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LIPSCHITZ EQUIVALENCE OF MCMULLEN SETS

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Abstract

Let sets E and F be McMullen sets with the same number of rectangles in each line. We show that E and F are Lipschitz equivalent if they are dust-like or they satisfy the horizontal block separation condition (HBSC).

Keywords: McMullen Set; Self-Affine; Lipschitz Equivalence; Fractal.

1. INTRODUCTION

Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces. If there exists a bijection $f : X_1 \to X_2$ which is bi-Lipschitz, i.e., there exists a constant $C > 0$ such that

$$C^{-1}\rho_1(x, y) \leqslant \rho_2(f(x), f(y)) \leqslant C\rho_1(x, y),$$

for all $x, y \in X_1$, we say X_1 and X_2 are *Lipschitz equivalent*, and denote this by $X_1 \sim X_2$. If the space X is $\mathbb{R}^n$, we take the metric ρ be the ordinary Euclidean metric.

An important topic in fractal geometry is to classify fractals under Lipschitz equivalence. Any two fractals in the same Lipschitz equivalence class may be considered to have the same geometric structure. There are many works on this. For example, Cooper and Pignataro[1] studied order-preserving bi-Lipschitz mappings between self-similar sets in $\mathbb{R}^1$. Falconer and Marsh[2,3] studied quasi-circles and dust-like self-similar sets, and obtained a necessary condition for Lipschitz equivalence between

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dust-like self-similar sets, which was further studied by Rao *et al.*,[4] and by Xi.[5] The Lipschitz equivalence of totally disconnected self-similar sets was discussed in Refs. 6–9. Wen *et al.*,[10] studied Lipschitz equivalence of a class of general Sierpinski carpets in $\mathbb{R}^2$ which have non-trivial connected components. Xi and Xiong[11] discussed the Lipschitz equivalence of Moran sets with a cubic pattern, which generalized results of Ref. 9. For other works on Lipschitz equivalence, see Refs. 12–18.

Most work on Lipschitz equivalence focuses on self-similar sets and self-conformal sets. One advantage of these sets is that there exists an isometry between the attractor and the corresponding sequence space, unfortunately, this property does not hold generally for self-affine sets due to the complexity of its geometric structure. Up till now, there has been little work on general self-affine sets. In the paper, we study the Lipschitz equivalence on a typical class of self-affine fractals, namely, McMullen sets.

Let m, n, r be positive integers such that $n > m \geqslant 2$ and $1 \leqslant r \leqslant mn$. Let r_j be nonnegative integers such that $0 \leqslant r_j \leqslant n-1$, $j = 0, \ldots, m-1$, and $r_0 + \cdots + r_{m-1} = r$. Let $R = \{d_0, \ldots, d_{r-1}\} \subset \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ be such that $\mathrm{card}\{i : (i,j) \in R, 0 \leqslant i \leqslant n-1\} = r_j, 0 \leqslant j \leqslant m-1$.

For each $d_k = (d_k^{(1)}, d_k^{(2)}) \in R$, we define a self-affine transformation by

$$S_k(x) = T(x + d_k), \quad x \in \mathbb{R}^2, \quad k = 0, 1, \ldots, r-1,$$

where $T = \mathrm{diag}(n^{-1}, m^{-1})$; then the family $\{S_k\}_{k=0}^{r-1}$ forms a self-affine iterated function system. According to Hutchinson,[19] there exists an attractor E, called a McMullen set,[20,21] such that $E = \bigcup_{k=0}^{r-1} S_k(E)$; the set E may also be written as

$$E = \left\{ \left(\sum_{k=1}^{\infty} \frac{d_{i_k}^{(1)}}{n^k}, \sum_{k=1}^{\infty} \frac{d_{i_k}^{(2)}}{m^k} \right) : i_k \in \{0, 1, \ldots, r-1\} \right\}.$$

Without loss of generality, we always assume that, for all $u, v \in \{0, \ldots, r-1\}$,

$$\text{if } u < v \quad \text{then } d_u^{(2)} < d_v^{(2)}, \quad \text{or}, \quad d_u^{(2)} = d_v^{(2)} \text{ and } d_u^{(1)} < d_v^{(1)}. \tag{1}$$

This just implies that the rectangles $S_0([0,1]^2)$, $S_1([0,1]^2), \ldots, S_{r-1}([0,1]^2)$ are numbered from left to right and from bottom to top.

As stated in Refs. 20 and 21, the Hausdorff dimension of E is

$$\dim_H E = \frac{\log \sum_{j=0}^{m-1} r_j^{\log m/\log n}}{\log m}.$$

The formula indicates that the Hausdorff dimension depends not only on r but also on r_i, that is to say, moving selected rectangles from a line to another may cause the change of dimension, which is very different to self-similar cases. We use $\mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$ to denote the collection of all such McMullen sets, i.e., with the number of rectangles in each line fixed. Obviously, $\dim_H E = \dim_H F$ for all $E, F \in \mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$.

We denote the unit square $[0,1]^2$ by Q. For $k = 0, 1, 2, \ldots$, let Ω^k be the set of all k-term sequences of integers $0, 1, 2, \ldots, r-1$, that is $\Omega^k = \{(\sigma_1 \cdots \sigma_k) : 0 \leq \sigma_j \leq r-1\}$; we regard Ω^0 as just containing the empty sequence, that is $\Omega^0 = \{\emptyset\}$; We abbreviate members of Ω^k by $\sigma = (\sigma_1 \cdots \sigma_k)$ and write $|\sigma| = k$ for the number of terms in σ. We write $\Omega = \bigcup_{k=0}^{\infty} \Omega^k$ for the set of all such finite sequences, and Ω^∞ for the corresponding set of infinite sequences, so $\Omega^\infty = \{(\sigma_1\sigma_2 \cdots \sigma_k \cdots) : 0 \leqslant \sigma_k \leqslant r-1\}$. For $\sigma = \sigma_1 \cdots \sigma_k \in \Omega^k$, $\tau = \tau_1 \cdots \tau_l \in \Omega^l$, write $\sigma * \tau = \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_l \in \Omega^{k+l}$. We write $\sigma|k = (\sigma_1 \cdots \sigma_k)$ for the *curtailment* after k terms of $\sigma = (\sigma_1\sigma_2 \cdots) \in \Omega^\infty$. We write $\sigma \preceq \tau$ if σ is a curtailment of τ. We call the set $[\sigma] = \{\tau \in \Omega^\infty : \sigma \preceq \tau\}$ the *cylinder* of σ, where $\sigma \in \Omega$. If $\sigma = \emptyset$, its cylinder is $[\sigma] = \Omega^\infty$.

Let $E \in \mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$. We call the McMullen set E *dust-like*, if $S_i(E) \cap S_j(E) = \emptyset$ for all $i \neq j \in \{0, \ldots, r-1\}$. We denote by $\mathcal{DR}(n, m, r, r_0, \ldots, r_{m-1})$ the collection of all dust-like McMullen sets in $\mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$. When E is dust-like, for every $x \in E$, there exists a unique sequence $\{\sigma_i\}_{i=1}^{\infty} \subset \Omega^\infty$, such that $x = \sum_{i=1}^{\infty} T^i d_{\sigma_i}$. We call $\{\sigma_i\}_{i=1}^{\infty}$ the Ω-sequence of x throughout this paper.

