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COLORFUL SYMMETRIC IMAGES IN THREE-DIMENSIONAL SPACE FROM DYNAMICAL SYSTEMS

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Abstract

Functions that are invariant with respect to the tetrahedral and cubic symmetries are determined. These invariant mappings are applied to serve as the density functions for automatic generation of the colorful images with such symmetries in three dimensional space from a dynamical system's point of view.

Keywords: Tetrahedron; Cube; Dynamical Systems; Invariant Mapping.

1. INTRODUCTION

The generation of aesthetic images with symmetry by means of dynamical systems has been the subject of considerable study. Such images are simultaneously complex and artistically appealing.[1,2] In Field *et al.*,[3] families of functions which can be used to create chaotic attractors with cyclic, dihedral and some of the planar wallpaper symmetries were investigated. Chaotic attractors with symmetries from each of the planar frieze and crystallographic groups were illustrated in Carter *et al.*[4] A group summation technique for generating chaotic attractors with cyclic and dihedral symmetry was developed in Jones *et al.*[5] Various colorful attractors with the symmetry of higher dimensional point groups based on the requirement that coefficients of general polynomial functions satisfy certain criterion in order to respect the desired symmetry group have also been studied. These have included the attractors with the symmetry of the cube,[6] n-cube,[7] tetrahedron[8] and dodecahedron.[9] In addition, a general method to generate chaotic attractors with crystallographic symmetries in n-dimensional Euclidean space was considered in Dumont *et al.*[10] On the other hand, the visualization of images in non-Euclidean space is uncommon. Automatic generation of aesthetic images in two- and three-dimensional hyperbolic spaces was investigated in Chung *et al.*,[2,11,12] and the created images are rather unusual but exotic. The key idea of the above papers was based on the construction of *equivariant* mapping f which satisfies the condition[3]

$$f \circ \delta = \delta \circ f \tag{1}$$

for any element δ of a symmetry group. The methods of coloring the attractors are based on the point-visiting frequency. Here, one always assumes that the attractor admits a Sinai-Rulle-Bowen (SRB) attractor and has an SRB measure. Thus it lends itself readily to computation of a density function that approximates the SRB measure.[13,14]

In this paper we consider an analogous three-dimensional version of some of the authors' previous work[15] for the creation of colorful images with the symmetries of the tetrahedron and cube from dynamics. We develop general forms for polynomial functions in three dimensional space that are invariant with respect to the required symmetries. These invariant functions are then used to determine the density functions. A map $F : \mathbb{R}^3 \to \mathbb{R}$ is *invariant* with respect to a symmetry group if it satisfies

$$F \circ \delta = F \tag{2}$$

for all elements δ of the group.

2. EQUIVARIANT POLYNOMIAL MAPS

2.1. Tetrahedron-Equivariant Maps

The group of rotational symmetries of the tetrahedron is isomorphic to the alternating group A_4.[16] This rotation group has twelve elements: two rotational symmetries about axes L labeled in Fig. 1 by $2\pi/3$ and $4\pi/3$ and one rotational symmetry about axes M labeled in Fig. 1 by π, which send the tetrahedron to itself. Since there are four axes like L and three axes like M, plus the identity symmetry which is equivariant to a full rotation by 2π about any of the symmetric axes, we have the twelve desired rotations. Suppose that the four vertices of the tetrahedron are $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$, then one can check by direct calculation that the transformations

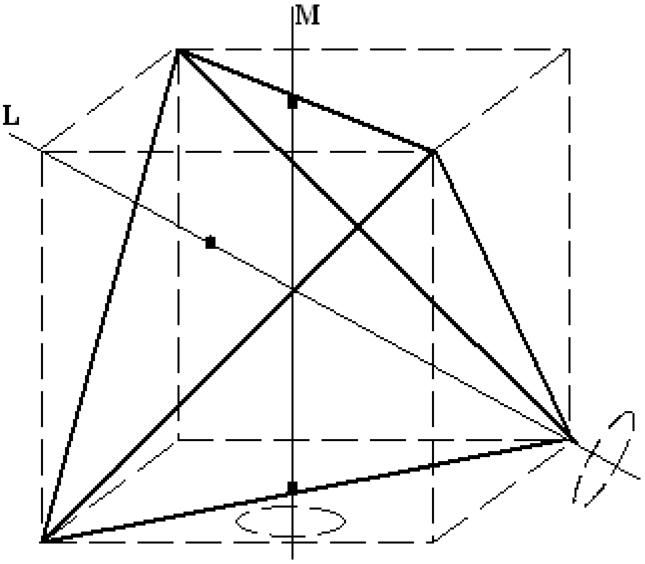

Fig. 1 A tetrahedron embedded in a cube and axes of rotation for a third-turn and a half-turn.

