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UNIQUE EXPANSION OF POINTS OF A CLASS OF SELF-SIMILAR SETS WITH OVERLAPS

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Abstract. For $q > 1$, the set F_q of real numbers which can be expanded in base q with respect to the digit set $\{0, 1, q\}$ is just a self-similar set with overlaps. We consider the subset of F_q whose elements have a unique expansion and calculate its Hausdorff dimension for the case where $q \geqslant (3+\sqrt{5})/2$.

§1. *Introduction.* We consider the iterated function system (IFS)

$$\phi_k(x) = \frac{1}{q}(x+k), \quad k \in \{0, 1, q\} \text{ and } q > 1.$$

We denote by F_q the *self-similar set* generated by the IFS $\{\phi_0, \phi_1, \phi_q\}$, i.e. F_q is the unique non-empty compact set invariant under the IFS $\{\phi_0, \phi_1, \phi_q\}$:

$$F_q = \phi_0(F_q) \cup \phi_1(F_q) \cup \phi_q(F_q).$$

It is well known that points in F_q can be encoded by digits from $\{0, 1, q\}$. This is done by the so-called *coding map* $\Pi : \{0, 1, q\}^{\mathbb{N}} \to F_q$ defined as follows. For $J = (j_i)_{i=1}^{\infty} \in \{0, 1, q\}^{\mathbb{N}}$,

$$\Pi(J) = \sum_{i=1}^{\infty} \frac{j_i}{q^i}.$$

We will write $(j_i)_{i=1}^{\infty}$ as simply (j_i) if no confusion is likely to arise. We have $\Pi(\{0, 1, q\}^{\mathbb{N}}) = F_q$. An alternative definition of the map Π is that for $J = (j_i) \in \{0, 1, q\}^{\mathbb{N}}$,

$$\Pi(J) = \bigcap_{n=1}^{\infty} \phi_{J|n}\left(\left[0, \frac{q}{q-1}\right]\right)$$

where $J|n = (j_i)_{i=1}^{n}$ and $\phi_{J|n} = \phi_{j_1} \circ \cdots \circ \phi_{j_n}$. Note that the right-hand side above is a singleton. Thus, for $(j_i) \in \{0, 1, q\}^{\mathbb{N}}$,

$$x = \Pi((j_i)) \iff x \in \phi_{j_1} \circ \cdots \circ \phi_{j_n}\left(\left[0, \frac{q}{q-1}\right]\right) \text{ for any } n \in \mathbb{N}.$$

For an $x \in F_q$ and a $(j_k) \in \{0, 1, q\}^{\mathbb{N}}$, (j_k) is called a *q-expansion* (with respect to $\{0, 1, q\}$) of x if $\Pi((j_k)) = x$. Clearly, some $x \in F_q$ may have multiple (even infinitely many) q-expansions.

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One can check (and it also follows directly from the "Pedicini condition" in [15]) that for $1 < q \leqslant (3+\sqrt{5})/2$,

$$[0, q/(q-1)] \\ = \phi_0([0, q/(q-1)]) \cup \phi_1([0, q/(q-1)]) \cup \phi_q([0, q/(q-1)]);$$

that is,

$$F_q = \left[0, \frac{q}{q-1}\right] \quad \text{for } 1 < q \leqslant \frac{3+\sqrt{5}}{2}. \tag{1}$$

This can be seen from the following observations: with $I = [0, q/(q-1)]$,

$$\begin{cases} \phi_0(I) \cap \phi_1(I) \neq \emptyset, \ \phi_1(I) \cap \phi_q(I) = \emptyset & \text{if } q > \dfrac{3+\sqrt{5}}{2}, \\ \phi_i(I) \cap \phi_j(I) \neq \emptyset & \text{if } 2 < q \leqslant \dfrac{3+\sqrt{5}}{2} \text{ and } (i, j) \in \{(0, 1), (1, q)\}, \\ \phi_i(I) \cap \phi_j(I) \neq \emptyset & \text{if } 1 < q \leqslant 2 \text{ and } (i, j) \in \{(0, 1), (1, q), (0, q)\}. \end{cases} \tag{2}$$

Ngai and Wang proved in [14, Example 5.4] that

$$\dim_H F_q = \frac{\log(3+\sqrt{5}) - \log 2}{\log q} \quad \text{for } q > \frac{3+\sqrt{5}}{2}. \tag{3}$$

From (1) and (3) it follows that the generating IFS $\{\phi_0, \phi_1, \phi_q\}$ fails to satisfy the open set condition (OSC) (so F_q is a self-similar set with overlaps), since $\dim_H F_q$ does not equal its similarity dimension $\log 3/\log q$ (see [5, 7, 17]). The reader can refer to [2, 11, 12, 16, 18, 20] for the Hausdorff dimensions of various self-similar sets with overlaps.

For $q > 1$, set

$$\mathcal{U}_q = \{x \in F_q : x \text{ has a unique } q\text{-expansion}\}. \tag{4}$$

The failure of the OSC implies that $\mathcal{U}_q$ has small size. In this paper we mainly investigate the size of $\mathcal{U}_q$. We establish the following result (see also Theorem 3.4).

THEOREM 1.1. *Let q_c $(= 2.32472\ldots)$ be the positive solution of the equation $x^3 - 3x^2 + 2x - 1 = 0$. Then:*

(a) *when $1 < q \leqslant q_c$, $\mathcal{U}_q$ just consists of the endpoints of $[0, q/(q-1)]$, i.e. $\mathcal{U}_q = \{0, q/(q-1)\}$;*
(b) *when $q_c < q < (3+\sqrt{5})/2$, $\mathcal{U}_q$ consists of $2^{\aleph_0}$ many points;*
(c) *when $q \geqslant (3+\sqrt{5})/2$, we have*

$$\dim_H \mathcal{U}_q = \dim_B \mathcal{U}_q = \frac{\log q_c}{\log q} =: \gamma \quad \text{and} \quad 0 < \mathcal{H}^\gamma(\mathcal{U}_q) < \infty.$$

Concerning (a) above, we remark that when $1 < q < 2$, each point of F_q has $2^{\aleph_0}$ q-expansions, except for the two endpoints; see Theorem 4.1.

In 2001, Glendinning and Sidorov [6] considered the set of points $x \in [0, 1/(q-1)]$ with $q \in (1, 2)$ for which there is only one $(\varepsilon_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ satisfying $x = \sum_{n=1}^{\infty} \varepsilon_n q^{-n}$. Our results are analogous to [6, Theorem 2]; however, they are non-trivial because we allow the digit set $\{0, 1, q\}$ to contain a non-integer real number q and we obtain the exact Hausdorff dimension of $\mathcal{U}_q$. Doing so requires new techniques. Unfortunately, we have not been able to give a uniform characterization of the Hausdorff dimension of $\mathcal{U}_q$ for $q_c < q < (3+\sqrt{5})/2$.