We are also interested in another class of McMullen sets, written as $\mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1}) \subset \mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$, which are not generally dust-like.

Let A be a subset of $\mathbb{R}^2$ and b a point in $\mathbb{R}^2$. We write $A + b = \{a + b : a \in A\} \subset \mathbb{R}^2$. Let $E \in \mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$. For each $\sigma \in \Omega^k$, we call $J_\sigma = S_\sigma(Q)$ *the basic rectangle* of level k for σ, and write $E_k = \bigcup_{\sigma \in \Omega^k} J_\sigma$ for the union of all basic rectangles of level k.

We say that the McMullen set E satisfies *horizontal block separation condition* ($HBSC$) if one of the following properties holds:

(I) Let $J \neq J' \in \{S_i(Q) : i = 0, \ldots, r-1\}$. Then J and J' are disjoint;

(II) Let $J \neq J' \in \{S_i(Q), S_i(Q) + (1, 0) : i = 0, \ldots, r-1\}$ such that $J \cap J' \neq \emptyset$. Then J and J' lie in the same horizontal line.

It is equivalent to say that, in each level, if two rectangles are connected, then they are located in the same line. In particularly, the second condition plays the essential role in Sec. 3, which enables us to divide the rectangles into blocks line by line.

We write $\mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1})$ for the collection of all McMullen sets in $\mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$ satisfying HBSC. It is easy to see that if the McMullen set E satisfies HBSC, then for all $k \geqslant 1$ and any two distinct $J, J' \in E_k$

$$J' = J + (n^{-k}, 0), \quad \text{or} \quad J' = J - (n^{-k}, 0), \quad \text{whenever } J \cap J' \neq \emptyset.$$

Obviously, every McMullen set in $\mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1})$ is totally disconnected, but may not be dust-like, see Example 4.1 in Sec. 4.

The main results in the paper are

Theorem 1.1. *Let E and F be two McMullen sets in $\mathcal{DR}(n, m, r, r_0, \ldots, r_{m-1})$. Then the sets E and F are Lipschitz equivalent.*

Theorem 1.2. *Let E and F be two McMullen sets in $\mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1})$. Then the sets E and F are Lipschitz equivalent.*

The proof of Theorem 1.1 is given in Sec. 2. Section 3 is devoted to the proof of Theorem 1.2. Afterwards, we gives some examples and further remarks in Sec. 4.

2. LIPSCHITZ EQUIVALENCE OF DUST-LIKE MCMULLEN SETS

The following lemma gives the geometric interpretation of dust-like set.

Lemma 2.1. *A McMullen set E is dust-like if and only if there exists a nonnegative integer M, such that*

$$\left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{i\sigma_1 \cdots \sigma_M}(Q) \right) \cap \left(\bigcup_{\tau_1 \cdots \tau_M \in \Omega^M} S_{j\tau_1 \cdots \tau_M}(Q) \right) = \emptyset, \tag{2}$$

for all $i \neq j \in \{0, 1, \ldots, r-1\}$.

Proof. Suppose that E is dust-like, and assume that for each integer $M > 0$

$$A_M = \bigcup_{i \neq j, 0 \leq i, j \leq r-1} \left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{i\sigma_1 \cdots \sigma_M}(Q) \right) \cap \left(\bigcup_{\tau_1 \cdots \tau_M \in \Omega^M} S_{j\tau_1 \cdots \tau_M}(Q) \right) \neq \emptyset,$$

we will obtain a contradiction. Since A_M is a non-empty compact set, and $A_{M+1} \subset A_M$, the intersection $\bigcap_{M=0}^{\infty} A_M$ is also a non-empty compact set.

Choose any $x \in \bigcap_{M=0}^{\infty} A_M$, we have there exist $i \neq j$ such that

$$x \in \bigcap_{M=0}^{\infty} \left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{i\sigma_1 \cdots \sigma_M}(Q) \right) = S_i(E),$$

and

$$x \in \bigcap_{M=0}^{\infty} \left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{j\sigma_1 \cdots \sigma_M}(Q) \right) = S_j(E).$$

This implies $S_i(E) \cap S_j(E) \neq \emptyset$, which contradicts the assumption that the set E is dust-like.

Conversely, we know that

$$E = \bigcap_{k=0}^{\infty} \bigcup_{\sigma_1 \cdots \sigma_k \in \Omega^k} S_{\sigma_1 \cdots \sigma_k}(Q),$$

obviously $E \subset \bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{\sigma_1 \cdots \sigma_M}(Q)$. Hence, for each $i = 0, 1, \ldots, r-1$,

$$S_i(E) \subset S_i \left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{\sigma_1 \cdots \sigma_M}(Q) \right) = \bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{i\sigma_1 \cdots \sigma_M}(Q).$$

Similarly, we have

$$S_j(E) \subset \bigcup_{\tau_1 \cdots \tau_M \in \Omega^M} S_{j\tau_1 \cdots \tau_M}(Q),$$

for each $j = 0, 1, \ldots, r-1$. By (2), we obtain $S_i(E) \cap S_j(E) = \emptyset$ for all $i \neq j \in \{0, 1, \ldots, r-1\}$, which implies the set E is dust-like. □

The next easily proved lemma is frequently used.

Lemma 2.2. *Let m, n, l, k be positive integers, such that $n > m \geqslant 2$ and $l = [k \frac{\log m}{\log n}]$, where $[\ ]$ denotes the floor function. Then $\frac{n^l}{m^k} \in (\frac{1}{n}, 1]$.*

Proof of Theorem 1.1. We first construct a bijection g from E to F.

Let E and F be two McMullen sets in $\mathcal{DR}(n, m, r, r_0, \ldots, r_{m-1})$, given by

$$E = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) \middle| (x_k, y_k) \in R \right\},$$

$$F = \left\{ \left(\sum_{k=1}^{\infty} \frac{x'_k}{n^k}, \sum_{k=1}^{\infty} \frac{y'_k}{m^k} \right) \middle| (x'_k, y'_k) \in R' \right\},$$

where

$$R = \{d_0, \ldots, d_{r-1}\} \subset \{0, \ldots, n-1\} \times \{0, \ldots, m-1\},$$
$$R' = \{d'_0, \ldots, d'_{r-1}\} \subset \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}.$$

Then the corresponding self-affine IFS $\{S_k(x)\}_{k=0}^{r-1}$ and $\{S'_k(x)\}_{k=0}^{r-1}$ are given by $S_k(x) = T(x + d_k)$, $k = 0, \ldots, r-1$, and $S'_k(x) = T(x + d'_k)$, $k = 0, \ldots, r-1$, respectively, and clearly, $E = \bigcup_{i=0}^{r-1} S_i(E)$ and $F = \bigcup_{i=0}^{r-1} S'_i(F)$.