$$\tau_1(x,y,z) = (x,y,z)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\tau_2(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{3}$$

generate A_4. If reflections are also allowed, then one can add

$$\tau_3(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

as a generator as well. This full symmetry group that includes reflections has 24 elements, and we designate it as $\bar{A}_4$.

Theorem 2.1. (i) *A polynomial mapping* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *is* A_4*-equivariant if it can be written as a* *linear combination of functions of the form*

$$T_{ijk}(x,y,z) = (x^i y^j z^k, x^k y^i z^j, x^j y^k z^i), \quad (5)$$

where j and k take the same parity (even or odd) but different from the parity of i.

(ii) *f is $\bar{A}_4$-equivariant if it can be written as the linear combination of functions of the form*

$$T_{ijk}(x,y,z) + T_{ikj}(x,y,z), \quad (6)$$

where j and k take the same parity but different from the parity of i.

For the proof of Theorem 2.1, one can refer to Reiter.[8] Based on Theorem 2.1, we here construct equivariant functions (together with the construction of invariant functions in Sec. 3.1) to generate artistic images with tetrahedral symmetry from dynamics. For example, in Figs. 3, 4 and 5, we take $f(x,y,z) = -1.6122962T_{100}(x,y,z) + 0.541232824T_{300}(x,y,z) + 0.420333344T_{255}(x,y,z) + T_{011}(x,y,z) + 0.0000004T_{033}(x,y,z)$, $f(x,y,z) = 0.68074587T_{100}(x,y,z) + 0.54829554T_{255}(x,y,z) + 1.67162084T_{300}(x,y,z) + 1.0200001T_{011}(x,y,z) - 1.49253822T_{013}(x,y,z)$ and $f(x,y,z) = -1.45139927T_{100}(x,y,z) - 0.35444197T_{211}(x,y,z) + 0.59932249T_{255}(x,y,z) + 0.56465346T_{122}(x,y,z) + 0.0000344T_{011}(x,y,z)$, respectively.

2.2. Cube-Equivariant Maps

Embedding the tetrahedron in a cube (Fig. 1) clearly shows that the symmetry group of the tetrahedron is a subgroup of the symmetry group of the cube, but not vice versa, since the cubic symmetry group has quarter-turns (and some half-turns) that do not preserve the tetrahedron. As shown in Fig. 2, there are twenty four rotations that preserve the eight vertices of a regular cube. These include three axes like M by $\pi/2$, π and $3\pi/2$, which together provide a total of nine rotations, six axes of type N with just one rotation each, and four principal diagonals like L about each of which the cube can be rotated by $2\pi/3$ and $4\pi/3$. Together with the identity symmetry which is equivariant to a full rotation by 2π about any of the symmetric axes. This accounts for all twenty four rotational symmetries. The group of rotational symmetries of the cube is isomorphic to S_4, which is generated by two elements,[6,16] such as

$$\tau_4(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\tau_5(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (7)$$

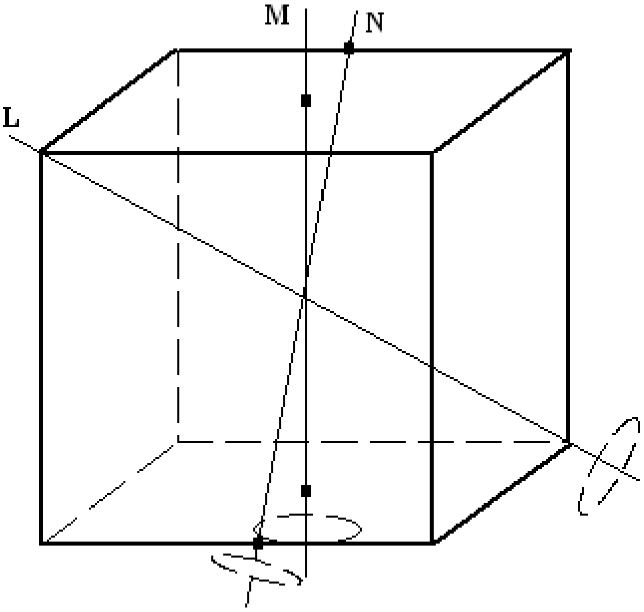

Fig. 2 Three different types of axis of rotation in a cube.