Theorem 1.1(c) tells us that points in F_q typically have multiple q-expansions, since $\dim_H(F_q \backslash \mathcal{U}_q) = \dim_H F_q$.

This paper is organized as follows. In order to characterize the set $\mathcal{U}_q$, we discuss the lazy, greedy and quasi-greedy q-expansions of $x \in F_q$ and give a characterization of the greedy q-expansion in §2. The proof of Theorem 1.1 (or Theorem 3.4) is presented in §3. Section 4 is devoted to a discussion of the q-expansions of 1 and $q-1$.

§2. *Characterization of the greedy expansions.* In this section, we characterize the greedy q-expansions of points in F_q. The reader can refer to [1, 3, 4, 6, 8–10, 15, 19] for studies of q-expansions in various settings.

We use the lexicographic order for elements in $\{0, 1, q\}^{\mathbb{N}}$: for two distinct $(u_i), (v_i) \in \{0, 1, q\}^{\mathbb{N}}$, we say that $(u_i) < (v_i)$ (or $(v_i) > (u_i)$) if there exists an $n \in \mathbb{N}$ such that $u_i = v_i$ for $i < n$ and $u_n < v_n$. The expression $(u_i) \leqslant (v_i)$ means $(u_i) < (v_i)$ or $(u_i) = (v_i)$.

For $(u_i) \in \{0, 1, q\}^{\mathbb{N}}$, we say that (u_i) is finite if there exists $n \in \mathbb{N}$ such that $u_i = 0$ for $i > n$. We say that (u_i) is infinite if it is not finite. For an $x \in F_q$, the *greedy* q-expansion of x is defined to be the biggest element in $\Pi^{-1}(x)$ (the set of all q-expansions of x); the *quasi-greedy* q-expansion of x is defined to be the biggest infinite element in $\Pi^{-1}(x)$; and the *lazy* q-expansion of x is defined to be the smallest element in $\Pi^{-1}(x)$. We will use $(\mathcal{G}_i^x)$, $(\mathcal{Q}_i^x)$ and $(\mathcal{L}_i^x)$ to denote the greedy, quasi-greedy and lazy q-expansions of x, respectively. Statements (I)–(VII) below are then clear.

(I) Any $x \in F_q$ has a unique greedy q-expansion and a unique lazy q-expansion, but it may have not a quasi-greedy q-expansion. Furthermore, the quasi-greedy q-expansion of x is unique if it exists, and $(\mathcal{L}_i^x) \leqslant (\mathcal{Q}_i^x) \leqslant (\mathcal{G}_i^x)$ always holds.

(II) $(\mathcal{G}_i^0) = (\mathcal{L}_i^0) = (0)$; that is, 0 has unique q-expansion (0) and has no quasi-greedy q-expansion. We also have $(\mathcal{G}_i^{q/(q-1)}) = (\mathcal{Q}_i^{q/(q-1)}) = (\mathcal{L}_i^{q/(q-1)}) = (q)$. If $(\mathcal{G}_i^x)$ is infinite, then $(\mathcal{G}_i^x) = (\mathcal{Q}_i^x)$.

(III) $x \in F_q$ has a unique q-expansion if and only if its greedy and lazy q-expansions coincide.

(IV) Let $(\tau_i) \in \{0, 1, q\}^{\mathbb{N}}$ be a q-expansion of $x \in F_q$. By the definitions of lazy, greedy and quasi-greedy q-expansions, we have

$$(\tau_i) = (\mathcal{G}_i^x) \text{ if and only if } \sum_{i=1}^{n-1} \frac{\tau_i}{q^i} + \frac{a}{q^n} > x$$
$$\text{when } \tau_n < q \text{ and } \tau_n < a \in \{0, 1, q\},$$

$$(\tau_i) = (\mathcal{Q}_i^x) \text{ if and only if } \sum_{i=1}^{n} \frac{\tau_i}{q^i} < x \text{ and } \sum_{i=1}^{n-1} \frac{\tau_i}{q^i} + \frac{a}{q^n} \geqslant x$$
$$\text{when } \tau_n < q \text{ and } \tau_n < a \in \{0, 1, q\},$$
$$(\tau_i) = (\mathcal{L}_i^x) \text{ if and only if } \sum_{i=1}^{n-1} \frac{\tau_i}{q^i} + \frac{a}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x$$
$$\text{when } \tau_n > 0 \text{ and } \tau_n > a \in \{0, 1, q\}.$$

(V) (This statement can be obtained directly from (IV).) Let $x \in F_q$ and let $k \in \mathbb{N}$. Then

$$(\mathcal{G}_i^y) = (\mathcal{G}_{k+i}^x) \quad \text{with } y = q^k x - \sum_{i=1}^{k} \mathcal{G}_i^x q^{k-i} \in F_q$$

and

$$(\mathcal{L}_i^y) = (\mathcal{L}_{k+i}^x) \quad \text{with } y = q^k x - \sum_{i=1}^{k} \mathcal{L}_i^x q^{k-i} \in F_q;$$

that is, $(\mathcal{G}_{k+i}^x)$ is still a greedy q-expansion of some element of F_q and $(\mathcal{L}_{k+i}^x)$ is still a lazy q-expansion of some element of F_q. In addition, if $(\mathcal{Q}_i^x)$ exists, then $y = q^k x - \sum_{i=1}^k \mathcal{Q}_i^x q^{k-i} \in F_q$ has quasi-greedy q-expansion $(\mathcal{Q}_i^y) = (\mathcal{Q}_{k+i}^x)$.

(VI) If $1 < q \leqslant (3+\sqrt{5})/2$, then each $x \in (0, q/(q-1)]$ has a quasi-greedy q-expansion. This can be seen from (1) or (2).

(VII) Let $x, y \in F_q$. Then $x < y$ if and only if $(\mathcal{G}_i^x) < (\mathcal{G}_i^y)$.

LEMMA 2.1. *Let $x, y \in F_q$. If $(\mathcal{G}_i^x) > (\mathcal{Q}_i^y)$, then $(\mathcal{G}_i^x) \geqslant (\mathcal{G}_i^y)$.*

Proof. Let $\ell \in \mathbb{N}$ be such that

$$\mathcal{G}_\ell^x > \mathcal{Q}_\ell^y \quad \text{and} \quad \mathcal{G}_i^x = \mathcal{Q}_i^y \text{ for } i < \ell.$$

Thus, by (IV),

$$x = \sum_{i=1}^{\infty} \frac{\mathcal{G}_i^x}{q^i} \geqslant \sum_{i=1}^{\ell} \frac{\mathcal{G}_i^x}{q^i} \geqslant y.$$

Hence $(\mathcal{G}_i^x) \geqslant (\mathcal{G}_i^y)$ directly from the definition of greedy q-expansions. □

The following lemma characterizes the greedy q-expansions. When $1 < q < (3+\sqrt{5})/2$, it is included in [**15**, Theorem 2.3] (in fact, it works in a more general setting). For a finite sequence $\varepsilon_1, \ldots, \varepsilon_k$ we use $(\varepsilon_1 \ldots \varepsilon_k)^\infty$ to denote repeating of $\varepsilon_1 \ldots \varepsilon_k$; for example, $(0qq)^\infty = 0qq0qq \ldots 0qq \ldots$ and $1^\infty = 111 \ldots$.

