

Assouad dimensions of Moran sets and Cantor-like sets

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Abstract We obtain the Assouad dimensions of Moran sets under suitable condition. Using the homogeneous set introduced in [J. Math. Anal. Appl., 2015, 432: 888–917], we also study the Assouad dimensions of Cantor-like sets.

Keywords Fractal, Assouad dimension, Moran set, Cantor-like set

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1 Introduction

Let (X, d) be a metric space. We say that X is *doubling* if there exists an integer $N > 0$ such that each ball in X can be covered by N balls of half the radius. Repeated applying this property, it gives that there exist constants $b, c, \alpha > 0$ such that for all r and R with $0 < r < R < b$, every ball $B(x, R)$ can be covered by $c(R/r)^\alpha$ balls of radius r . Let $N_{r,R}(X)$ denote the smallest number of balls with radii r which can cover a ball with radius R . The *Assouad dimension* of X , denoted by $\dim_A(X)$, is defined as

$$\dim_A(X) = \inf \left\{ \alpha \geq 0 \mid \exists b, c > 0 \text{ s.t. } N_{r,R}(X) \leq c \left(\frac{R}{r} \right)^\alpha, \forall 0 < r < R < b \right\},$$

which was introduced by Assouad [1–3]. Now, it plays a prominent role in the study of quasiconformal mappings on \mathbb{R}^d , and we refer the readers to [7,13] for more details. It is well known that

$$\dim_H(X) \leq \dim_P(X) \leq \dim_A X,$$

where $\dim_H(\cdot)$ and $\dim_P(\cdot)$ are Hausdorff and packing dimensions, respectively.

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Suppose that K is a compact subset of X and s is a non-negative real number. We say that K is *Ahlfors-David s -regular* if there exists a Borel measure μ supported on K and a constant $c \geq 1$ such that, for all $x \in K$ and $0 < r \leq |K|$,

$$c^{-1}r^s \leq \mu(B(x, r)) \leq cr^s, \quad (1)$$

where $B(x, r)$ is the closed ball centered at x with radius r and $|\cdot|$ denotes the diameter of set. Olsen [19] proved that for a class of fractals with some flexible graph-directed construction, their Assouad dimensions coincide with their Hausdorff and box dimensions. He also pointed out that the fractals in [19] are Ahlfors-David regular. It is well known that self-similar sets and self-conformal sets satisfying the open set condition (OSC) are always Ahlfors-David regular, see [16]. One advantage of such sets is that their dimensions coincide, namely, for Ahlfors-David s -regular set K ,

$$\dim_A K = \dim_H K = \dim_P K = s.$$

In general, it is difficult to compute the Assouad dimensions of sets which are not Ahlfors-David regular. Mackay [15] calculated the Assouad dimensions of two classes of self-affine fractals, namely, Bedford-McMullen carpets [4,17] and Lalley-Gatzouras sets [11], and he also solved the problem posed by Olsen [19]. Fraser [6] obtained Assouad dimensions for certain classes of self-affine sets and quasi-self-similar sets.

In this paper, we study the Assouad dimension formula of Moran sets, Cantor-like sets, and homogeneous sets. Moran set was first studied by Moran [18], where most cases are not Ahlfors-David regular. First, we recall the definition of Moran set.

Let $\{n_k (\geq 2)\}_{k \geq 1}$ be a sequence of positive integers. For each $k = 0, 1, \dots$, let

$$D_k = \{u_1 u_2 \cdots u_k : 1 \leq u_j \leq n_j, j \leq k\}$$

be the set of words of length k , with $D_0 = \{\emptyset\}$ containing only the empty word \emptyset . Let

$$D = \bigcup_{k=0}^{+\infty} D_k$$

be the set of all finite words. Suppose that $J \subset \mathbb{R}^d$ is a compact set with $\text{int}(J) \neq \emptyset$ (we always write $\text{int}(\cdot)$ for the interior of set). Let $\{\phi_k\}_{k \geq 1}$ be a sequence of positive real vectors, where

$$\phi_k = (c_{k,1}, c_{k,2}, \dots, c_{k,n_k}), \quad \sum_{j=1}^{n_k} (c_{k,j})^d \leq 1, \quad k \in \mathbb{N}.$$

We say that the collection $\mathcal{F} = \{J_{\mathbf{u}} : \mathbf{u} \in D\}$ of closed subsets of J fulfills the *Moran structure* if it satisfies the following Moran structure conditions (MSC).

(i) For each $\mathbf{u} \in D$, $J_{\mathbf{u}}$ is geometrically similar to J , i.e., there exists a similarity $S_{\mathbf{u}}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $J_{\mathbf{u}} = S_{\mathbf{u}}(J)$. We write $J_{\emptyset} = J$ for empty word \emptyset .

(ii) For all $k \in \mathbb{N}$ and $\mathbf{u} \in D_{k-1}$, the elements $J_{\mathbf{u}1}, J_{\mathbf{u}2}, \dots, J_{\mathbf{u}n_k}$ of \mathcal{F} are the subsets of $J_{\mathbf{u}}$ with disjoint interiors, i.e.,

$$\text{int}(J_{\mathbf{u}i}) \cap \text{int}(J_{\mathbf{u}i'}) = \emptyset, \quad i \neq i'.$$

Moreover, for all $1 \leq i \leq n_k$,

$$\frac{|J_{\mathbf{u}i}|}{|J_{\mathbf{u}}|} = c_{k,i}.$$

We call

$$E = E(\mathcal{F}) = \bigcap_{k=1}^{+\infty} \bigcup_{\mathbf{u} \in D_k} J_{\mathbf{u}}$$

a *Moran set* determined by \mathcal{F} . For all $\mathbf{u} \in D_k$, the elements $J_{\mathbf{u}}$ are called *kth-level basic sets* of E . Suppose that the set J and the sequences $\{n_k\}$ and $\{\phi_k\}$ are given. We denote by

$$\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\phi_k\})$$

the class of the Moran sets satisfying the MSC.