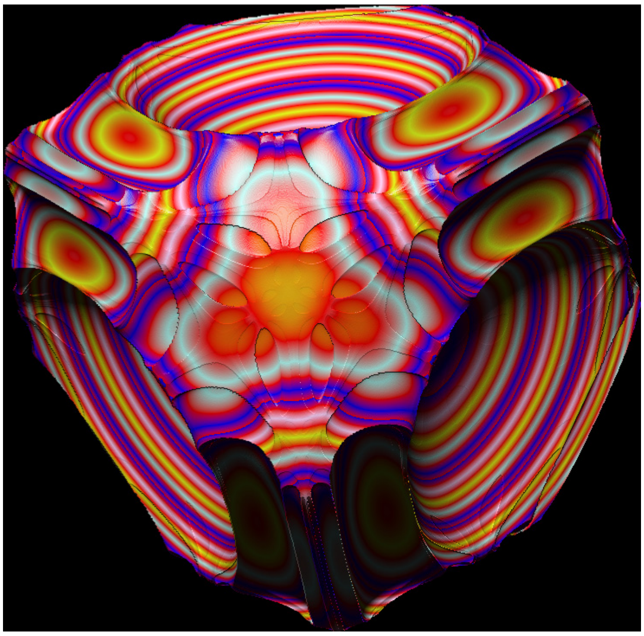

Fig. 3 A colorful image with tetrahedral shape and $\bar{A}_4$-symmetry.

Moreover, all the rotational and reflectional symmetries of the cube are generated by τ_4 and τ_5, along with the reflection τ_6 where

$$\tau_6(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8)$$

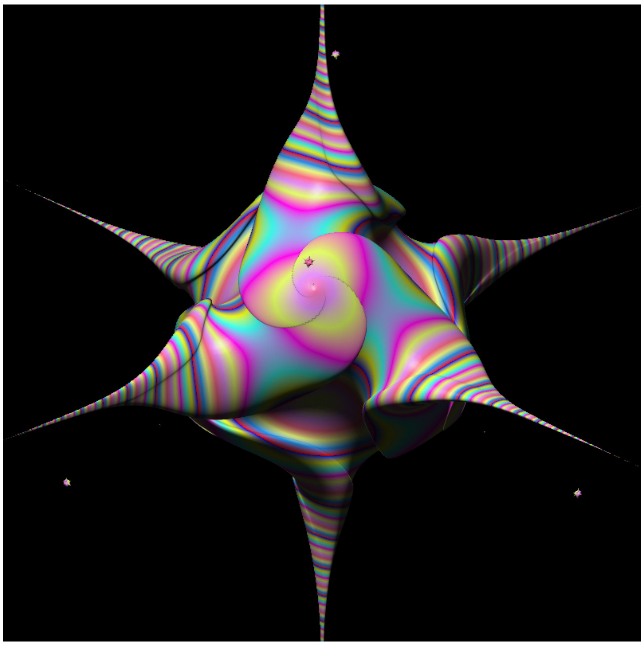

Fig. 4 A colorful image with A_4-symmetry.

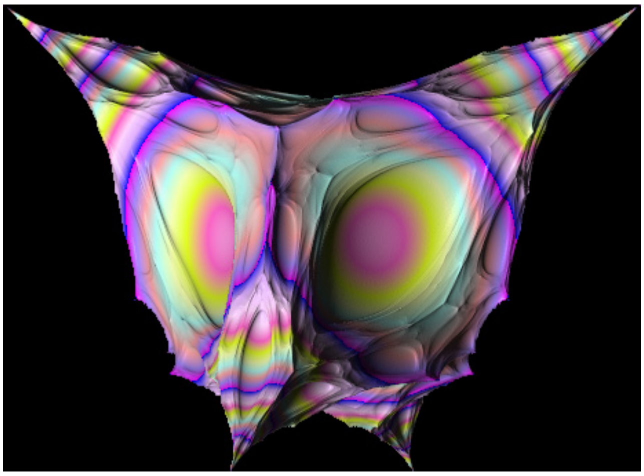

Fig. 5 A colorful image with $\bar{A}_4$-symmetry.

as can be verified by checking. These generate the desired forty eight elements and we denote it by $\bar{S}_4$.