LEMMA 2.2.

(a) *Let $1 < q \leqslant (3+\sqrt{5})/2$. For $x \in [0, q/(q-1)]$ and $(\varepsilon_i) \in \Pi^{-1}(x)$, we have that $(\varepsilon_i) = (\mathcal{G}_i^x)$ if and only if*

$$(\varepsilon_{n+i}) < \begin{cases} (\mathcal{Q}_i^1) & \text{if } \varepsilon_n = 0, \\ (\mathcal{Q}_i^{q-1}) & \text{if } \varepsilon_n = 1. \end{cases} \tag{5}$$

(b) *Let* $q > (3+\sqrt{5})/2$. *For* $x \in F_q$ *and* $(\varepsilon_i) \in \Pi^{-1}(x)$, *we have that* $(\varepsilon_i) = (\mathcal{G}_i^x)$ *if and only if*

$$(\varepsilon_{n+i}) < q0^\infty \quad \textit{if } \varepsilon_n = 0.$$

Proof. (a) When $1 < q < (3+\sqrt{5})/2$, this is given by [**15**, Theorem 2.3], so in the following we consider only the case where $q = (3+\sqrt{5})/2$.

Note that we have $1, q-1 \in (0, q/(q-1)]$. Suppose that $(\varepsilon_i) = (\mathcal{G}_i^x)$. Then by (IV) we have

$$\begin{cases} \displaystyle\sum_{i=1}^{n-1} \frac{\mathcal{G}_i^x}{q^i} + \frac{1}{q^n} > x = \sum_{i=1}^{n-1} \frac{\mathcal{G}_i^x}{q^i} + \sum_{i=n+1}^{\infty} \frac{\mathcal{G}_i^x}{q^i} & \text{if } \mathcal{G}_n^x = 0, \\ \displaystyle\sum_{i=1}^{n-1} \frac{\mathcal{G}_i^x}{q^i} + \frac{q}{q^n} > x = \sum_{i=1}^{n-1} \frac{\mathcal{G}_i^x}{q^i} + \frac{1}{q^n} + \sum_{i=n+1}^{\infty} \frac{\mathcal{G}_i^x}{q^i} & \text{if } \mathcal{G}_n^x = 1. \end{cases}$$

Thus

$$\begin{cases} \displaystyle 1 > \sum_{i=1}^{\infty} \frac{\mathcal{G}_{n+i}^x}{q^i} & \text{if } \mathcal{G}_n^x = 0, \\ \displaystyle q-1 > \sum_{i=1}^{\infty} \frac{\mathcal{G}_{n+i}^x}{q^i} & \text{if } \mathcal{G}_n^x = 1. \end{cases} \tag{6}$$

Note that $q = (3+\sqrt{5})/2$ implies $q-1 = q/(q-1)$, so that

$$(\mathcal{Q}_i^{q-1}) = (\mathcal{G}_i^{q-1}) = q^\infty, \qquad (\mathcal{G}_i^1) = q0^\infty \quad \text{and} \quad (\mathcal{Q}_i^1) = 1q^\infty. \tag{7}$$

Therefore, (5) follows from (6) and (7).

Now we prove the sufficiency. Suppose $(\varepsilon_i) < (\mathcal{G}_i^x)$. Let $n \in \mathbb{N}$ be such that

$$\varepsilon_n < \mathcal{G}_n^x \quad \text{and} \quad \varepsilon_i = \mathcal{G}_i^x \text{ for } i < n.$$

Case I: $\varepsilon_n = 0$. Then we have $\mathcal{G}_n^x = 1$ and

$$\sum_{i=1}^{n} \frac{\mathcal{G}_i^x}{q^i} \leqslant x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}.$$

So

$$\frac{1}{q} + \sum_{i=2}^{\infty} \frac{q}{q^i} = 1 \leqslant \sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i}.$$

Thus we have either $(\varepsilon_{n+i}) = 1q^\infty = (\mathcal{Q}_i^1)$ or $\varepsilon_{n+1} = q$, contradicting (5).

Case II: $\varepsilon_n = 1$. Then we have $\mathcal{G}_n^x = q$ and

$$\sum_{i=1}^{n} \frac{\mathcal{G}_i^x}{q^i} \leqslant x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}.$$

So

$$\sum_{i=1}^{\infty} \frac{q}{q^i} = q - 1 \leqslant \sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i}.$$

Thus we have $(\varepsilon_{n+i}) = q^\infty = (\mathcal{Q}_i^{q-1})$, contradicting (5).

(b) When $q > (3+\sqrt{5})/2$, we have $\phi_0(I) \cap \phi_1(I) \neq \emptyset$ but $\phi_1(I) \cap \phi_q(I) = \emptyset$ (also see (2)).

Suppose $\varepsilon_i = (\mathcal{G}_i^x)$. Then when $\varepsilon_n = 0$ we have

$$\sum_{i<n} \frac{\varepsilon_i}{q^i} + \frac{1}{q^n} = \sum_{i<n} \frac{\mathcal{G}_i^x}{q^i} + \frac{1}{q^n} > x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}.$$

This implies that

$$1 > \sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i},$$

yielding $\varepsilon_{n+1} \in \{0, 1\}$ and hence $(\varepsilon_{n+i}) < q0^{\infty}$.

Now we prove the sufficiency. Suppose $\varepsilon_i < (\mathcal{G}_i^x)$. Let $n \in \mathbb{N}$ be such that

$$\varepsilon_n < \mathcal{G}_n^x \quad \text{and} \quad \varepsilon_i = \mathcal{G}_i^x \text{ for } i < n.$$

Then it must be that $\varepsilon_n = 0$ and $\mathcal{G}_n^x = 1$. Hence

$$\sum_{i=1}^{n} \frac{\mathcal{G}_i^x}{q^i} \leqslant x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}.$$

This gives

$$1 \leqslant \sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i},$$

implying $\varepsilon_{n+1} = q$ and hence $(\varepsilon_{n+i}) \geqslant q0^{\infty}$. □

§3. *Proof of the main theorem.* In this section we give the proof of Theorem 1.1 (see also Theorem 3.4 below). We first give a characterization of points in $\mathcal{U}_q$. To do this, we consider the self-similar set $\widetilde{F}_q$ generated by the IFS

$$\{\phi_k(x) = (x+k)/q : k = 0, q-1, q\},$$

i.e. $\widetilde{F}_q$ is the unique non-empty compact set satisfying

$$\widetilde{F}_q = \phi_0(\widetilde{F}_q) \cup \phi_{q-1}(\widetilde{F}_q) \cup \phi_q(\widetilde{F}_q).$$