For any $k' > k$, let $s_{k,k'}$ be the unique real solution of the equation $\Delta_{k,k'}(s) = 1$, where

$$\Delta_{k,k'}(s) = \prod_{i=k+1}^{k'} \left(\sum_{j=1}^{n_i} (c_{i,j})^s \right). \tag{2}$$

If the sequence $\{\sup_k s_{k,k+m}\}_{m=1}^{+\infty}$ converges, we write

$$s^{**} = \lim_{m \rightarrow +\infty} \sup_k s_{k,k+m}.$$

In Section 2, we prove that the sequence $\{\sup_k s_{k,k+m}\}_{m=1}^{+\infty}$ is indeed convergent under the assumption

$$c_* = \inf_{i,j} c_{i,j} > 0.$$

Furthermore, The following theorem indicates that the limit is the Assouad dimension of Moran sets.

Theorem 1 *Suppose that $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\phi_k\})$ is a Moran class with $c_* = \inf_{i,j} c_{i,j} > 0$. Then, for all $E \in \mathcal{M}$,*

$$\dim_{\text{A}} E = s^{**}.$$

As an immediate consequence, we have the following corollary.

Corollary 1 Suppose that $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\phi_k\})$ is a Moran class with $c_* = \inf_{i,j} c_{i,j} > 0$. Let

$$c_{k,1} = c_{k,2} = \cdots = c_{k,n_k} = c_k, \quad \forall k \in \mathbb{N}.$$

Then, for all $F \in \mathcal{M}$,

$$\dim_A F = \lim_{m \rightarrow +\infty} \sup_k \frac{\log(n_k n_{k+1} \cdots n_{k+m})}{-\log(c_k c_{k+1} \cdots c_{k+m})}.$$

Let s_* and s^* be the upper and lower limits of the sequence $\{s_{0,m}\}_{m=1}^{+\infty}$, respectively, that is,

$$s_* = \varliminf_{m \rightarrow +\infty} s_{0,m}, \quad s^* = \overline{\lim}_{m \rightarrow +\infty} s_{0,m}.$$

It was shown in [8,9,20,21] that, for all $E \in \mathcal{M}$ with $c_* > 0$,

$$\dim_H E = s_*, \quad \dim_P E = s^*.$$

Remark 1 The Assouad dimension vs Hausdorff (or packing) dimension is similar to the situation of the Banach density [10] vs usual density of a subsequence of natural number. In some sense, the Assouad dimension is like the Banach density, and the Hausdorff (or packing) dimension is like the usual lower (or upper) density.

In the next example, we will construct a Moran set to satisfy

$$\dim_H E < \dim_P E < \dim_A E.$$

Note that it is also a counter-example to the conclusion in [12]. Hereby, Theorem 1 corrects their main conclusion.

Example 1 Let $\{p_i\}_i$ be an increasing sequence of integers such that $p_{i+1} - p_i > i$ for all i and

$$\lim_{i \rightarrow +\infty} \frac{p_{i-1}}{p_i - p_{i-1}} = \lim_{i \rightarrow +\infty} \frac{i}{p_i - p_{i-1}} = 0.$$

Let $J = [0, 1]$, $n_k \equiv 2$, and

$$c_{k,1} = c_{k,2} = \frac{1}{16} \begin{cases} 4, & k \in [p_i + 1, p_i + i] \text{ for some } i \in \mathbb{N}, \\ 2, & k \in [p_i + i + 1, p_{i+1}] \text{ for some even } i \in \mathbb{N}, \\ 1, & k \in [p_i + i + 1, p_{i+1}] \text{ for some odd } i \in \mathbb{N}. \end{cases}$$

Then we have

$$s_* = \frac{1}{4}, \quad s^* = \frac{1}{3}, \quad s^{**} = \frac{1}{2}.$$

Clearly, for all $E \in \mathcal{M}(J, \{n_k(\equiv 2)\}, \{(c_{k,1}, c_{k,2})\})$, the dimensions inequality strictly holds, that is,

$$\dim_H E = \frac{1}{4} < \dim_P E = \frac{1}{3} < \dim_A E = \frac{1}{2}.$$

Suppose that $\{a_n\}$ is a sequence of positive numbers with $\sum_n a_n < +\infty$. Give sequences $\{c_k\}_{k \geq 1}$ and $\{n_k\}_{k \geq 1}$ such that $c_k \in (0, 1)$ and $n_k \in \mathbb{N} \cap [2, +\infty)$ for all $k \in \mathbb{N}$. We always assume that $c_* = \inf_k c_k > 0$. Let I be the initial set such that $\text{int}(I) \neq \emptyset$. For each $i_1 i_2 \cdots i_{k-1} \in D_{k-1}$, suppose that

$$I_{i_1 i_2 \cdots i_{k-1} 1}, I_{i_1 i_2 \cdots i_{k-1} 2}, \dots, I_{i_1 i_2 \cdots i_{k-1} n_k} \subset I_{i_1 i_2 \cdots i_{k-1}}$$

are geometrically similar to $I_{i_1 i_2 \cdots i_{k-1}}$ such that

$$c_k(1 - a_k) \leq \frac{|I_{i_1 i_2 \cdots i_{k-1} j}|}{|I_{i_1 i_2 \cdots i_{k-1}}|} \leq c_k(1 + a_k), \quad j = 1, 2, \dots, n_k,$$

where the interiors of $I_{i_1 i_2 \cdots i_{k-1} j}$ are pairwise disjoint. We call

$$K = \bigcap_{k=1}^{+\infty} \bigcup_{i_1 i_2 \cdots i_k \in D_k} I_{i_1 i_2 \cdots i_k}$$

a *Cantor-like set*, and we write $\mathcal{C}(I, \{c_k\}_k, \{n_k\}_k, \{a_k\}_k)$ for the collection of such sets.