Theorem 2.2. (i) *A polynomial mapping* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *is* S_4*-equivariant if and only if it can be written as a linear combination of polynomial maps,* H_{ijk} *of the form*

$$H_{ijk}(x,y,z) = (x^iy^jz^k + (-1)^jx^iy^kz^j, x^ky^iz^j + (-1)^jx^jy^iz^k, x^jy^kz^i + (-1)^jx^ky^jz^i), \tag{9}$$

where j *and* k *have the same parity, which is different from the parity of* i.

(ii) *f is* $\bar{S}_4$*-equivariant if and only if it can be written as a linear combination of* H'_{ijk}, *where*

$$H'_{ijk}(x,y,z) = (x^iy^jz^k + x^iy^kz^j, x^ky^iz^j + x^jy^iz^k, x^jy^kz^i + x^ky^jz^i), \tag{10}$$

where both j *and* k *are even, and* i *is odd.*

Proof. For the proof of (i), one can refer to Ref. 6.

Let $f(x,y,z) = \sum_{i,j,k\geq 0}(a_{ijk}x^iy^jz^k, b_{ijk}x^iy^jz^k, c_{ijk}x^iy^jz^k), a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$ be an equivariant function with respect to the rotational and reflectional symmetries of the cube. From the definition of equivariance, $f(\tau_6(x,y,z)) = \tau_6 f(x,y,z)$ implies that

$$\begin{cases} \sum_{i,j,k\geq 0} a_{ijk}x^jy^iz^k = \sum_{i,j,k\geq 0} b_{ijk}x^iy^jz^k \\ \sum_{i,j,k\geq 0} b_{ijk}x^jy^iz^k = \sum_{i,j,k\geq 0} a_{ijk}x^iy^jz^k, \end{cases} \tag{11}$$

which shows that

$$a_{ijk} = b_{jik}. \tag{12}$$

On the other hand, the proof of (i) in Brisson *et al.*[6] shows that $a_{ijk} = (-1)^j b_{jik}$. Together with Eq. (12), the term j must be even. From the result of (i), we see that the terms j and k must be even whereas i is odd.

Direct computation shows that the functions H'_{ijk} are equivariant when j, k are even and i is odd. Thus the converse is proven. $\square$

From Theorem 2.2(i), in Figs. 6 and 8, we take $f(x,y,z) = 0.5600011H_{100}(x,y,z) - 0.8120001H_{031}(x,y,z) - 0.8120001H_{304}(x,y,z)$ and $f(x,y,z) = -0.6693931H_{100}(x,y,z) - 0.2092961H_{324}(x,y,z) - 0.5556201H_{013}(x,y,z)$. And according to the restriction of Theorem 2.2(ii), the image shown in Fig. 7 is created by $f(x,y,z) = 0.60786767H_{100}(x,y,z) - 0.6330149H_{122}(x,y,z) - 1.8815882H_{304}(x,y,z)$.

3. INVARIANT POLYNOMIAL FUNCTIONS

In this section we consider the general forms for tetrahedron- and cube-invariant polynomials respectively. These functions are used for creating our 3D images.