For each $x = \sum_{k=1}^{\infty} T^k d_{\sigma_k} \in E$, we define a mapping $g : E \to F$ by

$$g(x) = \sum_{k=1}^{\infty} T^k d'_{\sigma_k}, \tag{3}$$

where $\{\sigma_k\}_{k=1}^{\infty} \in \Omega^{\infty}$. As E and F are dust-like, the mapping g is well-defined and bijective from E to F.

We must prove that the bijection g is bi-Lipschitz. Since the sets E and F are dust-like, by Lemma 2.1, there exists a nonnegative integer M, such that

$$\left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S_{i\sigma_1 \cdots \sigma_M}(Q) \right) \cap \left(\bigcup_{\tau_1 \cdots \tau_M \in \Omega^M} S_{j\tau_1 \cdots \tau_M}(Q) \right) = \emptyset, \tag{4}$$

$$\left(\bigcup_{\sigma_1 \cdots \sigma_M \in \Omega^M} S'_{i\sigma_1 \cdots \sigma_M}(Q) \right) \cap \left(\bigcup_{\tau_1 \cdots \tau_M \in \Omega^M} S'_{j\tau_1 \cdots \tau_M}(Q) \right) = \emptyset, \tag{5}$$

for all $i \neq j \in \{0, 1, 2 \ldots, r-1\}$.

We choose any $x, y \in E$ such that $x \neq y$, and let $\{\sigma_i\}_{i=1}^{\infty}$ and $\{\tau_i\}_{i=1}^{\infty}$ be the Ω-sequences of x and y respectively. Suppose $\sigma_i = \tau_i$ for $i = 1, \ldots, l-M-1$, and $\sigma_{l-M} \neq \tau_{l-M}$, where $l \geqslant M+1$ is an integer. By (4), we have $S_{\sigma_{l-M}\sigma_{l-M+1}\cdots\sigma_l}(Q) \cap S_{\tau_{l-M}\tau_{l-M+1}\cdots\tau_l}(Q) = \emptyset$, so that

$$S_{\sigma_1 \cdots \sigma_{l-M-1}}(S_{\sigma_{l-M}\sigma_{l-M+1}\cdots\sigma_l}(Q)) \cap S_{\tau_1 \cdots \tau_{l-M-1}}(S_{\tau_{l-M}\tau_{l-M+1}\cdots\tau_l}(Q)) = \emptyset,$$

i.e., $S_{\sigma_1 \cdots \sigma_l}(Q) \cap S_{\tau_1 \cdots \tau_l}(Q) = \emptyset$. By the same argument, we have $S'_{\sigma_1 \cdots \sigma_l}(Q) \cap S'_{\tau_1 \cdots \tau_l}(Q) = \emptyset$.

Let $k = \min\{s \in \mathbb{N} : [s \frac{\log m}{\log n}] = l\}$, and write

$$(x_1, x_2) = S_{\sigma_1 \cdots \sigma_l}(0, 0),$$
$$(x_3, x_4) = S_{\sigma_1 \cdots \sigma_k}(0, 0).$$

Obviously, the sets $S_{\sigma_1 \cdots \sigma_k}(Q)$ and $S_{\sigma_1 \cdots \sigma_l}(Q)$ satisfy

$$S_{\sigma_1 \cdots \sigma_k}(Q) \subset [x_1, x_1 + n^{-l}] \times [x_4, x_4 + m^{-k}] \subset S_{\sigma_1 \cdots \sigma_l}(Q),$$

where the rectangle $[x_1, x_1 + n^{-l}] \times [x_4, x_4 + m^{-k}]$ is often named *the approximating square* of x, written as $\triangle_x$, and clearly $x \in \triangle_x$.

Similarly, we have

$$y \in \triangle_y = [y_1, y_1 + n^{-l}] \times [y_4, y_4 + m^{-k}],$$
$$g(x) \in \triangle_{g(x)} = [g(x)_1, g(x)_1 + n^{-l}] \times [g(x)_4, g(x)_4 + m^{-k}],$$
$$g(y) \in \triangle_{g(y)} = [g(y)_1, g(y)_1 + n^{-l}] \times [g(y)_4, g(y)_4 + m^{-k}],$$

where

$$(y_1, y_2) = S_{\tau_1 \cdots \tau_l}(0, 0),$$
$$(y_3, y_4) = S_{\tau_1 \cdots \tau_k}(0, 0),$$
$$(g(x)_1, g(x)_2) = S'_{\sigma_1 \cdots \sigma_l}(0, 0),$$
$$(g(x)_3, g(x)_4) = S'_{\sigma_1 \cdots \sigma_k}(0, 0),$$
$$(g(y)_1, g(y)_2) = S'_{\tau_1 \cdots \tau_l}(0, 0),$$
$$(g(y)_3, g(y)_4) = S'_{\tau_1 \cdots \tau_k}(0, 0).$$

Since $x_2 = \sum_{i=1}^{l} \frac{d_{\sigma_i}^{(2)}}{m^i}$, $x_4 = \sum_{i=1}^{k} \frac{d_{\sigma_i}^{(2)}}{m^i}$, $g(x)_2 = \sum_{i=1}^{l} \frac{d'^{(2)}_{\sigma_i}}{m^i}$, $g(x)_4 = \sum_{i=1}^{k} \frac{d'^{(2)}_{\sigma_i}}{m^i}$, and that $d_i^{(2)} = d'^{(2)}_i$ for all $i \in \{0, 1, 2, \ldots, r-1\}$, due to property (1), we have $g(x)_2 = x_2$, $g(x)_4 = x_4$. Similarly, $g(y)_2 = y_2$, $g(y)_4 = y_4$.

Recall that $S_{\sigma_1\cdots\sigma_l}(Q)\cap S_{\tau_1\cdots\tau_l}(Q)=\emptyset$, $S'_{\sigma_1\cdots\sigma_l}\times(Q)\cap S'_{\tau_1\cdots\tau_l}(Q)=\emptyset$, and $S_{\sigma_1\cdots\sigma_l}(Q), S_{\tau_1\cdots\tau_l}(Q)\subset S_{\sigma_1\cdots\sigma_{l-M-1}}(Q)$. To show g is bi-Lipschitz, we have to take account of the relative positions of $S_{\sigma_1\cdots\sigma_l}(Q)$ and $S_{\tau_1\cdots\tau_l}(Q)$, that is, the two cases $|x_2-y_2|>m^{-l}$ and $|x_2-y_2|\leqslant m^{-l}$.

For the case of $|x_2-y_2|>m^{-l}$, we are able to estimate the distances $\rho(x,y)$ and $\rho(g(x),g(y))$ by

$$\begin{aligned} m^{-l} &\le \rho(x,y)\\ &\leqslant \sqrt{(m^{-(l-M-1)})^2+(n^{-(l-M-1)})^2},\\ m^{-l} &\le \rho(g(x),g(y))\\ &\leqslant \sqrt{(m^{-(l-M-1)})^2+(n^{-(l-M-1)})^2}. \end{aligned}$$

Immediately,

$$\begin{aligned} \frac{1}{\sqrt{2m^{2(M+1)}}}\rho(x,y) &\leqslant \rho(g(x),g(y))\\ &\leqslant \sqrt{2m^{2(M+1)}}\rho(x,y). \end{aligned}$$

For the case of $|x_2-y_2|\leqslant m^{-l}$, according to the relative position of $\triangle_x$ and $\triangle_y$, we consider two different cases: $|x_4-y_4|>m^{-k}$ and $|x_4-y_4|\le m^{-k}$.