Remark 2 Cantor-like sets may not be Moran sets.

Theorem 2 Suppose that $K \in \mathcal{C}(I, \{c_k\}_k, \{n_k\}_k, \{a_k\}_k)$ is a Cantor-like set. Then

$$\dim_A K = \lim_{m \rightarrow +\infty} \sup_k \frac{\log(n_k n_{k+1} \cdots n_{k+m})}{-\log(c_k c_{k+1} \cdots c_{k+m})}.$$

In fact, for Ahlfors-David regular set, using (1), there exist constants $0 < \eta < 1 \leq \lambda$ and $1 < \delta \leq \Delta < +\infty$ such that, for all $x, x' \in K$ and $r \leq |K|$,

$$\lambda^{-1} \leq \frac{\mu(B(x, r))}{\mu(B(x', r))} \leq \lambda, \tag{3}$$

$$\delta \leq \frac{\mu(B(x, r))}{\mu(B(x, \eta r))} \leq \Delta. \tag{4}$$

It follows from (4) that the measure μ and set K are doubling, and K is uniformly perfect [14]. We say that a compact subset K of X is *homogeneous* if there exists a Borel measure μ supported on K satisfying (3) and (4), and we refer the readers to [14] for details.

Remark 3 All Ahlfors-David regular sets are homogeneous, but homogeneous sets may not be Ahlfors-David regular.

Given a point $x \in K$, we write

$$\alpha_x(r) = \frac{\log \mu(B(x, r))}{\log r}, \quad 0 < r \leq |K|.$$

Here, $\alpha_x(r)$ is like the function with respect to pointwise dimension of measure.

Given $\varepsilon > 0$, we write

$$\Omega = \{g(r): (0, \varepsilon) \rightarrow \mathbb{R}^+ \mid 0 < \inf_{r < \varepsilon} g(r) \leq \sup_{r < \varepsilon} g(r) < +\infty\}.$$

For each $g \in \Omega$, we focus on the behavior of function $g(r)$ as r tends to 0. If a mapping $h \in \Omega$ satisfies that, for all $r < \varepsilon$,

$$|h(r) - g(r)| \leq C |\log r|^{-1} \quad (5)$$

for some constant C , we say that h and g are *equivalent*, denoted by $g \sim h$, and we write equivalence class $[g] = \{h: g \sim h\}$. By the result of [14], we have $\alpha_x(r) \in \Omega$. Notice that $\alpha_x(r) \sim \alpha_{x'}(r)$ by (3). We use $h(r)$ to denote any function in the equivalence class $[\alpha_x(r)]$ with $x \in K$, and $h(r)$ is called a *scale function* of K .

Remark 4 For Ahlfors-David s -regular set, we can take $h(r) \equiv s$.

It is easy to check that

$$\dim_{\mathbb{H}} K = \liminf_{r \rightarrow 0} h(r), \quad \dim_{\mathbb{P}} K = \limsup_{r \rightarrow 0} h(r),$$

see [14]. Similarly, scale functions also play an important role in the Assouad dimension formula of homogeneous sets.

Theorem 3 Suppose that K is homogeneous with a scale function $h(r)$. Then

$$\dim_{\mathbb{A}} K = \limsup_{\rho \rightarrow 0} \sup_R \left| \frac{h(R) \log R - h(\rho R) \log(\rho R)}{\log \rho} \right|.$$

Remark 5 Suppose that $h(r)$ is defined on $(0, \varepsilon)$. Using (4), we can obtain that for all $\varepsilon_1, \varepsilon_2 \leq \varepsilon$,

$$\limsup_{\rho \rightarrow 0} \sup_{R < \varepsilon_1} \psi(R, \rho) = \limsup_{\rho \rightarrow 0} \sup_{R < \varepsilon_2} \psi(R, \rho),$$

where

$$\psi(R, \rho) = \left| \frac{h(R) \log R - h(\rho R) \log(\rho R)}{\log \rho} \right|.$$

For each Cantor-like set $K \in \mathcal{C}(I, \{c_k\}_k, \{n_k\}_k, \{a_k\}_k)$, using the approach in [14], it is clear that K is homogeneous with a scale function

$$h(r) = \frac{\log(n_1 n_2 \cdots n_k)}{-\log(c_1 c_2 \cdots c_k)}, \quad c_1 c_2 \cdots c_k |I| < r \leq c_1 c_2 \cdots c_{k-1} |I|.$$

Therefore, Theorem 2 follows immediately from Theorem 3.

Remark 6 Using the result in [14], for every Cantor-like set K as above, we have

$$\dim_{\mathbb{H}} K = \varliminf_{k \rightarrow +\infty} \frac{\log(n_1 n_2 \cdots n_k)}{-\log(c_1 c_2 \cdots c_k)}, \quad \dim_{\mathbb{P}} K = \varlimsup_{k \rightarrow +\infty} \frac{\log(n_1 n_2 \cdots n_k)}{-\log(c_1 c_2 \cdots c_k)}.$$

For the rest of the paper, we will prove Theorems 1 and 3 in Sections 2 and 3, respectively.

2 Assouad dimension of Moran set

Suppose that $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\phi_k\})$, where

$$\phi_k = (c_{k,1}, c_{k,2}, \dots, c_{k,n_k}), \quad k = 1, 2, \dots$$

Without loss of generality, we assume that $|J| = 1$.