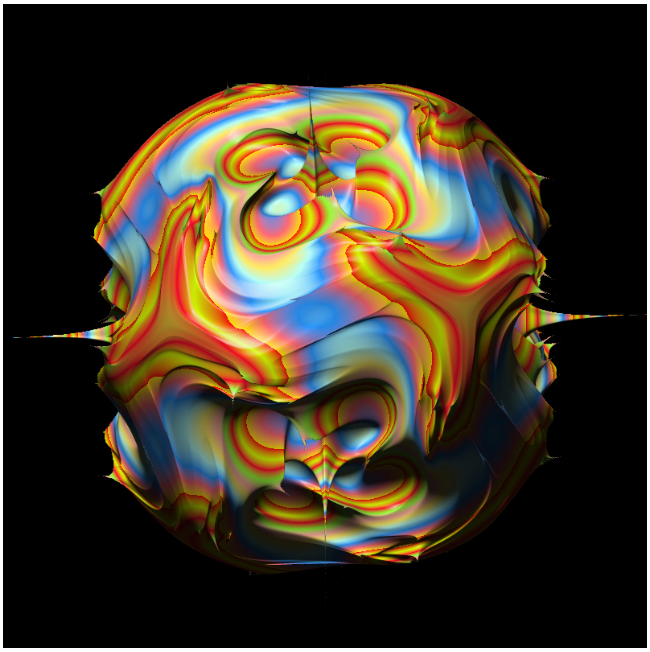

Fig. 6 A colorful image with S_4-symmetry.

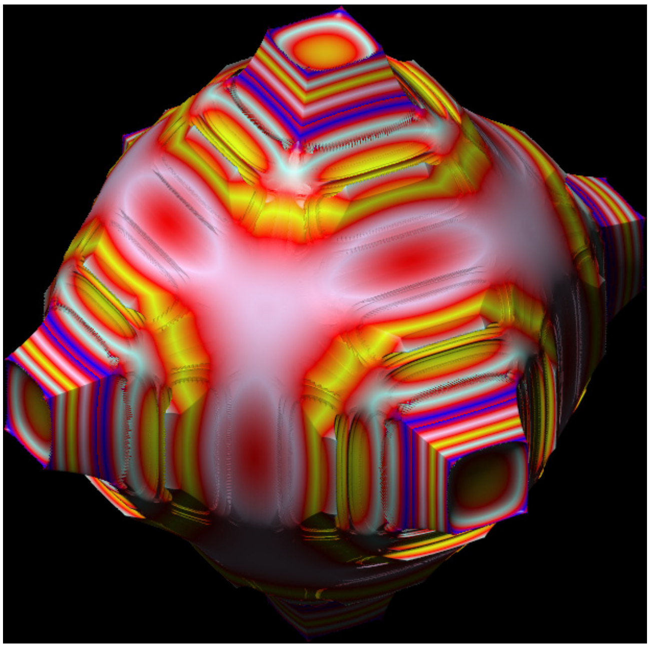

Fig. 7 A colorful image with $\bar{S}_4$-symmetry.

3.1. Tetrahedron-Invariant Functions

Similar to the equivariance, if a function is invariant with respect to the generators of a group then it is invariant with respect to every element of the group. So we only need to develop a function invariant with respect to τ_1 and τ_2 (and τ_3).

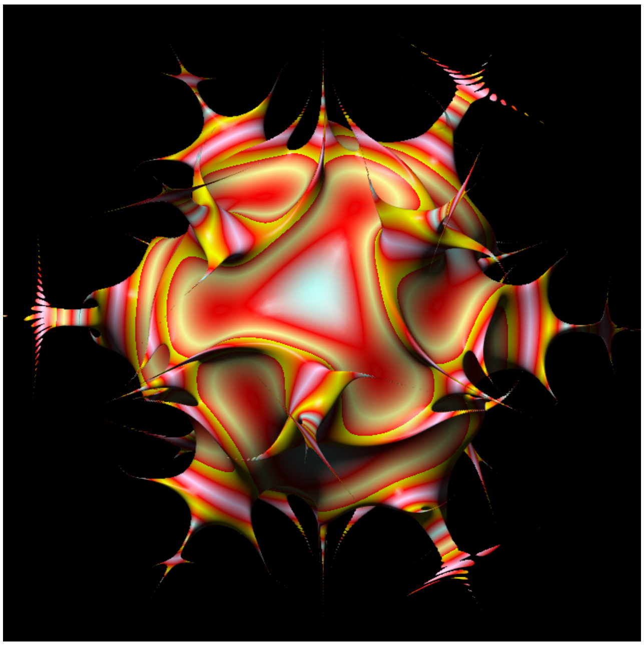

Fig. 8 A colorful image with S_4-symmetry.