For the former case, we know

$$\begin{aligned} &\sqrt{((a-2)m^{-k})^2+(n^{-l})^2}\\ &\qquad\le \rho(x,y)\leqslant \sqrt{(am^{-k})^2+(n^{-(l-M-1)})^2},\\ &\sqrt{((a-2)m^{-k})^2+(n^{-l})^2}\\ &\qquad\le \rho(g(x),g(y))\leqslant \sqrt{(am^{-k})^2+(n^{-(l-M-1)})^2}, \end{aligned}$$

where the constant a is an integer such that $3\le a\le 2m^{k-l}$. By Lemma 2.2, we have

$$\frac{\sqrt{(am^{-k})^2+(n^{-(l-M-1)})^2}}{\sqrt{((a-2)m^{-k})^2+(n^{-l})^2}}\leqslant \sqrt{9+n^{2(M+2)}}.$$

Hence, we have

$$\begin{aligned} \frac{1}{\sqrt{9+n^{2(M+2)}}}\rho(x,y) &\le \rho(g(x),g(y))\\ &\leqslant \sqrt{9+n^{2(M+2)}}\rho(x,y). \end{aligned}$$

For the latter case, we know

$$\begin{aligned} n^{-l} &\le \rho(x,y)\leqslant \sqrt{(2m^{-k})^2+(n^{-(l-M-1)})^2},\\ n^{-l} &\le \rho(g(x),g(y))\leqslant \sqrt{(2m^{-k})^2+(n^{-(l-M-1)})^2}. \end{aligned}$$

By Lemma 2.2, this gives

$$\begin{aligned} \frac{1}{\sqrt{4+n^{2(M+1)}}}\rho(x,y) &\leqslant \rho(g(x),g(y))\\ &\leqslant \sqrt{4+n^{2(M+1)}}\rho(x,y). \end{aligned}$$

Finally, we combine all cases, and obtain that

$$C^{-1}\rho(x,y)\leqslant \rho(g(x),g(y))\leqslant C\rho(x,y),$$

for all $x,y\in E$ with $x\neq y$, where

$$\begin{aligned} C=\max\{&\sqrt{2m^{2(M+1)}},\sqrt{9+n^{2(M+2)}},\\ &\sqrt{4+n^{2(M+1)}}\}. \end{aligned}$$

As the constant C only depends on n,m,M, the bijection g is bi-Lipschitz, i.e., the McMullen sets E and F are Lipschitz equivalent. □

3. LIPSCHITZ EQUIVALENCE OF MCMULLEN SETS SATISFYING THE HBSC

To prove Theorem 1.2, we must find an appropriate bijection from E to F. Normally, the natural projection mapping from the sequence space to the attractors plays an important role. Unfortunately, this does not work in our case, since the projection is not bijective. More precisely, given $E\in\mathcal{SR}(n,m,r,r_0,\ldots,r_{m-1})$, let $\{S_k\}_{k=0}^{r-1}$ be the generating IFS of E. For each $k\in\mathbb{N}$, there exists a one-to-one correspondence Π_k between E_k and Ω^k given by

$$\Pi_k(S_\sigma(Q))=\sigma\quad\text{for all }\sigma\in\Omega^k.$$

Recall that E_k is the union of all basic rectangles of level k. The projection mapping $\Pi:\Omega^\infty\to E$ is given by

$$\Pi((\sigma_k)_{k=1}^\infty)=\bigcap_{\ell=1}^{\infty}S_{\sigma_1\cdots\sigma_\ell}(Q).$$

For each $x\in E$, we denote by $U_{k,x}$ the collection of all k-level basic rectangles containing x. Let $\mathcal{U}_{k,x}=\{\Pi_k(V):V\in U_{k,x}\}$. Then

$$\Pi^{-1}(x)=\bigcap_{k=1}^{\infty}[\mathcal{U}_{k,x}].$$

Since the cardinality of $\Pi^{-1}(x)$ may be greater than 1, the mapping Π need not be injective although we have $\Pi(\Omega^\infty)=E$. Therefore, the projection Π is not the proper candidate for our purpose.

Instead of the projection Π, we define new bijections $f_E : E \to \Omega^\infty$, for each $E \in \mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1})$. Then by the composition of two bijections f_E and f_F^{-1}, we are able to set up a bijection g from E to F. Then, we use the approximating squares to show that it is bi-Lipschitz. The idea to construct bijections $f_E : E \to \Omega^\infty$ is inspired by Xi and Xiong's work,[11] and we believe their idea is very important to solve such Lipschitz equivalent questions for the fractals with connected patterns.

To define the bijection f_E, we make following conventions and notations. Let $E \in \mathcal{R}(n, m, r, r_0, \ldots, r_{m-1})$. For $k \geqslant 1$, a *block* $\mathcal{B}$ of level k is a sub-collection of basic rectangles of level k such that $\bigcup_{J_\sigma \in \mathcal{B}} J_\sigma$ is a connected component of E_k. The cardinality of a block $\mathcal{B}$ is written as $\sharp\mathcal{B}$, and we denote by $\mathfrak{B}_k$ the collection of all blocks of level k. From the horizontal block separation condition, especially the property (II) in the definition on page 3, all basic rectangles in the same block $\mathcal{B}$ of level k lie in the same line of E_k, and $\sharp\mathcal{B} \leqslant n-1$, whenever $E \in \mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1})$. (Recall that $0 \leqslant r_j \leqslant n-1$, $j = 0, \ldots, m-1$, and $r_0 + \cdots + r_{m-1} = r$.)

For $W \subset \Omega^k$, we write $[W] = \bigcup_{\sigma \in W} [\sigma]$, where $[\sigma]$ is the cylinder set of $\sigma \in \Omega^k$. For every collection $\mathcal{A}$ of sets, we write $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ for the union of all sets in $\mathcal{A}$. For each $\mathcal{B} \in \mathfrak{B}_k$, we write $W^{\mathcal{B}}$ as the collection of sequences of the basic rectangles of level k contained in $\mathcal{B}$.