Thus, we have

$$\widetilde{F}_q = \left\{ \sum_{i=1}^{\infty} \frac{j_i}{q^i} : (j_i) \in \{0, q-1, q\}^{\mathbb{N}} \right\},$$

and so for each $x \in \widetilde{F}_q$ there exists at least one $(j_i) \in \{0, q-1, q\}^{\mathbb{N}}$, called the q-expansion of x with respect to the digit set $\{0, q-1, q\}$, such that $x = \sum_{i=1}^{\infty} j_i/q^i$. For each $x \in \widetilde{F}_q$ one can define its lazy, greedy and quasi-greedy q-expansions in the same way as was done for points of F_q. One can check (and it also follows directly from the "Pedicini condition" in [15]) that for $1 < q \leqslant (3+\sqrt{5})/2$,

$$[0, q/(q-1)] = \phi_0([0, q/(q-1)]) \cup \phi_{q-1}([0, q/(q-1)]) \\ \cup \phi_q([0, q/(q-1)]);$$

that is,

$$\widetilde{F}_q = \left[0, \frac{q}{q-1}\right] \quad \text{if } 1 < q \leqslant \frac{3+\sqrt{5}}{2}.$$

Note that $1, q-1 \in \widetilde{F}_q = F_q = [0, q/(q-1)]$ when $1 < q \leqslant (3+\sqrt{5})/2$.

We denote by $(\widetilde{\mathcal{Q}}_i^1)$ and $(\widetilde{\mathcal{Q}}_i^{q-1})$ the quasi-greedy q-expansions of 1 and $q-1$ with respect to the digit set $\{0, q-1, q\}$. Recall that we use $(\mathcal{Q}_i^1)$ and $(\mathcal{Q}_i^{q-1})$ to denote the quasi-greedy q-expansions of 1 and $q-1$ with respect to the digit set $\{0, 1, q\}$.

The following lemma characterizes points that have a unique q-expansion. When $1 < q < (3+\sqrt{5})/2$, this characterization was given by Pedicini in [15] (in fact, he studied a more general setting).

LEMMA 3.1.

(a) *Let $1 < q \leqslant (3+\sqrt{5})/2$. For $x \in [0, q/(q-1)]$ and $(\varepsilon_i) \in \Pi^{-1}(x)$, (ε_i) is the unique q-expansion of x if and only if*

$$(\varepsilon_{n+i}) < \begin{cases} (\mathcal{Q}_i^1) & \textit{if } \varepsilon_n = 0, \\ (\mathcal{Q}_i^{q-1}) & \textit{if } \varepsilon_n = 1 \end{cases}$$

and (8)

$$(q - \varepsilon_{n+i}) < \begin{cases} (\widetilde{\mathcal{Q}}_i^1) & \textit{if } \varepsilon_n = 1, \\ (\widetilde{\mathcal{Q}}_i^{q-1}) & \textit{if } \varepsilon_n = q. \end{cases}$$

(b) *Let $q > (3+\sqrt{5})/2$. For $x \in F_q$ and $(\varepsilon_i) \in \Pi^{-1}(x)$, (ε_i) is the unique q-expansion of x if and only if*

$$(\varepsilon_{n+i}) \begin{cases} < q0^\infty & \textit{if } \varepsilon_n = 0, \\ > 1^\infty & \textit{if } \varepsilon_n = 1. \end{cases}$$

Proof. (a) When $1 < q < (3+\sqrt{5})/2$, this statement is included in [15, Theorem 3.1], so we will only consider the case where $q = (3+\sqrt{5})/2$.

Suppose that (ε_i) is the unique q-expansion of x. Then the first part of (8) holds by Lemma 2.2. Since (ε_i) is the lazy q-expansion of x, by (IV) we have

$$\begin{cases} \displaystyle\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{0}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} & \text{when } \varepsilon_n = 1, \\ \displaystyle\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{1}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} & \text{when } \varepsilon_n = q; \end{cases}$$

that is,

$$\begin{cases} \sum_{i=1}^{\infty} \frac{q-\varepsilon_{n+i}}{q^i} < 1 & \text{when } \varepsilon_n = 1, \\ \sum_{i=1}^{\infty} \frac{q-\varepsilon_{n+i}}{q^i} < q-1 & \text{when } \varepsilon_n = q. \end{cases}$$

Note that $(q-\varepsilon_{n+i})_{i=1}^{\infty} \in \{0, q-1, q\}^{\mathbb{N}}$ is a q-expansion of $y := \sum_{i=1}^{\infty}(q-\varepsilon_{n+i})/q^i \in [0, q/(q-1)]$ with respect to the digit set $\{0, q-1, q\}$.

Case I: $y < 1$. We claim it must be true that $(q-\varepsilon_{n+i}) < (\widetilde{\mathcal{Q}}_i^1) = (q-1)^{\infty}$. Otherwise, we would have $(q-\varepsilon_{n+i}) = (q-1)^{\ell-1}q(q-\varepsilon_{\ell+i})_{i=1}^{\infty}$ for some $\ell \in \mathbb{N}$, so that

$$y \geqslant \sum_{i<\ell} \frac{q-1}{q^i} + \frac{q}{q^{\ell}} = 1,$$

which is a contradiction.

Case II: $y < q-1$. Clearly, we have $(q-\varepsilon_{n+i}) < (\widetilde{\mathcal{Q}}_i^{q-1}) = q^{\infty}$ (note that $q-1 = q/(q-1)$).

Now we turn to proving the sufficiency. We will prove that the second part of (8) implies that (ε_i) is the lazy q-expansion of x.

Suppose that $(q-\varepsilon_{n+i}) < (\widetilde{\mathcal{Q}}_i^1) = (q-1)^{\infty}$ and $\varepsilon_n = 1$. Then $(q-\varepsilon_{n+i}) = (q-1)^{\ell-1}0(q-\varepsilon_{n+\ell+i})_{i=1}^{\infty}$ for some $\ell \in \mathbb{N}$, and so (also note that $q-1 = q/(q-1)$)

$$\sum_{i=1}^{\infty} \frac{q-\varepsilon_{n+i}}{q^i} \leqslant \sum_{i<\ell} \frac{q-1}{q^i} + \sum_{i>\ell} \frac{q}{q^i} = 1 - \frac{1}{q^{\ell}} < 1.$$

Thus

$$\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{0}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} \quad \text{when } \varepsilon_n = 1. \tag{9}$$

Suppose that $(q-\varepsilon_{n+i}) < (\widetilde{\mathcal{Q}}_i^{q-1}) = q^{\infty}$ and $\varepsilon_n = q$.