For each word $\mathbf{u} = u_1 u_2 \cdots u_k \in D_k$, we write $|\mathbf{u}|$ ($= k$) for the length of \mathbf{u} . Given $k, k' \in \mathbb{N}$, we write

$$D_{k,k'} = \{\mathbf{v} = v_k v_{k+1} \cdots v_{k'} : 1 \leq v_j \leq n_j, k \leq j \leq k'\}, \quad k \leq k',$$

otherwise, $D_{k,k'} = \{\emptyset\}$. Note that $D_{1,k} = D_k$. For $\mathbf{v} = v_k \cdots v_{k'} \in D_{k,k'}$, we write

$$c_{\mathbf{v}} = c_{k,v_k} c_{k+1,v_{k+1}} \cdots c_{k',v_{k'}},$$

with $c_{\emptyset} = 1$. For

$$\mathbf{u} = u_1 u_2 \cdots u_{k-1} \in D_{k-1}, \quad \mathbf{v} = v_k v_{k+1} \cdots v_{k'} \in D_{k,k'},$$

we write

$$\mathbf{u} * \mathbf{v} = u_1 u_2 \cdots u_{k-1} v_k v_{k+1} \cdots v_{k'} \in D_{k'}.$$

For $\mathbf{v} \in D_{k,k'}$, we denote by \mathbf{v}^- the word obtained by deleting the last letter of \mathbf{v} . Note that $\mathbf{v}^- = \emptyset$ (the empty word) if $k = k' - 1$. Given $\mathbf{u} \in D$, for $0 < \delta < c_*$, we write

$$\mathcal{A}_{\mathbf{u}}(\delta) = \{\mathbf{u} * \mathbf{v} \in D : c_{\mathbf{v}} \leq \delta < c_{\mathbf{v}^-}\}.$$

For $\mathbf{u} = \emptyset$, we write $\mathcal{A}(\delta)$ for $\mathcal{A}_{\emptyset}(\delta)$.

Let

$$\Lambda = \{u_1 u_2 \cdots u_k \cdots : u_1 u_2 \cdots u_k \in D_k, \forall k\}$$

be the symbolic system composed of infinite words. Given a word $\mathbf{i} = i_1 i_2 \cdots i_n \in D_n$, we call

$$[\mathbf{i}] = \{u_1 u_2 \cdots u_n \cdots \in \Lambda : u_1 u_2 \cdots u_n = i_1 i_2 \cdots i_n\}$$

the cylinder with respect to the word \mathbf{i} .

Lemma 1 Given $\mathbf{u} \in D$, we have

$$1 = \sum_{\mathbf{u} * \mathbf{v} \in \mathcal{A}_{\mathbf{u}}(\delta)} \frac{(c_{\mathbf{v}})^s}{\prod_{p=|\mathbf{u}|+1}^{|\mathbf{u}|+|\mathbf{v}|} \sum_{q=1}^{n_p} c_{p,q}^s}.$$

Proof Fix $\mathbf{u} \in D$. Then we have a probability measure μ supported on $[\mathbf{u}]$ such that

$$\mu([\mathbf{u} * \mathbf{v}]) = \frac{(c_{\mathbf{v}})^s}{\prod_{p=|\mathbf{u}|+1}^{|\mathbf{u}|+|\mathbf{v}|} \sum_{q=1}^{n_p} c_{p,q}^s}, \quad \forall \mathbf{u} * \mathbf{v} \in D.$$

Since $[\mathbf{u}] = \cup_{\mathbf{u} * \mathbf{v} \in \mathcal{A}_{\mathbf{u}}(\delta)} [\mathbf{u} * \mathbf{v}]$ is a disjoint union, we obtain

$$1 = \sum_{\mathbf{u} * \mathbf{v} \in \mathcal{A}_{\mathbf{u}}(\delta)} \mu([\mathbf{u} * \mathbf{v}]) = \sum_{\mathbf{u} * \mathbf{v} \in \mathcal{A}_{\mathbf{u}}(\delta)} \frac{(c_{\mathbf{v}})^s}{\prod_{p=|\mathbf{u}|+1}^{|\mathbf{u}|+|\mathbf{v}|} \sum_{q=1}^{n_p} c_{p,q}^s}. \quad \square$$

The following lemma can be obtained directly by using [5, Lemma 9.2].

Lemma 2 Suppose $c_* > 0$. Then there exists a positive integer l such that for all $\delta \in (0, c_*)$, $\mathbf{u} \in D$, and $x \in E \cap J_{\mathbf{u}}$, we have

$$\#\{\mathbf{u} * \mathbf{v} \in \mathcal{A}_{\mathbf{u}}(\delta) \mid B(x, c_{\mathbf{u}}\delta) \cap J_{\mathbf{u} * \mathbf{v}} \neq \emptyset\} \leq l.$$

In particular, if \mathbf{u} is the empty word, we have

$$\#\{\mathbf{v} \in \mathcal{A}(\delta) \mid B(x, \delta) \cap J_{\mathbf{v}} \neq \emptyset\} \leq l.$$

Lemma 3 Suppose $c_* > 0$. Let $s_{k,k+m}$ be defined by (2). Then the sequence $\{\sup_k s_{k,k+m}\}_{m=1}^{+\infty}$ is convergent.

Proof Suppose $E \subset \mathbb{R}^d$. Denote by \mathcal{L} the Lebesgue measure on \mathbb{R}^d . Recall that $c_* > 0$ and $n_k \geq 2$. Since for each $\mathbf{u} \in D_{k-1}$,

$$\text{int}(J_{\mathbf{u} * i}) \cap \text{int}(J_{\mathbf{u} * j}) = \emptyset,$$

for all $i \neq j \leq n_k$, we have

$$\sum_{i=1}^{n_k} \mathcal{L}(\text{int}(J_{\mathbf{u} * i})) \leq \mathcal{L}(\text{int}(J_{\mathbf{u}})),$$

that is,

$$\sum_{i=1}^{n_k} c_{k,i}^d \leq 1.$$

It implies

$$\max_i c_{k,i} \leq (1 - c_*^d)^{1/d}, \quad \sup_k n_k \leq (c_*)^{-d}. \tag{6}$$

For every m , we write

$$\theta_m = \sup_k s_{k,k+m}.$$

Fix an integer $m \in \mathbb{N}$. Let $s > \theta_m$. For each $n \in \mathbb{Z} \cap [0, m - 1]$, we have

$$\Delta_{t,t+pm+n}(s) = \left(\prod_{i=0}^{p-1} \Delta_{t+im,t+(i+1)m}(s) \right) \cdot \Delta_{t+pm,t+pm+n}(s).$$

