedford T 1984 Crinkly curves, Markov partitions and box dimension in self-similar sets, *PhD Thesis* University of Warwick

[3] Billingsley P 1965 *Ergodic Theory and Information* (New York: Wiley)

[4] Chow Y S and Teicher H 1978 *Probability Theory* (New York: Springer)

[5] Davis G J and Hu T Y 1995 On the structure of the intersection of two middle third Cantor sets *Pub. Math.* **39** 43–60

[6] Dekking F M and Grimmett G R 1988 Superbranching processes and projections of random Cantor sets *Probab. Theory Relat. Fields* **78** 335–55

[7] Deng G-T, He X-G and Wen Z-X 2008 Self-similar structure on intersection of triadic Cantor sets *J. Math. Anal. Appl.* **337** 617–31

[8] Falconer K J 1990 *Fractal Geometry—Mathematical Foundations and Applications* (Chichester: Wiley)

[9] Falconer K J 1986 Random fractals *Math. Proc. Camb. Phil. Soc.* **100** 559–82

[10] Falconer K J 1989 Projections of random Cantor sets *J. Theor. Probab.* **2** 65–70

[11] Gatzouras D and Lalley S P 1992 Hausdorff and box dimensions of certain self-affine fractals *Indiana Univ. Math. J.* **41** 533–68

[12] Gatzouras D and Lalley S P 1994 Statistically self-affine sets: Hausdorff and box dimensions *J. Theor. Probab.* **7** 437–68

[13] Gui Y X and Li W X 2007 Hausdorff dimension of fiber-coding sub-Sierpinski carpets *J. Math. Anal. Appl.* **331** 62–8

[14] Hua S 1994 On the Hausdorff dimension of generalized self-similar sets *Acta Math. Appl. Sin.* **17** 551–8 (in Chinese)

[15] Hua S and Li W X 1996 Packing dimension of generalized moran sets *Prog. Nat. Sci.* **2** 148–52

[16] Igudesman K B 2003 Lacunary self-similar fractal sets and intersection of Cantor sets *Lobachevskii J. Math.* **12** 41–5

[17] Kenyon R and Peres Y 1996 Measures of full dimension on affine-invariant sets *Erg. Theory Dyn. Syst.* **16** 307–23

[18] Kenyon R and Peres Y 1996 Hausdorff dimensions of sofic affine-invariant sets *Isr. J. Math.* **122** 540–74

[19] Kenyon R and Peres Y 1991 Intersecting random translates of invariant Cantor sets *Invent. Math.* **104** 601–29

[20] King J 1995 The singular spectrum for general Sierpinski carpets *Adv. Math.* **116** 1–8

[21] Kraft R L 1992 Intersection of thick Cantor sets *Mem. Am. Math. Soc.* **97** 1–119

[22] Kraft R L 1994 What's the difference between Cantor sets? *Am. Math. Mon.* **101** 640–50

[23] Kraft R L 1999 Random intersections of thick Cantor sets *Trans. Am. Math. Soc.* **352** 1315–28

[24] Li J and Nekka F 2004 Intersection of triadic Cantor sets with their translates: II. Hausdorff measure spectrum function and its introduction for the classification of Cantor sets *Chaos Solitons Fractals* **19** 35–46

[25] Li W X and Xiao D M 1998 A note on generalized Moran set *Acta Math. Sci. (Supp.)* **18** 88–93

[26] Li W and Xiao D 1999 Intersections of translations of Cantor triadic sets *Acta. Math. Sci.* **19** 214–9

[27] Li W and Xiao D 1998 On the intersection of translation of middle-α Cantor sets *Fractals and Beyond: Complexities in the Sciences. Proc. 5th Int. Multidisciplinary Conf., Fractal'98* ed M M Novak (Malta) (Singapore: World Scientific) pp 137–48

[28] Mandelbrot B 1983 *The Fractal Geometry of Nature* (San Francisco, CA: Freeman)

[29] Mattila P 1995 *Geometry of Sets and Measures in Euclidean Space: Fractals and Rectifiability* (Cambridge: Cambridge University Press)

[30] Mauldin R D and Williams S C 1986 Random recursive constructions: asymptotic geometric and topological properties *Trans. Am. Math. Soc.* **295** 325–46

[31] McMullen C 1984 The Hausdorff dimension of general Sierpinski carpets *Nagoya Math. J.* **96** 1–9

[32] Nekka F and Li J 2002 Intersection of triadic Cantor sets with their translates: I. Fundamental properties *Chaos Solitons Fractals* **13** 1807–17

[33] Nielsen O A 1999 The Hausdorff and packing dimensions of some sets related to Sierpinski carpets *Can. J. Math.* **51** 1073–88

[34] Olsen L 1998 Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$ *Pacific J. Math.* **183** 143–99

[35] Peres Y 1994 The packing measure of self-affine carpets *Math. Proc. Camb. Phil. Soc.* **115** 437–50

[36] Peres Y 1994 The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure *Math. Proc. Camb. Phil. Soc.* **116** 513–26

[37] Walters P 1982 *An Introduction to Ergodic Theory* (New York: Springer)

[38] Wen Z Y 2001 Moran sets and Moran classes *Chin. Sci. Bull.* **46** 1849–56

[39] Williams R F 1991 How big is the intersection of two thick Cantor sets *Contemporary Mathematics* ed M Brown *Proc. 1989 Joint Summer Research Conf. on Continua and Dynamical Systems* (Providence, RI: American Mathematical Society)