Then $(q-\varepsilon_{n+i}) = q^{\ell-1}a(q-\varepsilon_{n+\ell+i})_{i=1}^{\infty}$ for some $\ell \in \mathbb{N}$ and $a \in \{0, q-1\}$; hence (also note that $q-1 = q/(q-1)$)

$$\sum_{i=1}^{\infty} \frac{q-\varepsilon_{n+i}}{q^i} \leqslant \sum_{i<\ell} \frac{q}{q^i} + \frac{q-1}{q^{\ell}} + \sum_{i>\ell} \frac{q}{q^i} < q-1.$$

Thus

$$\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{1}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} \quad \text{when } \varepsilon_n = q. \tag{10}$$

So (ε_i) is the lazy q-expansion of x by (9), (10) and (IV).

(b) Suppose that (ε_i) is the unique q-expansion of x. Then Lemma 2.2 tells us that $(\varepsilon_{n+i}) < q0^{\infty}$ if $\varepsilon_n = 0$. Note that (ε_i) is the lazy q-expansion of x. Thus, when $\varepsilon_n = 1$,

$$\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{0}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}$$

by (IV). This is equivalent to

$$\sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i} > \frac{1}{q-1} = \sum_{i=1}^{\infty} \frac{1}{q^i}.$$

We claim that $(\varepsilon_{n+i}) > 1^\infty$. Otherwise, we would have

$$(\varepsilon_{n+i}) = 1^{\ell-1}0(\varepsilon_{n+\ell+i})_{i=1}^{\infty}$$

for some $\ell \in \mathbb{N}$, so that

$$\sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i} \leqslant \sum_{i<\ell} \frac{1}{q^i} + \sum_{i>\ell} \frac{q}{q^i} = \frac{1}{q-1},$$

which is a contradiction.

For the sufficiency part we need to check that the condition "$(\varepsilon_{n+i}) > 1^\infty$ if $\varepsilon_n = 1$" implies that (ε_i) is the lazy q-expansion of x. Let $(\varepsilon_{n+i}) = 1^{\ell-1}q(\varepsilon_{n+\ell+i})_\infty$ for some $\ell \in \mathbb{N}$. Then

$$\sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}}{q^i} \geqslant \sum_{i<\ell} \frac{1}{q^i} + \frac{q}{q^\ell} = \frac{q^{\ell-1}+q-2}{q^{\ell-1}(q-1)} > \frac{1}{q-1},$$

which implies that

$$\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{0}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} \quad \text{when } \varepsilon_n = 1.$$

On the other hand, when $q > (3+\sqrt{5})/2$ we have $q^2 - 3q + 1 > 0$. This gives

$$\frac{1}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < \frac{q}{q^n}$$

or, equivalently,

$$\sum_{i=1}^{n-1} \frac{\varepsilon_i}{q^i} + \frac{1}{q^n} + \sum_{i=n+1}^{\infty} \frac{q}{q^i} < x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} \quad \text{when } \varepsilon_n = q.$$

Therefore, (ε_i) is the lazy q-expansion of x. □

For the purpose of establishing the following lemma, we would like to project the coding spaces $\{0, 1, q\}^{\mathbb{N}}$ and $\{0, q-1, q\}^{\mathbb{N}}$ into the coding space $\{1, 2, 3\}^{\mathbb{N}}$. For $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$ (or $(\varepsilon_i) \in \{0, q-1, q\}^{\mathbb{N}}$) let

$$\hbar((\varepsilon_i)) = (\varepsilon_i^{\text{proj}}) \quad \text{where } 0^{\text{proj}} = 1,\ 1^{\text{proj}} = (q-1)^{\text{proj}} = 2 \text{ and } q^{\text{proj}} = 3.$$

This allows us to compare elements taken from different coding spaces: for $(\varepsilon_i) \in \{0, 1, q_1\}^{\mathbb{N}}$ and $(\varepsilon_i^*) \in \{0, 1, q_2\}^{\mathbb{N}}$ (or $(\varepsilon_i) \in \{0, q_1 - 1, q_1\}^{\mathbb{N}}$ and $(\varepsilon_i^*) \in \{0, q_2 - 1, q_2\}^{\mathbb{N}}$) we say that $(\varepsilon_i) \prec (\varepsilon_i^*)$ if $\hbar((\varepsilon_i)) < \hbar((\varepsilon_i^*))$ in the lexicographic order, and we write $(\varepsilon_i) \asymp (\varepsilon_i^*)$ if $\hbar((\varepsilon_i)) = \hbar((\varepsilon_i^*))$.

LEMMA 3.2. *Let $q > 1$; then the following properties hold.*

(A) *We always have that* $(\widetilde{\mathcal{Q}}_i^1) = (q-1)^\infty$.

(B) *When* $1 < q \leqslant (3+\sqrt{5})/2$ *we have* $(\mathcal{Q}_i^1) = 1(\mathcal{Q}_i^{q-1})$, *while* $(\mathcal{G}_i^1) = q0^\infty$ *is the unique q-expansion of* 1 *when* $q > (3+\sqrt{5})/2$.

(C) *Let* $1 < q \leqslant (3+\sqrt{5})/2$. *Then* $(\mathcal{Q}_i^{q-1})$ *(or* $(\widetilde{\mathcal{Q}}_i^{q-1})$*) is strictly increasing in q in the sense that* $(\mathcal{Q}_i^{q_2-1}) \prec (\mathcal{Q}_i^{q_1-1})$ *(or* $(\widetilde{\mathcal{Q}}_i^{q_2-1}) \prec (\widetilde{\mathcal{Q}}_i^{q_1-1})$*) if* $q_2 < q_1$ *where, for* $\ell = 1, 2$, $(\mathcal{Q}_i^{q_\ell-1})$ *(or* $(\widetilde{\mathcal{Q}}_i^{q_\ell-1})$*) is the quasi-greedy q_ℓ-expansion of* $q_\ell - 1$ *with respect to the digit set* $\{0, 1, q_\ell\}$ *(or* $\{0, q_\ell - 1, q_\ell\}$*).*

Proof. (A) and (B) can be verified directly. We now prove (C). Note that for $1 < q \leqslant (3+\sqrt{5})/2$ we have $q - 1 \in F_q = \widetilde{F}_q = [0, q/(q-1)]$ so that $q-1$ has quasi-greedy q-expansions with respect to the digit sets $\{0, 1, q\}$ and $\{0, q-1, q\}$ (one can easily check that $q - 1 \notin F_q \cup \widetilde{F}_q$ if $q > (3+\sqrt{5})/2$).

We first consider the digit set $\{0, 1, q\}$. The desired result is implied by the following discussions.

Case I: suppose that $(\mathcal{Q}_i^{q_1-1}) \asymp (\mathcal{Q}_i^{q_2-1})$. In this case we always have $\mathcal{Q}_i^{q_1-1}/q_1^i \leqslant \mathcal{Q}_i^{q_2-1}/q_2^i$, and strict inequality holds for infinitely many i. Thus

$$q_1 - 1 = \sum_{i=1}^{\infty} \frac{\mathcal{Q}_i^{q_1-1}}{q_1^i} < \sum_{i=1}^{\infty} \frac{\mathcal{Q}_i^{q_2-1}}{q_2^i} = q_2 - 1,$$

which is a contradiction.