Hence, by (6),

$$\Delta_{t+pm,t+pm+n}(s) \leq \left(\sup_k n_k \right)^n \leq (c_*)^{-nd}$$

and

$$\begin{aligned} \Delta_{t+im,t+(i+1)m}(s) &\leq \Delta_{t+im,t+(i+1)m}(\theta_m) \cdot \left(\sup_{k,i} c_{k,i} \right)^{s-\theta_m} \\ &\leq 1 \cdot (1 - c_*^d)^{(s-\theta_m)/d} \\ &= (1 - c_*^d)^{(s-\theta_m)/d}. \end{aligned}$$

Therefore, for all $t \in \mathbb{N}$, we have

$$\Delta_{t,t+pm+n}(s) \leq (1 - c_*^d)^{p(s-\theta_m)/d} c_*^{-nd},$$

which means that there exists an integer $p_0(s)$ such that for all $p \geq p_0(s)$,

$$\Delta_{t,t+pm+n}(s) \leq 1,$$

that is,

$$s_{t,t+pm+n} \leq s$$

for all $p \geq p_0(s)$ and $t \geq 0$. Hence,

$$\overline{\lim}_{p \rightarrow +\infty} \theta_{pm+n} = \overline{\lim}_{p \rightarrow +\infty} \sup_t s_{t,t+pm+n} \leq s.$$

Since it holds for all $s > \theta_m$, we obtain

$$\overline{\lim}_{p \rightarrow +\infty} \theta_{pm+n} \leq \theta_m.$$

Thus,

$$\overline{\lim}_{p \rightarrow +\infty} \theta_{pm+n} \leq \inf_m \theta_m \leq \overline{\lim}_{p \rightarrow +\infty} \theta_m,$$

which implies that $\lim_{m \rightarrow +\infty} \theta_m$ exists. □

Proof of Theorem 1 We first prove that s^{**} is an upper bound of $\dim_A E$. It suffices to verify that the inequality $\dim_A E \leq s$ holds for all $s > s^{**}$.

Since

$$s > \lim_{m \rightarrow +\infty} \sup_k s_{k,k+m},$$

there exists a positive integer N such that, for all $m > N$, we have $s > s_{k,k+m}$. Therefore, for all $m > N$,

$$\prod_{i=k+1}^{k+m} \left(\sum_{j=1}^{n_i} c_{i,j}^s \right) \leq \prod_{i=k+1}^{k+m} \left(\sum_{j=1}^{n_i} c_{i,j}^{s_{k,k+m}} \right) = 1. \quad (7)$$

Fix a word $\mathbf{i} \in D$ and $\delta \in (0, c_{i-})$. The fact that $c_* > 0$ implies that the sequence $\{n_k\}$ is bounded, say $\varpi > 1$, that is, $n_k \leq \varpi$, $k = 1, 2, \dots$. Thus, for all $0 < m \leq N$,

$$\prod_{i=k+1}^{k+m} \left(\sum_{j=1}^{n_i} c_{i,j}^s \right) \leq \varpi^N. \quad (8)$$

By (7) and (8), for all $\mathbf{j} \in D_p$, we have

$$\prod_{p=|\mathbf{i}|+1}^{|\mathbf{i}|+|\mathbf{j}|} \sum_{q=1}^{n_p} c_{p,q}^s \leq \varpi^N. \quad (9)$$

Combining Lemma 1 with (9), we have

$$\begin{aligned} 1 &= \sum_{\mathbf{i}*\mathbf{j} \in \mathcal{A}_1(\delta)} \frac{(c_{\mathbf{j}})^s}{\prod_{p=|\mathbf{i}|+1}^{|\mathbf{i}|+|\mathbf{j}|} \sum_{q=1}^{n_p} c_{p,q}^s} \\ &\geq \varpi^{-N} \sum_{\mathbf{i}*\mathbf{j} \in \mathcal{A}_1(\delta)} (c_{\mathbf{j}})^s \\ &\geq \varpi^{-N} \sum_{\mathbf{i}*\mathbf{j} \in \mathcal{A}_1(\delta)} (c_* c_{\mathbf{j}-})^s \\ &\geq (\varpi^{-N} c_*^s) \cdot \delta^s \cdot \#\mathcal{A}_1(\delta). \end{aligned}$$

It follows that

$$\#\mathcal{A}_1(\delta) \leq \frac{\varpi^N}{c_*^s \delta^s}, \quad \forall \mathbf{i} \in D, \forall \delta \in (0, c_*). \quad (10)$$

Fix a point $x \in E$ and r, R with $0 < r < R$. Since E is doubling, without loss of generality, we may assume that

$$0 < r < c_* R < R < c_*.$$

It is clear that

$$B(x, R) \cap E \subset \bigcup_{\mathbf{i} \in \mathcal{A}(R), B(x, R) \cap J_{\mathbf{i}} \neq \emptyset} J_{\mathbf{i}} \cap E. \quad (11)$$

For each $\mathbf{i} \in \mathcal{A}(R)$ with $B(x, R) \cap J_{\mathbf{i}} \neq \emptyset$, we have

$$J_{\mathbf{i}} \cap E \subset \bigcup_{\mathbf{i}*\mathbf{j} \in \mathcal{A}_1(r/R)} J_{\mathbf{i}*\mathbf{j}}.$$

Now, taking $x_{\mathbf{i},\mathbf{j}} \in J_{\mathbf{i}*\mathbf{j}} \cap E$, we have

$$J_{\mathbf{i}*\mathbf{j}} \subseteq B(x_{\mathbf{i},\mathbf{j}}, r),$$

due to

$$c_{\mathbf{i}*\mathbf{j}} = c_{\mathbf{i}}c_{\mathbf{j}} \leq R \cdot \frac{r}{R} = r, \quad |J| = 1.$$