We first establish the following lemma which is a self-affine version of Lemma 11 from Ref. 11. The lemma is essential preparation for the construction of f_E, and its proof relies on the geometry of McMullen sets. We shall redefine the mapping $\Pi_k : E_k \to \Omega^k$, $k \in \mathbb{N}$ in the lemma, which sets up a one-to-one correspondence between all basic rectangles of level k and Ω^k so that each basic rectangle of level k is endowed a unique code in Ω^k. The key idea to it is the "equal grouping" which originated from Rao *et al.* work.[6]

Lemma 3.1. *There exist one-to-one correspondences* $\Pi_k : E_k \to \Omega^k$, $k \in \mathbb{N}$ *such that*

$$W^{\mathcal{B}} := \{\Pi_k(J) : J \in \mathcal{B} \in \mathfrak{B}_k\} \tag{6}$$

satisfying

(a) $W^{\mathcal{B}'} \cap W^{\mathcal{B}''} = \emptyset$ *whenever* $\mathcal{B}' \neq \mathcal{B}'' \in \mathfrak{B}_k$;
(b) *for each* $\mathcal{B} \in \mathfrak{B}_k$ *and each* $\mathcal{B}' \in \mathfrak{B}_{k+1}$ *such that* $\bigcup \mathcal{B}' \subset \bigcup \mathcal{B}$, *there exists some* $\sigma \in W^{\mathcal{B}}$ *such that*

$$[W^{\mathcal{B}'}] \subset [\sigma].$$

Proof. We construct Π_k by induction.

Let $k = 1$. Then $E_1 = \{S_k(Q) : 0 \leq k \leq r-1\}$. The correspondence Π_1 between E_1 and Ω^1 is defined by

$$\Pi_1(S_k(Q)) = k \quad \text{for } k = 0, 1, \ldots, r-1.$$

Thus for each $\mathcal{B} \in \mathfrak{B}_1$ with $\bigcup \mathcal{B} = S_s(Q) \cup S_{s+1}(Q) \cup \cdots \cup S_{s+(\sharp\mathcal{B}-1)}(Q)$ for some $s \in \{0, 1, 2, \ldots, r-1\}$, we have

$$W^{\mathcal{B}} = \{s, s+1, \ldots, s + (\sharp\mathcal{B} - 1)\}.$$

(See (a) in Fig. 1, where $\sigma = \emptyset$.)

Suppose that for $k-1$ ($k \geq 2$), we have established a one-to-one correspondence Π_{k-1} between E_{k-1} and Ω^{k-1}. Then for each $\mathcal{B} \in \mathfrak{B}_{k-1}$, we have obtained

$$W^{\mathcal{B}} = \{\Pi_{k-1}(J) : J \in \mathcal{B}\}.$$

Fix a basic rectangle $U \in E_k$, then there exists a unique k-level block $\mathcal{B}_U \in \mathfrak{B}_k$ containing U. Correspondingly there exists a unique $(k-1)$-level block $\mathcal{B}_{UU} \in \mathfrak{B}_{k-1}$ containing $\mathcal{B}_U$. We define the mapping $\Pi_k(U)$ as follows.

Case 1. $\sharp\mathcal{B}_{UU} = 1$. By induction there exists a unique finite sequence $\tau \in \Omega^{k-1}$ such that $W^{\mathcal{B}_{UU}} = \{\tau\}$. For the time being, the block $\mathcal{B}_{UU}$ just consists of a single $(k-1)$-level rectangle, say J. Hence $\Pi_{k-1}(J) = \tau$, keep in mind that $J = S_\tau(Q)$ may not hold. Let $\tau' \in \Omega^{k-1}$ be such that $J = S_{\tau'}(Q)$. Then J contains r basic rectangles of level k, namely, $S_{\tau'}(S_k(Q)), k = 0, 1, \ldots, r-1$. We define

$$\Pi_k(S_{\tau'}(S_k(Q))) = \tau * k \quad \text{for } k = 0, 1, \ldots, r-1.$$

Note that U is one of $S_{\tau'}(S_k(Q)), k = 0, 1, \ldots, r-1$. This completes the definition of $\Pi_k(U)$. On the other hand, for each k-level block $\mathcal{B}$ contained in J, we have a unique index set $I_{\mathcal{B}} \subset \{0, 1, \ldots, r-1\}$ such that

$$W^{\mathcal{B}} = \{\tau * k : k \in I_{\mathcal{B}}\} \quad \text{and so } [W^{\mathcal{B}}] \subset [\tau].$$

Moreover, any two distinct k-level blocks $\mathcal{B}', \mathcal{B}''$ contained in J satisfy

$$W^{\mathcal{B}'} \cap W^{\mathcal{B}''} = \emptyset.$$

(See (a) in Fig. 1).

Case 2. $\sharp\mathcal{B}_{UU} \geqslant 2$. The rectangles in $\mathcal{B}_{UU}$ are connected, and we assume that

$$\bigcup \mathcal{B}_{UU} = \bigcup_{\ell=1}^{\sharp\mathcal{B}_{UU}} J_\ell,$$

where $J_\ell \in E_{k-1}$ are numbered from left to right. Each J_ℓ contains r basic rectangles of level k, so the

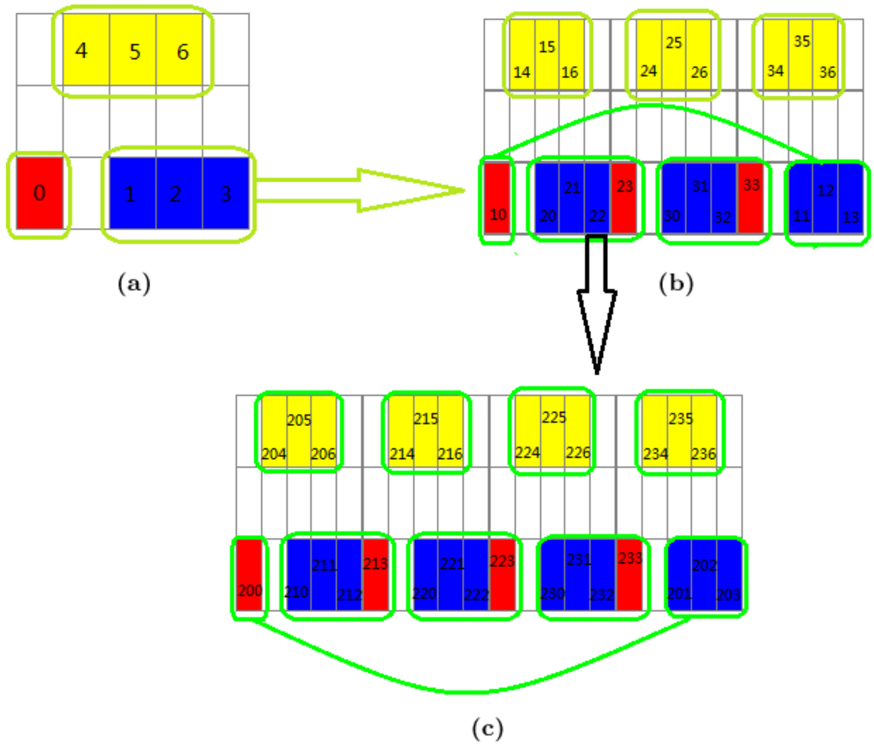

Fig. 1 Assume the square in **(a)** is a block in $\mathfrak{B}_{k-1}$ consisting of a rectangle, the corresponding sequence is σ. The graph indicates rules in Lemma 3.1 to assign sequences to blocks. Note that σ is omitted in the figure, i.e., σ should be added in front of all numbers in **(a)**, **(b)** and **(c)**.

block $\mathcal{B}_{UU}$ contains $r\sharp\mathcal{B}_{UU}$ basic rectangles of level k, which are located in m horizontal lines.