Theorem 3.1. (i) *A polynomial function* $F : \mathbb{R}^3 \to \mathbb{R}$ *is invariant with respect to the rotational symmetries of the tetrahedron if and only if it can be written as a linear combination of the functions defined by* h_{ijk}, *where*

$$\begin{aligned} h_{ijk}(x, y, z) &\\ = \alpha(x^i y^j z^k &+ x^k y^i z^j + x^j y^k z^i) + \beta(x^i y^k z^j \\ &+ x^j y^i z^k + x^k y^j z^i), \quad \alpha, \beta \in \mathbb{R} \end{aligned} \tag{13}$$

and i, j, k *have the same parity.*

(ii) F *is invariant with respect to the rotational and reflectional symmetries of the tetrahedron if and only if it can be written as a linear combination of* h_{ijk}, *where* i, j, k *have the same parity and* $\alpha = \beta$.

Proof. (i) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a polynomial function invariant with respect to the rotational symmetries of the tetrahedron. Since F is a polynomial, we can write F in form

$$F(x, y, z) = \sum_{i,j,k \geq 0} b_{ijk} x^i y^j z^k, \tag{14}$$

where $b_{ijk} \in \mathbb{R}$.

Thus the invariance corresponding to

$$\begin{cases} F(x, y, z) = F(\tau_1(x, y, z)) \\ F(x, y, z) = F(\tau_2(x, y, z)) \end{cases} \tag{15}$$

implies that

$$\sum_{i,j,k\geq 0} b_{ijk}x^i y^j z^k = \sum_{i,j,k\geq 0} b_{ijk}(-x)^i(-y)^j z^k$$

$$= \sum_{i,j,k\geq 0} b_{kij}x^i y^j z^k. \tag{16}$$

Compare coefficients for the $x^i y^j z^k$ term which appears in each sum in Eq. (16), then

$$b_{ijk}x^i y^j z^k = b_{ijk}(-1)^{i+j}x^i y^j z^k$$

$$= b_{kij}x^i y^j z^k. \tag{17}$$

and therefore

$$b_{ijk} = (-1)^{i+j}b_{ijk} = b_{kij}. \tag{18}$$

We repeat this procedure for all six possible permutations of the exponents, then

$$\begin{cases} b_{ikj} = (-1)^{i+k}b_{ikj} = b_{jik}, \\ b_{kij} = (-1)^{i+k}b_{kij} = b_{jki}, \\ b_{jik} = (-1)^{i+j}b_{jik} = b_{ikj}, \\ b_{jki} = (-1)^{j+k}b_{jki} = b_{ijk}, \\ b_{kji} = (-1)^{k+j}b_{kij} = b_{jik}. \end{cases} \tag{19}$$

Simplifying Eqs. (18) and (19), we have

$$\begin{cases} b_{ijk} = b_{kij} = b_{jki}, \\ b_{ikj} = b_{jik} = b_{kji}, \end{cases} \tag{20}$$

and

$$(-1)^{i+j} = (-1)^{i+k} = (-1)^{j+k} = 1. \tag{21}$$

Equation (21) shows that non-zero terms i, j, k have the same parity. Thus any invariant function that satisfies Eq. (20) can be written as a linear combination of the function h_{ijk}.

Conversely, direct calculation shows that h_{ijk} is invariant when i, j, k have the same parity, thus completing the proof of (i).

(ii) Let F be an invariant function with respect to the rotational symmetry and reflectional symmetry of the tetrahedron, with the form of Eq. (14). Then, besides Eqs. (20) and (21) for the rotational symmetries, F satisfies $F(x, y, z) = F(\tau_3(x, y, z))$ for the reflectional symmetry. That is,

$$b_{ijk}x^i y^j z^k = b_{jik}x^i y^j z^k \quad \text{and hence}$$

$$b_{ijk} = b_{jik}. \tag{22}$$

Together with Eq. (20), we see that

$$b_{ijk} = b_{kij} = b_{jki} = b_{ikj} = b_{jik} = b_{kji} \tag{23}$$

which completes the necessity of (ii).