Thus, by (11), we obtain that

$$B(x, R) \cap E \subset \bigcup_{\mathbf{i} \in \mathcal{A}(R), B(x,R) \cap J_{\mathbf{i}} \neq \emptyset} \bigcup_{\mathbf{i}*\mathbf{j} \in \mathcal{A}_i(r/R)} B(x_{\mathbf{i},\mathbf{j}}, r).$$

By (10) and Lemma 2, we have

$$\begin{aligned} N_{r,R}(E) &\leq \sum_{\mathbf{i} \in \mathcal{A}(R), B(x,R) \cap J_{\mathbf{i}} \neq \emptyset} \#\mathcal{A}_{\mathbf{i}}\left(\frac{r}{R}\right) \\ &\leq \sum_{\mathbf{i} \in \mathcal{A}(R), B(x,R) \cap J_{\mathbf{i}} \neq \emptyset} \frac{\varpi^N}{c_*^s} \left(\frac{R}{r}\right)^s \\ &\leq \frac{\varpi^N}{c_*^s} \left(\frac{R}{r}\right)^s \cdot \#\{\mathbf{i} \in \mathcal{A}(R) : B(x, R) \cap J_{\mathbf{i}} \neq \emptyset\} \\ &\leq \frac{l\varpi^N}{c_*^s} \left(\frac{R}{r}\right)^s. \end{aligned}$$

Hence, s is an upper bound, and the arbitrariness implies that

$$\dim_{\Lambda} E \leq s^{**}.$$

For the rest of the proof, we will verify that s^{**} is also a lower bound.

Since s^{**} is the limit of $\{\sup_k s_{k,k+m}\}$, there exists a sequence $\{(m_k, m'_k)\}_{k=1}^{+\infty}$ of integer pairs with $(m'_k - m_k)$ tending to $+\infty$ such that

$$\lim_{k \rightarrow +\infty} s_{m_k, m'_k} = s^{**}.$$

Arbitrarily choose $s < s^{**}$. Without loss of generality, we assume that, for all $k \in \mathbb{N}$,

$$s_{m_k, m'_k} > s.$$

Hence, it is clear that

$$\Delta_{m_k, m'_k}(s) > \Delta_{m_k, m'_k}(s_{m_k, m'_k}) = 1.$$

Fixing an integer k , we have

$$\sum_{\mathbf{j} \in D_{m_k+1, m'_k}} c_{\mathbf{j}}^s = \Delta_{m_k, m'_k}(s) > 1. \tag{12}$$

For each $p \in \mathbb{N} \cup \{0\}$, let

$$\mathcal{B}_{p,k} = \{\mathbf{j} \in D_{m_k+1,m'_k} : 2^{-p-1} < c_{\mathbf{j}} \leq 2^{-p}\}, \tag{13}$$

and we write

$$p_k = \min\{p : \mathcal{B}_{p,k} \neq \emptyset\}.$$

Since

$$2^{-p-1} \leq c_{\mathbf{j}} \leq (1 - c_*^d)^{(m'_k - m_k)/d},$$

it is obvious that the sequence $\{p_k\}$ tends to infinity, that is,

$$\lim_k p_k = +\infty,$$

Thus, by (12) and (13), we obtain that

$$\sum_{p=0}^{+\infty} \#\mathcal{B}_{p,k} 2^{-ps} > 1. \tag{14}$$

Hence, for any $\varepsilon > 0$, there exists an integer $q_k (\geq p_k)$ such that

$$2^{-\varepsilon q_k} (1 - 2^{-\varepsilon}) \leq \#\mathcal{B}_{q_k,k} (2^{-q_k})^s, \tag{15}$$

otherwise,

$$\sum_{p=0}^{+\infty} \#\mathcal{B}_{p,k} 2^{-ps} < \sum_{p=0}^{+\infty} 2^{-\varepsilon p} (1 - 2^{-\varepsilon}) = 1,$$

which contradicts (14). Since p_k tends to $+\infty$ and $q_k \geq p_k$, we have

$$\lim_k q_k = +\infty.$$

Given $\mathbf{i} \in D_{m_k}$, we take

$$R_k = |J_{\mathbf{i}}|, \quad r_k = \min_{\mathbf{j} \in \mathcal{B}_{q_k,k}} |J_{\mathbf{i}*\mathbf{j}}| \in [2^{-q_k-1}|J_{\mathbf{i}}|, 2^{-q_k}|J_{\mathbf{i}}|].$$

Since

$$|J_{\mathbf{i}*\mathbf{j}}| \in [2^{-q_k-1}|J_{\mathbf{i}}|, 2^{-q_k}|J_{\mathbf{i}}|], \quad \forall \mathbf{j} \in \mathcal{B}_{q_k,k},$$

and

$$\text{int}(J_{\mathbf{i}*\mathbf{j}}) \cap \text{int}(J_{\mathbf{i}*\mathbf{j}'}) = \emptyset$$

for all $\mathbf{j} \neq \mathbf{j}' \in \mathcal{B}_{q_k,k}$, by Lemma 2 again, there exists a positive integer l' independent of k such that each ball with radius $r_k (\in [2^{-q_k-1}|J_{\mathbf{i}}|, 2^{-q_k}|J_{\mathbf{i}}|])$ intersects at most l' elements in $\{J_{\mathbf{i}*\mathbf{j}}\}_{\mathbf{j} \in \mathcal{B}_{q_k,k}}$.