Suppose that U is located at the $(j+1)$th horizontal line with $0 \leq j \leq m-1$ (counted from bottom to top). Let $J_1 = S_\sigma(Q)$ with $\sigma \in \Omega^{k-1}$. (Note that $\Pi_{k-1}(J_1)$ may not equal σ.) Note that $S_i(Q) \in E_1, i = r_0 + \cdots + r_{j-1}, r_0 + \cdots + r_{j-1} + 1, \ldots, r_0 + \cdots + r_{j-1} + r_j - 1$ are all rectangles of level 1, located at the $(j+1)$th horizontal line. Here and below we adopt the convention $r_0 + \cdots + r_{j-1} = 0$ if $j = 0$. Let $T_\ell \subset J_1, \ell = 1, \ldots, r_j$ be all the k-level basic rectangles, from left to right, located at the $(j+1)$th horizontal line. Then

$$T_\ell = S_\sigma(S_{r_0+\cdots+r_{j-1}+\ell-1}(Q)), \quad \ell = 1, \ldots, r_j.$$

Thus all the k-level basic rectangles in the same line of $\bigcup \mathcal{B}_{UU}$ are

$$T_\ell + (sn^{-(k-1)}, 0), \quad \ell = 1, \ldots, r_j,$$
$$s = 0, 1, \ldots, \sharp\mathcal{B}_{UU} - 1.$$

We now regroup these rectangles into $\sharp\mathcal{B}_{UU}$ collections as follows. Let us focus on rectangles $T_\ell \subset J_1, \ell = 1, \ldots, r_j$, which can be divided into γ connected parts, denoted by $\mathcal{Q}_i, i = 1, \ldots, \gamma$ from left to right. Obviously, the sets $\mathcal{Q}_i, i = 1, \ldots, \gamma - 1$ belong to $\mathfrak{B}_k$. We let

$$\mathcal{G}_1 = \left(\bigcup_{i=1}^{\gamma-1} \mathcal{Q}_i\right) \cup (\mathcal{Q}_\gamma + ((\sharp\mathcal{B}_{UU} - 1) \times n^{-(k-1)}, 0)) \quad \text{and so } \sharp\mathcal{G}_1 = r_j,$$

where $\mathcal{Q}_\gamma + ((\sharp\mathcal{B}_{UU} - 1)n^{-(k-1)}, 0) \in \mathfrak{B}_k$ is just the last connected part contained in $J_{\sharp\mathcal{B}_{UU}}$. For the remaining rectangles in $\{T_\ell + (sn^{-(k-1)}, 0), \ell = 1, \ldots, r_j, s = 0, 1, \ldots, \sharp\mathcal{B}_{UU} - 1\} \backslash \mathcal{G}_1$, we regroup them equally into $\sharp\mathcal{B}_{UU} - 1$ collections from left to right, denoted by $\mathcal{G}_2, \ldots, \mathcal{G}_{\sharp\mathcal{B}_{UU}}$. In this way, we can see that $\mathcal{G}_i$ has cardinality r_j and each connected part in $\mathcal{G}_i$ belongs to $\mathfrak{B}_k$. For $1 \leqslant i \leqslant \sharp\mathcal{B}_{UU}$ and a k-level basic rectangle $J \in \mathcal{G}_i$ define

$$\Pi_k(J) = \Pi_{k-1}(J_i) * (r_0 + \cdots + r_{j-1} + \ell - 1),$$

if J is the ℓth (note that $1 \leqslant \ell \leqslant r_j$) k-level basic rectangle from left to right. This completes the definition of $\Pi_k(U)$. (See (b) and (c) in Fig. 1)

With the above definitions for Π_k, one can check that (a) and (b) are satisfied. □

Using this Lemma, we are ready to construct a bijection from E to Ω^∞. For each $x \in E$, there exists a unique sequence of blocks $\mathcal{B}_k \in \mathfrak{B}_k$, $k = 1, 2, \ldots$, such that

$$\{x\} = \bigcap_{k\geq 1} \cup \mathcal{B}_k.$$

For each block $\mathcal{B}$, let the collection $W^{\mathcal{B}}$ consist of those sets defined in Lemma 3.1. The bijection $f_E : E \to \Omega^\infty$ is defined by

$$\{f_E(x)\} = \bigcap_{k\geq 1} [W^{\mathcal{B}_k}].$$

Obviously, the mapping f_E is well-defined and bijective.

We now define the prospective bi-Lipschiz mapping. Let $g : E \to F$ be given by $g(x) = f_F^{-1} f_E(x)$, which is well-defined and bijective for all McMullen sets $E, F \in \mathcal{SR}(n, m, r, r_0, \ldots, r_{m-1})$.

In the rest of the paper, we use $\mathcal{B}$ and $\mathfrak{B}_k$ to denote a block and the collection of all blocks of level k corresponding to E, respectively. And we use $\widetilde{\mathcal{B}}$ and $\widetilde{\mathfrak{B}}_k$ to denote a block and the collection of all blocks of level k corresponding to F, respectively.

The following straightforward lemma gives us the geometric relationship between $x \in E$ and $g(x) \in F$, which plays an important role in the proof of Theorem 1.2.

Lemma 3.2. *Given $x \in E$, suppose that for each integer $k \geqslant 1$ there exists $\mathcal{B} \in \mathfrak{B}_k, \widetilde{\mathcal{B}} \in \widetilde{\mathfrak{B}}_k$ such that $x \in \bigcup \mathcal{B}$ and $g(x) \in \bigcup \widetilde{\mathcal{B}}$. Then $x_2 = g(x)_2$, where x_2 and $g(x)_2$ are the vertical coordinates of the left-lower vertexes of $\bigcup \mathcal{B}$ and $\bigcup \widetilde{\mathcal{B}}$ respectively.*

Proof. Suppose $W^{\mathcal{B}} = \{\tau * \sigma^1, \ldots, \tau * \sigma^{\sharp\mathcal{B}}\}$, where $\tau = \tau_1 \cdots \tau_{k-1} \in \Omega^{k-1}$ and $\sigma^1, \ldots, \sigma^{\sharp\mathcal{B}} \in \{0, 1, \ldots, r-1\}$. From the proof of Lemma 3.1, we know $d^{(2)}_{\sigma^1} = \cdots = d^{(2)}_{\sigma^{\sharp\mathcal{B}}}$. So the vertical coordinate x_2 is given by

$$x_2 = \sum_{i=1}^{k-1} \frac{d^{(2)}_{\tau_i}}{m^i} + \frac{d^{(2)}_{\sigma^1}}{m^k}.$$

By the definition of g, we have $W^{\widetilde{\mathcal{B}}} = \{\tau * \omega^1, \ldots, \tau * \omega^{\sharp\widetilde{\mathcal{B}}}\}$, where $\omega^1, \ldots, \omega^{\sharp\widetilde{\mathcal{B}}} \in \{0, 1, \ldots, r-1\}$, and $d^{(2)}_{\omega^1} = \cdots = d^{(2)}_{\omega^{\sharp\widetilde{\mathcal{B}}}} = d^{(2)}_{\sigma^1}$.