Moreover, we can directly check that such functions are invariant with respect to τ_1, τ_2 and τ_3 giving the desired result. □

It follows directly from the definition of invariant mapping in Eq. (2) that for any real function $g : \mathbb{R} \to \mathbb{R}$, $g \circ F$ is also an invariant function if F is an invariant function. So, together with Theorem 3.1(i), we construct $F(x, y, z) = |\cos(x^3yz^5 + x^5y^3z + xy^5z^3 + x^3y^5z + xy^3z^5 + x^5yz^3)| + x^2 + y^2 + z^2$ to create the image shown in Fig. 4. Figure 4 appears to roughly form six conch-like components about a cube. However, notice that quarter-turn of each of the conch-like components cause the oscillations to be misoriented. And four surfaces each of which is represented by three conch-components contain all rotational symmetries of the tetrahedron. So, although there is a cubical underlying form at first glance, the image is of the symmetries of the tetrahedron and not the symmetries of the cube. By Theorem 3.1(ii), we take $F(x, y, z) = 0.5|x^2y^4 + y^2z^4 + x^4z^2 + x^2z^4 + x^4y^2 + y^4z^2| + 3\cos^2(xyz + x^2y^4z^2 + x^2y^2z^4 + x^4y^2z^2)$ and $F(x, y, z) = |xyz + 3\sin(x^2y^4z^2 + x^2y^2z^4 + x^4y^2z^2)| + x^2 + y^2 + z^2 + \cos^2(xy^3z^5 + x^5yz^3 + x^3y^5z + xy^5z^3 + x^3yz^5 + x^5y^3z)$ to generate the images shown in Figs. 3 and 5, respectively. Besides rotational symmetries of the tetrahedron, these figures also have reflectional symmetry. Notice that the underlying shape of a tetrahedron in Fig. 3 consisting of four concave surfaces is apparent. In Fig. 5, we take the four horns as the vertices of the tetrahedron shown as in Fig. 1, it is definitely of the symmetries of the tetrahedron.

3.2. Cube-Invariant Functions

As discussed in Sec. 2.2, the cube-symmetry is isomorphic to S_4 group (or $\bar{S}_4$ group for plus reflectional symmetries). We require the functions invariant with respect to τ_4 and τ_5 (and τ_6).

Theorem 3.2. (i) *A polynomial function* $F : \mathbb{R}^3 \to \mathbb{R}$ *is invariant with respect to group* S_4 *if and only if it can be written as a linear combination of the functions defined by* g_{ijk}, *where*

$$\begin{aligned} g_{ijk}(x, y, z) = x^i y^j z^k &+ (-1)^j x^i y^k z^j \\ &+ x^k y^i z^j + (-1)^j x^j y^i z^k \\ &+ x^j y^k z^i + (-1)^j x^k y^j z^i \end{aligned} \tag{24}$$

and i, j, k *have the same parity.*

(ii) *F is invariant with respect to $\bar{S}_4$ if and only if it can be written as a linear combination of g'_{ijk}, where*

$$\begin{aligned} g'_{ijk}(x,y,z) = x^iy^jz^k &+ x^iy^kz^j \\ &+ x^ky^iz^j + x^jy^iz^k \\ &+ x^jy^kz^i + x^ky^jz^i \end{aligned} \tag{25}$$

and i, j, k are all even.

Proof. The proof of (i) appears in Ref. 6.

Let $F(x,y,z) = \sum_{i,j,k\geq 0} a_{ijk}x^iy^jz^k, a_{ijk} \in \mathbb{R}$ be an invariant function with respect to the rotational and reflectional symmetries of the cube. Notice that the invariance corresponding to $F(x,y,z) = F(\tau_4(x,y,z))$ and $F(x,y,z) = F(\tau_6(x,y,z))$ implies that

$$\begin{aligned} \sum_{i,j,k\geq 0} a_{ijk}x^iy^jz^k &= \sum_{i,j,k\geq 0} a_{ijk}(-y)^ix^jz^k \\ &= \sum_{i,j,k\geq 0} a_{ijk}y^ix^jz^k. \end{aligned} \tag{26}$$

Compare coefficients for the $x^iy^jz^k$ term which appears in each sum in Eq. (26), then

$$a_{ijk} = (-1)^j a_{jik} = a_{jik}, \tag{27}$$

which shows that the term j must be even. Together with the results of (i) discussed in Ref. 6, the terms i, j, k all must be even. Conversely, direct calculation checks that g'_{ijk} is invariant when i, j, k are all even, thus completing the proof of (ii). □