To prove the lower bound, we need the following inequality:

$$\frac{\#\mathcal{B}_{q_k,k}}{l'} \leq N_{r_k,R_k}(E). \tag{16}$$

Notice that $J_{\mathbf{i}} \subset B(z, R_k)$ for all $z \in J_{\mathbf{i}}$. We assume that there exists a smallest number t such that $B(z, R_k)$ can be covered by t balls of radius r_k , e.g.,

$$B(z, R_k) \subset B(x_1, r_k) \cup B(x_2, r_k) \cup \dots \cup B(x_t, r_k).$$

Notice that $t \leq N_{r_k, R_k}(E)$ and

$$\bigcup_{\mathbf{j} \in \mathcal{B}_{q_k, k}} J_{\mathbf{i}*\mathbf{j}} \subset J_{\mathbf{i}} \subset B(z, R_k) \subset B(x_1, r_k) \cup B(x_2, r_k) \cup \dots \cup B(x_t, r_k).$$

Then for any $\mathbf{j} \in \mathcal{B}_{q_k, k}$, there exists at least a ball $B(x_i, r_k)$, $1 \leq i \leq t$, such that

$$J_{\mathbf{i}*\mathbf{j}} \cap B(x_i, r_k) \neq \emptyset,$$

that is,

$$\mathcal{B}_{q_k, k} \subset \bigcup_{i=1}^t \{\mathbf{j} \in \mathcal{B}_{q_k, k} : J_{\mathbf{i}*\mathbf{j}} \cap B(x_i, r_k) \neq \emptyset\}.$$

Therefore, we have

$$\begin{aligned} \#\mathcal{B}_{q_k, k} &\leq \sum_{i=1}^t \#\{\mathbf{j} \in \mathcal{B}_{q_k, k} : J_{\mathbf{i}*\mathbf{j}} \cap B(x_i, r_k) \neq \emptyset\} \\ &\leq t \cdot l' \\ &\leq N_{r_k, R_k}(E) \cdot l', \end{aligned}$$

which completes the proof of inequality (16).

For any $\zeta > 0$, there exists C_ζ such that for any k ,

$$N_{r_k, R_k}(E) \leq C_\zeta \left(\frac{R_k}{r_k}\right)^{\dim_A E + \zeta}. \tag{17}$$

Therefore, using (15)–(17), we have

$$\begin{aligned} \frac{2^{q_k(s-\varepsilon)}(1-2^{-\varepsilon})}{l'} &\leq \frac{\#\mathcal{B}_{q_k, k}}{l'} \\ &\leq N_{r_k, R_k}(E) \\ &\leq C_\zeta \left(\frac{R_k}{r_k}\right)^{\dim_A E + \zeta} \\ &\leq C_\zeta (2^{q_k+1})^{\dim_A E + \zeta}. \end{aligned}$$

Since $\lim_k q_k = +\infty$, by letting $k \rightarrow +\infty$, it gives

$$\dim_A E + \zeta \geq s - \varepsilon.$$

By taking $\varepsilon \rightarrow 0$ and $\zeta \rightarrow 0$, we have $\dim_A E \geq s$ for all $s < s^{**}$, and thus, $\dim_A E \geq s^{**}$. □

3 Assouad dimension of homogeneous set

In this section, we will prove the dimension formula for homogeneous sets.

Lemma 4 *Suppose that $K \subset X$ is doubling. Then*

$$\dim_A K = \limsup_{\rho \rightarrow 0} \sup_{R < \varepsilon} \frac{\log N_{\rho R, R}(K)}{-\log \rho}, \quad \forall \varepsilon < |K|. \quad (18)$$

Proof First, we prove that

$$\dim_A K = \overline{\lim}_{\rho \rightarrow 0} \sup_{R < |K|} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)}. \quad (19)$$

Arbitrarily choose a real α such that

$$\alpha > \overline{\lim}_{\rho \rightarrow 0} \sup_{R < |K|} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)}.$$

Then, there exists $\delta \in (0, 1)$ such that, for all $\rho < \delta$, we have

$$\sup_{R < |K|} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} < \alpha,$$

that is,

$$N_{r, R}(K) \leq \left(\frac{R}{r}\right)^\alpha, \quad 0 < r < \delta R < R < |K|.$$

On the other hand, there exists a constant $c_\delta > 0$ such that

$$N_{r, R}(K) \leq c_\delta, \quad \forall r \in (\delta R, R).$$

Hence, for all $0 < r < R < |K|$, we have

$$N_{r, R}(K) \leq c_\delta \left(\frac{R}{r}\right)^\alpha,$$

which implies that $\alpha \geq \dim_A K$. Since α is arbitrarily chosen, we have

$$\overline{\lim}_{\rho \rightarrow 0} \sup_{R < |K|} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} \geq \dim_A K.$$

Suppose that α is a fixed number such that, for all $R < b$,

$$N_{\rho R, R}(K) \leq c \left(\frac{R}{\rho R}\right)^\alpha,$$

where b is a constant. Then

$$\sup_{R < b} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} \leq \frac{\log c}{\log R - \log(\rho R)} + \alpha.$$

Taking limit on both sides, we have

$$\overline{\lim}_{\rho \rightarrow 0} \sup_{R < b} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} \leq \alpha.$$

Using the doubling property of K , we have

$$N_{\rho R, R}(K) \leq N_{\rho b/2, b/2}(K) \cdot N_{b/2, |K|}(K), \quad \forall R \in [b, |K|],$$

which implies

$$\overline{\lim}_{\rho \rightarrow 0} \sup_{R < |K|} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} = \overline{\lim}_{\rho \rightarrow 0} \sup_{R < b} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} \leq \alpha.$$

Since the inequality holds for all $\alpha > \dim_A K$, it follows that

$$\overline{\lim}_{\rho \rightarrow 0} \sup_{R < |K|} \frac{\log N_{\rho R, R}(K)}{\log R - \log(\rho R)} \leq \dim_A K,$$

which finishes the proof of (19).

We write

$$t(\rho) = \sup_R \frac{\log N_{\rho R, R}(K)}{-\log \rho}.$$

To obtain the formula (18), by (19), it is sufficient to show that the limit of $t(\rho)$ exists as ρ tends to 0.