Hence, $g(x)_2 = x_2$. □

Proof of Theorem 1.2. Given two distinct points $x, y \in E$, there exists a positive integer l such that

$$x, y \in \bigcup \mathcal{B}_{x,y}, \quad x \in \bigcup \mathcal{B}_x, \ \ y \in \bigcup \mathcal{B}_y,$$

where $\mathcal{B}_{x,y} \in \mathfrak{B}_{l-1}$, and $\mathcal{B}_x, \mathcal{B}_y \in \mathfrak{B}_l$ with $\mathcal{B}_x \neq \mathcal{B}_y$.

For the corresponding points $g(x)$ and $g(x) \in F$, we assume that

$$g(x) \in \bigcup \widetilde{\mathcal{B}}_{g(x)}, \quad g(y) \in \bigcup \widetilde{\mathcal{B}}_{g(y)},$$
$$\widetilde{\mathcal{B}}_{g(x)}, \widetilde{\mathcal{B}}_{g(y)} \in \widetilde{\mathfrak{B}}_l.$$

Recall that it is possible to have $\widetilde{\mathcal{B}}_{g(x)} = \widetilde{\mathcal{B}}_{g(y)}$.

Let x_2, y_2 be the vertical coordinates of the left-lower vertexes of the rectangles $\bigcup \mathcal{B}_x, \bigcup \mathcal{B}_y$ respectively. We divide the argument into three cases to prove g is bi-Lipschitz according to geometric position of x, y and $g(x), g(y)$.

(1) $|x_2 - y_2| > m^{-l}$.

Since $\sharp\mathcal{B}_{x,y} \leqslant n-1$, $\mathcal{B}_x \neq \mathcal{B}_y$, we have

$$m^{-l} \leq \rho(x, y) \leq \sqrt{(m^{-(l-1)})^2 + ((n-1)n^{-(l-1)})^2}.$$

If $\sharp\mathcal{B}_{x,y} = 1$, write $W^{\mathcal{B}_{x,y}} = \{\sigma\} \subset \Omega^{l-1}$. So $g(x), g(y) \in \bigcup \widetilde{\mathcal{B}}_{g(x),g(y)}$, where $\widetilde{\mathcal{B}}_{g(x),g(y)} \in \widetilde{\mathfrak{B}}_{l-1}$, and by Lemma 3.2 and since $\sharp\widetilde{\mathcal{B}}_{g(x),g(y)} \leqslant n-1$, we have

$$\begin{aligned} m^{-l} &\leqslant \rho(g(x), g(y)) \\ &\leqslant \sqrt{(m^{-(l-1)})^2 + ((n-1)n^{-(l-1)})^2}. \end{aligned}$$

If $\sharp\mathcal{B}_{x,y} \geqslant 2$, write $W^{\mathcal{B}_{x,y}} = \{\sigma * \tau_1, \ldots, \sigma * \tau_{\sharp\mathcal{B}_{x,y}}\}$, where $\sigma \in \Omega^{l-2}, \tau_1, \ldots, \tau_{\sharp\mathcal{B}_{x,y}} \in \{0, 1, \ldots, r-1\}$. So $g(x), g(y) \in \bigcup \widetilde{\mathcal{B}}_{g(x),g(y)}$, where $\widetilde{\mathcal{B}}_{g(x),g(y)} \in \widetilde{\mathcal{B}}_{l-2}$, and by Lemma 3.2 and since $\sharp\widetilde{\mathcal{B}}_{g(x),g(y)} \leq n-1$, we have

$$\begin{aligned} m^{-l} &\leqslant \rho(g(x), g(y)) \\ &\leqslant \sqrt{(m^{-(l-1)})^2 + ((n-1)n^{-(l-2)})^2}. \end{aligned}$$

Since

$$\frac{\sqrt{(m^{-(l-1)})^2 + (n^{-(l-3)})^2}}{m^{-l}} \leqslant \sqrt{m^2 + n^6},$$

we have

$$\begin{aligned} \frac{1}{\sqrt{m^2+n^6}} \rho(x, y) &\leq \rho(g(x), g(y)) \\ &\leqslant \sqrt{m^2+n^6} \rho(x, y). \end{aligned}$$

(2) $|x_2 - y_2| \leqslant m^{-l}$, and $\widetilde{\mathcal{B}}_{g(x)} \neq \widetilde{\mathcal{B}}_{g(y)}$.

Let $k = \min\{s \in \mathbb{N} : [s\frac{\log m}{\log n}] = l\}$. Suppose $x \in \bigcup \mathcal{B}^{(k)}_x, y \in \bigcup \mathcal{B}^{(k)}_y$, $\mathcal{B}^{(k)}_x, \mathcal{B}^{(k)}_y \in \mathfrak{B}_k$. Let x_4, y_4 be the vertical coordinates of the left-lower vertexes of the rectangles $\bigcup \mathcal{B}^{(k)}_x, \bigcup \mathcal{B}^{(k)}_y$ respectively, and let x_1, y_1 be the horizontal coordinates of the left-lower vertexes of the rectangles $\bigcup \mathcal{B}_x, \bigcup \mathcal{B}_y$. Then the approximating squares $\triangle_x$ and $\triangle_y$ of the points x and y are given by

$$\triangle_x = [x_1, x_1 + (\sharp\mathcal{B}_x)n^{-l}] \times [x_4, x_4 + m^{-k}],$$
$$\triangle_y = [y_1, y_1 + (\sharp\mathcal{B}_y)n^{-l}] \times [y_4, y_4 + m^{-k}].$$

Clearly, we have

$$x \in \bigcup \mathcal{B}_x^{(k)} \subset \triangle_x \subset \bigcup \mathcal{B}_x,$$
$$y \in \bigcup \mathcal{B}_y^{(k)} \subset \triangle_y \subset \bigcup \mathcal{B}_y.$$

Depending on the relative positions of $\triangle_x$ and $\triangle_y$, we consider two different cases: $|x_4 - y_4| > m^{-k}$ and $|x_4 - y_4| \leqslant m^{-k}$.

For the first case, we have

$$\sqrt{((a-2)m^{-k})^2 + (n^{-l})^2} \leqslant \rho(x,y) \leqslant \sqrt{(am^{-k})^2 + (n^{-(l-3)})^2},$$

where a is an integer and $a \geqslant 3$.