Case II: suppose that $(\mathcal{Q}_i^{q_1-1}) \prec (\mathcal{Q}_i^{q_2-1})$. Then $(\alpha_i) := \hbar((\mathcal{Q}_i^{q_1-1})) < (\beta_i) := \hbar((\mathcal{Q}_i^{q_2-1}))$. Let $n \in \mathbb{N}$ be such that $\alpha_n < \beta_n$ and $\alpha_i = \beta_i$ for $i < n$. Let $a \in \{1, q_1\}$ with $a^{\text{proj}} = \beta_n$. Then $a > \mathcal{Q}_n^{q_1-1}$ and so

$$q_1 - 1 \leqslant \sum_{i<n} \frac{\mathcal{Q}_i^{q_1-1}}{q_1^i} + \frac{a}{q_1^n} \leqslant \sum_{i\leqslant n} \frac{\mathcal{Q}_i^{q_2-1}}{q_2^i} < q_2 - 1,$$

which is a contradiction.

For the digit set $\{0, q-1, q\}$, the corresponding result follows from a similar argument.

Now suppose that $(\widetilde{\mathcal{Q}}_i^{q_1-1}) \asymp (\widetilde{\mathcal{Q}}_i^{q_2-1})$. In this case we always have that $\widetilde{\mathcal{Q}}_i^{q_1-1}/(q_1^i(q_1-1)) \leqslant \widetilde{\mathcal{Q}}_i^{q_2-1}/(q_2^i(q_2-1))$, and strict inequality holds for infinitely many i. Thus

$$1 = \sum_{i=1}^{\infty} \frac{\widetilde{\mathcal{Q}}_i^{q_1-1}}{q_1^i(q_1-1)} < \sum_{i=1}^{\infty} \frac{\widetilde{\mathcal{Q}}_i^{q_2-1}}{q_2^i(q_2-1)} = 1,$$

giving a contradiction. Suppose that $(\widetilde{\mathcal{Q}}_i^{q_1-1}) \prec (\widetilde{\mathcal{Q}}_i^{q_2-1})$. We can choose $a \in \{q_1 - 1, q_1\}$ in the same way as in Case II above. Then

$$1 \leqslant \sum_{i<n} \frac{\widetilde{\mathcal{Q}}_i^{q_1-1}}{q_1^i(q_1-1)} + \frac{a}{q_1^n(q_1-1)} \leqslant \sum_{i\leqslant n} \frac{\widetilde{\mathcal{Q}}_i^{q_2-1}}{q_2^i(q_2-1)} < 1,$$

which also leads to a contradiction. □

We let $|A|$ denote the cardinality of a set A.

PROPOSITION 3.3. *Let $\mathcal{U}_q$ be defined by (4) with $q > 1$. If $q_1 > q_2$, then* $|\mathcal{U}_{q_1}| \geqslant |\mathcal{U}_{q_2}|$.

Proof. We divide the proof into three cases.

Case I: $q_1, q_2 \in (1, (3+\sqrt{5})/2]$. Let $x \in \mathcal{U}_{q_2}$, with (ε_i) being the unique q_2-expansion of x. Then

$$(\varepsilon_{n+i}) < \begin{cases} (\mathcal{Q}_i^1) = 1(\mathcal{Q}_i^{q_2-1}) & \text{if } \varepsilon_n = 0, \\ (\mathcal{Q}_i^{q_2-1}) & \text{if } \varepsilon_n = 1 \end{cases}$$

and

$$(q_2 - \varepsilon_{n+i}) < \begin{cases} (\widetilde{\mathcal{Q}}_i^1) = (q_2-1)^\infty & \text{if } \varepsilon_n = 1, \\ (\widetilde{\mathcal{Q}}_i^{q_2-1}) & \text{if } \varepsilon_n = q_2 \end{cases}$$

by Lemmas 3.1 and 3.2(A),(B). Take $(\delta_i) \in \{0, 1, q_1\}^{\mathbb{N}}$ such that $\hbar((\delta_i)) = \hbar((\varepsilon_i))$. Then $\Pi((\delta_i)) \in \mathcal{U}_{q_1}$ by Lemmas 3.1 and 3.2(C). Thus $|\mathcal{U}_{q_1}| \geqslant |\mathcal{U}_{q_2}|$.

Case II: $q_2 \in (1, (3+\sqrt{5})/2]$ and $q_1 > (3+\sqrt{5})/2$. Let x and (ε_i) be as in Case I. Then

$$(\varepsilon_{n+i}) < 1(\mathcal{Q}_i^{q_2-1}) \quad \text{if } \varepsilon_n = 0, \qquad (q_2 - \varepsilon_{n+i}) < (q_2-1)^\infty \quad \text{if } \varepsilon_n = 1;$$

equivalently,

$$(\varepsilon_{n+i}) < 1(\mathcal{Q}_i^{q_2-1}) \quad \text{if } \varepsilon_n = 0, \qquad (\varepsilon_{n+i}) > 1^\infty \quad \text{if } \varepsilon_n = 1.$$

Also take $(\delta_i) \in \{0, 1, q_1\}^{\mathbb{N}}$ such that $\hbar((\delta_i)) = \hbar((\varepsilon_i))$. Then

$$(\delta_{n+i}) < 1(\mathcal{Q}_i^{q_1-1}) < q_1 0^\infty \quad \text{if } \delta_n = 0, \qquad (\delta_{n+i}) > 1^\infty \quad \text{if } \delta_n = 1,$$

so that $\Pi((\delta_i)) \in \mathcal{U}_{q_1}$ by Lemma 3.1, implying that $|\mathcal{U}_{q_1}| \geqslant |\mathcal{U}_{q_2}|$.

Case III: $q_1, q_2 \in ((3+\sqrt{5})/2, \infty)$. We have $|\mathcal{U}_{q_1}| = |\mathcal{U}_{q_2}|$ by Lemma 3.1(b). This completes the proof of the proposition. □

Now we are ready to prove our main theorem.