Give $\rho > 0$. For any $\rho' < \rho$, there exists an integer m such that

$$\rho^{m+1} \leq \rho' < \rho^m.$$

Since

$$N_{r_1, r_3}(K) \leq N_{r_1, r_2}(K) N_{r_2, r_3}(K), \quad r_1 < r_2 < r_3,$$

it follows that

$$N_{(\rho')R, R}(K) \leq N_{\rho^{m+1}R, R}(K) \leq \left(\sup_r N_{\rho r, r}(K) \right)^{m+1}.$$

Hence, we have

$$\left| \frac{\log N_{(\rho')R, R}(K)}{\log \rho'} \right| \leq \left| \frac{\log(\sup_r N_{\rho r, r}(K))^{m+1}}{\log(\rho'/\rho^{m+1}) + (m+1)\log \rho} \right|,$$

which implies that

$$\overline{\lim}_{\rho' \rightarrow 0} t(\rho') \leq \lim_{m \rightarrow +\infty} \left| \frac{\log(\sup_r N_{\rho r, r}(K))^{m+1}}{\log(\rho'/\rho^{m+1}) + (m+1)\log \rho} \right| = t(\rho)$$

due to $1 \leq \rho'/\rho^{m+1} \leq \rho^{-1}$. Therefore, we obtain that

$$\overline{\lim}_{\rho' \rightarrow 0} t(\rho') \leq \inf_{\rho} t(\rho) \leq \underline{\lim}_{\rho' \rightarrow 0} t(\rho'),$$

that is,

$$\lim_{\rho \rightarrow 0} t(\rho) = \inf_{\rho} t(\rho).$$

On the other hand, since K is doubling, for any $\varepsilon_1, \varepsilon_2 < |K|$, we have

$$\lim_{\rho \rightarrow 0} \sup_{R < \varepsilon_1} \frac{\log N_{\rho R, R}(K)}{-\log \rho} = \lim_{\rho \rightarrow 0} \sup_{R < \varepsilon_2} \frac{\log N_{\rho R, R}(K)}{-\log \rho}. \quad \square$$

Proof of Theorem 3 Fix a point $x_0 \in K$. It is clear that $h \sim \alpha_{x_0}$. By (5), for $r < R$, we have

$$\begin{aligned} |\alpha_{x_0}(r) \log r - h(r) \log r| &\leq C, \\ |\alpha_{x_0}(R) \log R - h(R) \log R| &\leq C. \end{aligned}$$

Hence,

$$\begin{aligned} &\left| \frac{h(R) \log R - h(r) \log r}{\log R - \log r} - \frac{\alpha_{x_0}(R) \log R - \alpha_{x_0}(r) \log r}{\log R - \log r} \right| \\ &\leq \left| \frac{\alpha_{x_0}(R) \log R - h(R) \log R}{\log R - \log r} \right| + \left| \frac{\alpha_{x_0}(r) \log r - h(r) \log r}{\log R - \log r} \right| \\ &\leq \frac{2C}{|\log(R/r)|}. \end{aligned} \quad (20)$$

Suppose that k is the smallest number of balls with radius r needed to cover $B(x, R)$, i.e., suppose that $B(x, R)$ is covered by $B(y_1, r), B(y_2, r), \dots, B(y_k, r)$. In fact, we can choose

$$k = N_{r, R}(K). \quad (21)$$

Then

$$\mu(B(x, R)) \leq \sum_{i=1}^k \mu(B(y_i, r)),$$

which implies

$$\frac{\mu(B(x, R))}{\max_{y \in K} \mu(B(y, r))} \leq k.$$

Using (3), we have

$$\lambda^{-2} \frac{\mu(B(x_0, R))}{\mu(B(x_0, r))} \leq k. \quad (22)$$

We also assume that p is the largest number of disjoint $(r/2)$ -balls with centers in $B(x, R)$, for example, $B(z_1, r/2), B(z_2, r/2), \dots, B(z_p, r/2)$ are pairwise disjoint. By the routine argument, we have

$$k \leq p.$$

In the same way,

$$p \min_{y \in K} \mu\left(B\left(y, \frac{r}{2}\right)\right) \leq \sum_{i=1}^p \mu(B(z_i, r)) \leq \mu(B(x, R+r)) \leq \mu(B(x, 2R)).$$

Therefore, using (3), we have

$$k \leq p \leq \frac{\mu(B(x, 2R))}{\min_y \mu(B(y, r/2))} \leq \lambda^2 \frac{\mu(B(x_0, 2R))}{\mu(B(x_0, r/2))}. \quad (23)$$

Using (4), the measure μ is doubling, i.e., there is a constant $D > 0$ such that

$$\begin{aligned} \mu(B(x_0, 2R)) &\leq D\mu(B(x_0, R)), \\ \mu\left(B\left(x_0, \frac{r}{2}\right)\right) &\geq D^{-1}\mu(B(x_0, r)). \end{aligned}$$

Then (23) shows that

$$k \leq (\lambda D)^2 \frac{\mu(B(x_0, R))}{\mu(B(x_0, r))}. \quad (24)$$

Combining (21), (22), and (24), we obtain that

$$\begin{aligned} &\frac{\log \lambda^{-2}}{\log R - \log r} + \frac{\alpha_{x_0}(R) \log R - \alpha_{x_0}(r) \log r}{\log R - \log r} \\ &\leq \frac{\log N_{r,R}(K)}{\log R - \log r} \\ &\leq \frac{\log(\lambda D)^2}{\log R - \log r} + \frac{\alpha_{x_0}(R) \log R - \alpha_{x_0}(r) \log r}{\log R - \log r}. \end{aligned} \quad (25)$$

By Lemma 4, (20), and (25), we obtain that

$$\dim_A K = \limsup_{\rho \rightarrow 0} \sup_R \frac{h(R) \log R - h(\rho R) \log(\rho R)}{-\log \rho}. \quad \square$$

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