We estimate the distance of $g(x)$ and $g(y)$ in the same way, and by Lemma 3.2, we obtain that

$$\sqrt{((a-2)m^{-k})^2 + (n^{-l})^2} \leqslant \rho(g(x),g(y)) \leqslant \sqrt{(am^{-k})^2 + (n^{-(l-3)})^2}.$$

By Lemma 2.2, we have following inequality

$$\frac{\sqrt{(am^{-k})^2 + (n^{-(l-3)})^2}}{\sqrt{((a-2)m^{-k})^2 + (n^{-l})^2}} \leqslant \sqrt{9+n^8},$$

which gives

$$\frac{1}{\sqrt{9+n^8}}\rho(x,y) \leqslant \rho(g(x),g(y)) \leqslant \sqrt{9+n^8}\rho(x,y).$$

For the second case, we follow the same argument and obtain

$$n^{-l} \leqslant \rho(x,y) \leqslant \sqrt{(2m^{-k})^2 + ((n-1)n^{-(l-1)})^2} \leqslant \sqrt{(2m^{-k})^2 + (n^{-(l-3)})^2},$$
$$n^{-l} \leqslant \rho(g(x),g(y)) \leqslant \sqrt{(2m^{-k})^2 + ((n-1)n^{-(l-2)})^2} \leqslant \sqrt{(2m^{-k})^2 + (n^{-(l-3)})^2}.$$

Hence, we have

$$\frac{1}{\sqrt{4+n^6}}\rho(x,y) \leqslant \rho(g(x),g(y)) \leqslant \sqrt{4+n^6}\rho(x,y).$$

(3) $x_2 = y_2$, and $\widetilde{\mathcal{B}}_{g(x)} = \widetilde{\mathcal{B}}_{g(y)}$.

Consider

$$x \in \bigcup \mathcal{B}_x^{(l+1)}, \quad y \in \bigcup \mathcal{B}_y^{(l+1)}, \quad g(x) \in \bigcup \widetilde{\mathcal{B}}_{g(x)}^{(l+1)}, \quad g(y) \in \bigcup \widetilde{\mathcal{B}}_{g(y)}^{(l+1)},$$

where

$$\mathcal{B}_x^{(l+1)}, \mathcal{B}_y^{(l+1)} \in \mathfrak{B}_{l+1}, \quad \widetilde{\mathcal{B}}_{g(x)}^{(l+1)}, \widetilde{\mathcal{B}}_{g(y)}^{(l+1)} \in \widetilde{\mathfrak{B}}_{l+1}.$$

Since $\mathcal{B}_x \neq \mathcal{B}_y$, we know

$$\mathcal{B}_x^{(l+1)} \neq \mathcal{B}_y^{(l+1)}, \widetilde{\mathcal{B}}_{g(x)}^{(l+1)} \neq \widetilde{\mathcal{B}}_{g(y)}^{(l+1)}.$$

By considering the approximating squares in the same way as in case 1 and case 2, we can find $C_1 > 0$ such that

$$C_1^{-1}\rho(x,y) \leqslant \rho(g(x),g(y)) \leqslant C_1\rho(x,y).$$

Finally, combining cases 1, 2 and 3, we have

$$C^{-1}\rho(x,y) \leqslant \rho(g(x),g(y)) \leqslant C\rho(x,y),$$

for $x, y \in E$, where C is a constant. Therefore $E \sim F$. □

4. EXAMPLES AND FURTHER REMARKS

Let $\mathcal{TR}(n,m,r,r_0,\ldots,r_{m-1})$ be the collection of all totally disconnected McMullen sets in $\mathcal{R}(n,m,r,r_0,\ldots,r_{m-1})$. We give an example here to illustrate the difference of these four class of McMullen sets.

Example 4.1. Let $n = 7$, $m = 3$ and $r = 6$. We take $r_0 = 3$, $r_1 = 1$ and $r_2 = 2$, and choose four different subsets of $\{0,\ldots,6\} \times \{0,1,2\}$ (see Fig. 2) as following

$$R_1 = \{(0,0),(2,0),(5,0),(6,1),(0,2),(4,2)\},$$
$$R_2 = \{(0,0),(1,0),(6,0),(3,1),(0,2),(6,2)\},$$
$$R_3 = \{(0,0),(2,0),(6,0),(4,1),(0,2),(6,2)\},$$
$$R_4 = \{(0,0),(1,0),(3,0),(1,1),(4,2),(6,2)\}.$$

These give four McMullen sets, and we write them as

$$E_i = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{7^k}, \sum_{k=1}^{\infty} \frac{y_k}{3^k} \right) : (x_k, y_k) \in R_i \right\}, \quad i = 1,2,3,4.$$

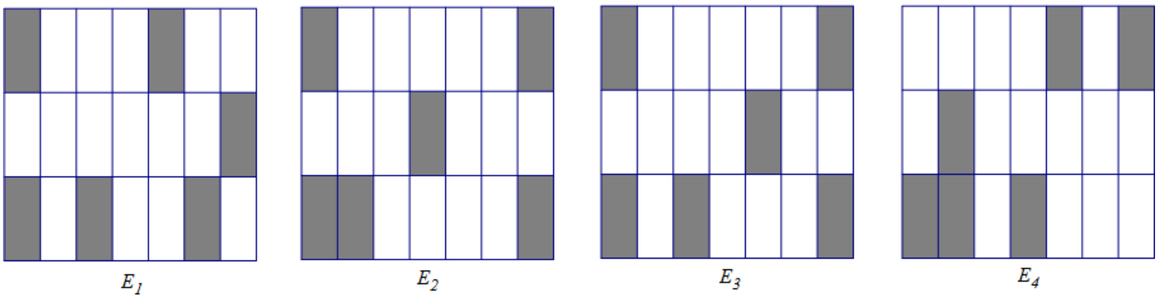

Fig. 2 E_1 and E_3 are dust-like; E_2 and E_3 satisfy HBSC; E_4 is totally disconnected.

It is easy to check that

$$E_1 \in \mathcal{DR}(7,3,6,3,1,2),$$

$$E_2 \in \mathcal{SR}(7,3,6,3,1,2),$$

$$E_3 \in \mathcal{SR}(7,3,6,3,1,2) \cap \mathcal{DR}(7,3,6,3,1,2),$$

$$E_4 \in \mathcal{TR}(7,3,6,3,1,2) \backslash (\mathcal{SR}(7,3,6,3,1,2) \cup \mathcal{DR}(7,3,6,3,1,2)).$$

By Theorem 1.1 and Theorem 1.2, any two of $\{E_1, E_2, E_3\}$ are Lipschitz equivalent.

There are still many open problems. For example, are the sets E_1 and E_4 Lipschitz equivalent? Our result does not cover such cases.

Furthermore, does there exist a necessary and sufficient condition for two totally disconnected McMullen sets in $\mathcal{TR}(n, m, r, r_0, \ldots, r_{m-1})$ to be Lipschitz equivalent? Unfortunately, the method we used does not work for this general question, since the ratio of the height to the width is unbounded with respect to the positive integer k.

Alternatively, one can consider general self-affine sets. Let $S_i = T_i + a_i$ be affine contractions of the plane, where $\|T_i\| < 1/2$, for $i = 1, \ldots, N$. For fixed T_i, let $E(a_1, \ldots, a_N)$ be the attractor. Are $E(a_1, \ldots, a_N)$ bi-Lipschitz equivalent for almost all $(a_1, \ldots, a_N)$ for which strong separation condition holds?

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