THEOREM 3.4. *Let q_c be the positive solution of the equation $x^3 - 3x^2 + 2x - 1 = 0$. Then:*

(a) *when $1 < q \leqslant q_c$, $\mathcal{U}_q$ just consists of the endpoints of* $[0, q/(q-1)]$, *i.e.* $\mathcal{U}_q = \{0, q/(q-1)\}$;

(b) *when $q_c < q < (3+\sqrt{5})/2$, $\mathcal{U}_q$ consists of $2^{\aleph_0}$ many points;*

(c) *when $q \geqslant (3+\sqrt{5})/2$, we have*

$$\dim_H \mathcal{U}_q = \dim_B \mathcal{U}_q = \frac{\log q_c}{\log q} =: \gamma \quad \textit{and} \quad 0 < \mathcal{H}^\gamma(\mathcal{U}_q) < \infty.$$

Proof. (a) We first consider the case where $q = q_c$, the unique real root of the equation $x^3 - 3x^2 + 2x - 1 = 0$. Recall that $q_c = 2.32472\ldots \in (2, (3+\sqrt{5})/2)$. Recall that $(\mathcal{Q}_i^{q_c-1})$ and $(\widetilde{\mathcal{Q}}_i^{q_c-1})$ are the quasi-greedy q_c-expansions of $q_c - 1$ with respect to the digit sets $\{0, 1, q_c\}$ and $\{0, q_c - 1, q_c\}$, respectively, and $(\mathcal{Q}_i^1)$ and $(\widetilde{\mathcal{Q}}_i^1)$ are the quasi-greedy q_c-expansions of 1 with respect to the digit sets $\{0, 1, q_c\}$ and $\{0, q_c - 1, q_c\}$, respectively.

Note that $q_c^3 - 3q_c^2 + 2q_c - 1 = 0$. Thus we have

$$q_c - 1 = \frac{q_c^2 - q_c + 1}{q_c^2 - q_c} = \frac{q_c}{q_c} + \sum_{n=2}^{\infty} \frac{1}{q_c^n}.$$

It is easy to check that $q_c/q_c + q_c/q_c^2 > q_c - 1$. And, for $k \geqslant 2$,

$$\frac{q_c}{q_c} + \sum_{n=2}^{k} \frac{1}{q_c^n} + \frac{q_c}{q_c^{k+1}} = 1 + \frac{q_c^{k-1} + q_c - 2}{q_c^{k+1} - q_c^k} \geqslant q_c - 1,$$

where the last inequality is equivalent to $q_c - 2 \geqslant q_c^{k-1}(q_c^3 - 3q_c^2 + 2q_c - 1) = 0$. By (IV) we have $(\mathcal{Q}_i^{q_c-1}) = q_c 1^\infty$. Moreover, we have

$$\begin{aligned} q_c - 1 = \frac{4q_c^2 - 3q_c + 2}{3q_c^2 - 2q_c + 1} &= \frac{q_c^3 + q_c^2 - q_c + 1}{q_c^3} \\ &= \frac{q_c}{q_c} + \frac{q_c - 1}{q_c^2} + \frac{0}{q_c^3} + \sum_{n=4}^{\infty} \frac{q_c - 1}{q_c^n} \end{aligned}$$

and

$$1 = \frac{2q_c^2 - 2q_c + 1}{q_c^3 - q_c^2} = \frac{1}{q_c} + \frac{q_c}{q_c^2} + \sum_{n=3}^{\infty} \frac{1}{q_c^n} = \sum_{n=1}^{\infty} \frac{q_c - 1}{q_c^n}.$$

One can verify in the same way as above that these, in fact, give $(\widetilde{\mathcal{Q}}_i^{q_c-1}) = q_c(q_c - 1)0(q_c - 1)^\infty$, $(\mathcal{Q}_i^1) = 1q_c1^\infty$ and $(\widetilde{\mathcal{Q}}_i^1) = (q_c - 1)^\infty$.

Let $x \in \mathcal{U}_{q_c}$, and let (ε_i) be its unique q_c-expansion. Then

$$\begin{aligned} (\varepsilon_{n+i}) < 1q_c1^\infty \quad \text{if } \varepsilon_n = 0, \qquad 1^\infty < (\varepsilon_{n+i}) < q_c1^\infty \quad \text{if } \varepsilon_n = 1, \\ (\varepsilon_{n+i}) > 01q_c1^\infty \quad \text{if } \varepsilon_n = q_c \end{aligned} \tag{11}$$

by Lemma 3.1(a). This implies that the sequence (ε_i) cannot contain the words $0q_c$ or 10. We claim that the sequence (ε_i) cannot contain the digit 1. Otherwise, without loss of generality we can assume $\varepsilon_1 = 1$. By the second condition in (11) we would have $(\varepsilon_i) = 1^\ell q_c(\varepsilon_{\ell+1+i})$ with $\ell \in \mathbb{N}$ and $01q_c1^\infty < (\varepsilon_{\ell+1+i}) < 1^\infty$.

Case I: $\varepsilon_{\ell+2} = 1$. This forces $(\varepsilon_{\ell+1+i})$ to contain the word 10, which gives a contradiction.

Case II: $\varepsilon_{\ell+2} = 0$. In this case, we have $\varepsilon_{\ell+2}\varepsilon_{\ell+3} = 01$ (note that $0q_c$ is a forbidden word). Thus $(\varepsilon_{\ell+1+i}) = 01q_c(\varepsilon_{\ell+4+i})$ and $(\varepsilon_{\ell+4+i}) > 1^\infty$, which contradict the first condition in (11).

Therefore $(\varepsilon_i) \in \{0, q_c\}^{\mathbb{N}}$. This forces $(\varepsilon_i) = 0^\infty$ or q_c^∞. For $1 < q < q_c$, the desired result just follows from Proposition 3.3.

(b) Fix a $q \in (q_c, (3 + \sqrt{5})/2)$. Then $(\widetilde{\mathcal{Q}}_i^{q-1}) > q0^\infty$. By Lemma 3.2(C) we have $(\mathcal{Q}_i^{q-1}) \succ (\mathcal{Q}_i^{q_c-1}) = q_c1^\infty$, which implies that $(\mathcal{Q}_i^{q-1}) > q1^kq0^\infty$ for some positive integer k. Thus we have $(\mathcal{Q}_i^1) = 1(\mathcal{Q}_i^{q-1}) > 1q1^kq0^\infty$.

Lemma 3.2(A) shows that $(\widetilde{\mathcal{Q}}_i^1) = (q-1)^\infty$. So it follows from Lemma 3.1(a) that for an $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$, we have $\Pi((\varepsilon_i)) \in \mathcal{U}_q$ provided that

$$\begin{cases} (\varepsilon_{n+i}) < 1q1^k q0^\infty & \text{if } \varepsilon_n = 0, \\ 1^\infty < (\varepsilon_{n+i}) < q1^k q0^\infty & \text{if } \varepsilon_n = 1, \\ (q - \varepsilon_{n+i}) < q0^\infty & \text{if } \varepsilon_n = q. \end{cases} \tag{12}$$

Set

$$\Delta = \{q^m(p_i),\, 1^n(p_i) : (p_i) = q1^{\ell_1}q1^{\ell_2}q1^{\ell_3}\ldots,\ \ell_j > k,\ m \geqslant 0,\ n \geqslant 1\}.$$

Then Δ is uncountable and all its elements satisfy (12).

(c) As in [**14**, Example 5.4], we will show that the set $\mathcal{U}_q$ is actually a graph-directed self-similar set removing a countable set.