From Theorem 3.2, we construct $F(x,y,z) = |2\cos(x^3yz^5 - x^3y^5z + x^5y^3z - xy^3z^5 + xy^5z^3 - x^5yz^3)| + x^2 + y^2 + z^2$ and $F(x,y,z) = |\sin(x^2y^2z^2) + x^2 + y^2 + z^2 + \cos(x^2y^2z^2)| + \sin^2(x^4y^2z^6 + x^4y^6z^2 + x^6y^4z^2 + x^2y^4z^6 + x^2y^6z^4 + x^6y^2z^4)$ to create the images shown in Figs. 6 and 7, respectively. The image in Fig. 6 contains all rotational symmetries of the cube and its six Eiffel Tower-like components can be regarded as the central axes for rotating around, just like M labeled in Fig. 2. Besides rotational symmetries of the cube, the image in Fig. 7 also has reflectional symmetry. Notice that the underlying shape of a cube in Fig. 7 consisting of six bulging components is apparent.

Theorem 3.3. *Let $P : \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary function and G be a finite group realized by 3 by 3 matrices acting on $\mathbb{R}^3$ by multiplication on the right. Then $F(x,y,z) = \sum_{\tau\in G} P(\tau(x,y,z))$ is invariant with respect to G.*

Proof. Let $\gamma \in G$. Then

$$\begin{aligned} F(\gamma(x,y,z)) &= \sum_{\tau\in G} P(\tau(\gamma(x,y,z))) \\ &= \sum_{\tau\in G} P((\tau\gamma)(x,y,z)) \\ &= F(x,y,z). \end{aligned} \tag{28}$$

by the linearity of γ and the fact that $\tau\gamma$ runs through as τ does. □

From Theorem 3.3, we take the function $P(x,y,z) = |\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} a_{ijk}x^ky^jz^i|$, where the coefficients a_{ijk} are selected randomly between -0.005 and 0.005. As described in Sec. 2.2, the S_4-symmetry group is represented by twenty four 3 by 3 matrices denoted by I_i for $i = 1, \ldots, 24$, which are generated by using matrix multiplication to compose the two generators τ_4 and τ_5 and repeating that on the resulting matrices until the set stabilizes. Therefore, we create our function with the desired symmetry by $F(x,y,z) = \sum_{i=1}^{24} P(I_i(x,y,z))$. By the same argument, we can create the invariant function for $\bar{S}_4$-symmetry group by $F(x,y,z) = \sum_{i=1}^{48} P(I'_i(x,y,z))$, where $I'_i, i = 1, \ldots, 48$, are generated by the three generators τ_4, τ_5 and τ_6. As an example, we use $F(x,y,z) = \sum_{i=1}^{24} P(I_i(x,y,z))$ to produce the image with S_4-symmetry as shown in Fig. 8. Just like Fig. 6, six tower-like components in Fig. 8 can be taken as central axes for rotating around.

4. COLOR SCHEME

We consider the density value of the orbit $\{z_k \in \mathbb{R}^3 | k \geq 0\}$. Let X denote a tetrahedral/cubic images in three-dimensional space, $\psi(X)$ be the associated symmetry group, the density $\hat{F}(z)$ is defined by $\hat{F}(z) = \frac{F(z)}{r}$ where F is an invariant function, r is an empirically determined constant and $r \in \mathbb{R}^+$. For an orbit, we define $\rho^k(z_0) = \hat{F}(f^k(z_0))$ where f is an equivariant mapping with respect to $\psi(X)$ and f^k is the kth iteration of f. For given $n \in \mathbb{Z}^+$, we compute the ρ-values $\{\rho^k(z_0)\}_{k=1}^{n}$. If $\rho^k(z_0) < 1$ for some $k \leq n$, the iteration exits and the $\rho^k(z_0)$ is used to determine the color at z_0. Otherwise, the pixel z_0 is colored by the background color. As the function ρ is invariant under $\psi(X)$, i.e. $\rho(z) = \rho(\gamma z)$ for any $\gamma \in \psi(X)$, the γ-symmetrical points have the same color. It follows that the generated image exhibits the required symmetry. Outstanding features of this color scheme are described in Lu *et al.*[17]

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