Fix a $q > (3+\sqrt{5})/2$. From Lemma 3.1(b) it follows that for an $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$, $\Pi((\varepsilon_i)) \in \mathcal{U}_q$ if and only if

$$(\varepsilon_{n+i}) < q0^\infty \quad \text{if } \varepsilon_n = 0, \qquad (\varepsilon_{n+i}) > 1^\infty \quad \text{if } \varepsilon_n = 1. \tag{13}$$

Set $G = \{00, 01, 11, 1q, q0, q1, qq\}$. Then the condition (13) is equivalent to

$$\varepsilon_i\varepsilon_{i+1} \in G \text{ for } i \geqslant 1 \quad \text{and } (\varepsilon_i) \text{ does not end with } 1^\infty.$$

Let

$$S = (s_{i,j})_{\{0,1,q\}\times\{0,1,q\}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$\mathcal{V}_q = \Pi(\{(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}} : s_{\varepsilon_i,\varepsilon_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}). \tag{14}$$

Then $\mathcal{V}_q$ is just the graph-directed self-similar set (see [**13**]); that is,

$$\mathcal{V}_q = A_0 \cup A_1 \cup A_q \quad \text{where } A_i = \bigcup_{j : s_{i,j}=1} \phi_j(A_j).$$

Note that the incidence matrix S is irreducible and the corresponding graph-directed IFS satisfies the open set condition with the open sets $O_0 = (0, 1)$, $O_1 = (0, q-1)$ and $O_q = (0, q/(q-1))$. Thus we have

$$\dim_H \mathcal{V}_q = \dim_B \mathcal{V}_q = \frac{\log q_c}{\log q} =: \gamma \quad \text{and} \quad 0 < \mathcal{H}^\gamma(\mathcal{V}_q) < \infty$$

where q_c is the spectral radius of S, which is the positive root of $x^3 - 3x^2 + 2x - 1 = 0$.

Note that $\mathcal{U}_q = \mathcal{V}_q \backslash W$, where $W = \{x = \Pi((\varepsilon_i)) : (\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$ are the sequences ending with $1^\infty\}$, is countable. Thus

$$\dim_H \mathcal{U}_q = \dim_B \mathcal{U}_q = \frac{\log q_c}{\log q} =: \gamma \quad \text{and} \quad 0 < \mathcal{H}^\gamma(\mathcal{U}_q) < \infty. \tag{15}$$

For the case where $q = (3+\sqrt{5})/2$, we have

$$\mathcal{Q}_i^1 = 1q^\infty, \qquad \widetilde{\mathcal{Q}}_i^1 = (q-1)^\infty, \qquad \mathcal{Q}_i^{q-1} = \widetilde{\mathcal{Q}}_i^{q-1} = q^\infty.$$

From Lemma 3.1(a) it follows that for an $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$, $\Pi((\varepsilon_i)) \in \mathcal{U}_q$ if and only if it satisfies

$$\begin{cases} (\varepsilon_{n+i}) < 1q^\infty & \text{if } \varepsilon_n = 0, \\ 1^\infty < (\varepsilon_{n+i}) < q^\infty & \text{if } \varepsilon_n = 1, \\ (\varepsilon_{n+i}) > 0^\infty & \text{if } \varepsilon_n = q. \end{cases}$$

Let $W = \{x = \Pi((\varepsilon_i)) : (\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$ are the sequences ending with 1^∞, $1q^\infty$ or $q0^\infty\}$. Thus we have $\mathcal{U}_q = \mathcal{V}_q \backslash W$ where $\mathcal{V}_q$ is defined by (14), and so (15) holds. □

§4. *q-expansions of* 1 *and* $q-1$. In this section we discuss the q-expansions of 1 and $q-1$. Theorem 4.1 is devoted to the q-expansions of 1 and Theorem 4.2 to the q-expansions of $q-1$.

THEOREM 4.1.

(a) 1 *has a unique q-expansion* $q0^\infty$ *with respect to* $\{0, 1, q\}$ *when* $q > (3+\sqrt{5})/2$.
(b) 1 *has two q-expansions with respect to* $\{0, 1, q\}$, *which are* $1q^\infty$ *and* $q0^\infty$ *when* $q = (3+\sqrt{5})/2$.
(c) *Each* $x \in (0, q/(q-1))$ *has* $2^{\aleph_0}$ *q-expansions with respect to* $\{0, 1, q\}$ *when* $1 < q < 2$.

Proof. The first two assertions are clear. Let us now prove (c); in the case where $q < (1+\sqrt{5})/2$ this is covered by [**4**, Theorem 3].

We have $F_q = [0, q/(q-1)]$ if $1 < q < 2$. Thus each $x \in [0, q/(q-1)]$ has a q-expansion with respect to the digit set $\{0, 1, q\}$. It is clear that 0 and $q/(q-1)$ have unique q-expansions 0^∞ and q^∞, respectively, with respect to the digit set $\{0, 1, q\}$.

As before, let $I = [0, q/(q-1)]$. Let

$$I_1 = \left[0, \frac{1}{q}\right), \qquad I_2 = \left[\frac{1}{q}, 1\right), \qquad I_3 = \left[1, \frac{1}{q-1}\right],$$
$$I_4 = \left(\frac{1}{q-1}, \frac{2q-1}{q(q-1)}\right] \quad \text{and} \quad I_5 = \left(\frac{2q-1}{q(q-1)}, \frac{q}{q-1}\right].$$

Then we have $I = \bigcup_{i=1}^5 I_i$ with disjoint union and

$$\phi_0(I) = I_1 \cup I_2 \cup I_3, \qquad \phi_1(I) = I_2 \cup I_3 \cup I_4, \qquad \phi_q(I) = I_3 \cup I_4 \cup I_5.$$

Let us make a geometrical observation: if x lies in the interior of $\phi_k(I)$ with $k \in \{0, 1, q\}$, then x has a q-expansion of the form $k(\varepsilon_i)$ with $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}} \backslash \{0^\infty, q^\infty\}$, i.e. $x = k/q + \sum_{i=1}^\infty \varepsilon_i/q^{i+1}$. In fact, we have $qx - k = \phi_k^{-1}(x) \in I^\circ = (0, q/(q-1))$, and $qx - k$ has a q-expansion $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}} \backslash \{0^\infty, q^\infty\}$.

The desired result can be obtained by repeating the following claim.

CLAIM. *For each* $x \in (0, q/(q-1))$ *there exist* $k \in \mathbb{N}$, *a finite sequence* $(\tau_i) \in \{0, 1, q\}^{k-1}$, *distinct* $\alpha, \beta \in \{0, 1, q\}$ *and distinct*

$$(\alpha_i), (\beta_i) \in \{0, 1, q\}^{\mathbb{N}} \backslash \{0^\infty, q^\infty\}$$

such that

$$x = \sum_{i \leqslant k-1} \frac{\tau_i}{q^i} + \frac{\alpha}{q^k} + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{i+k}} = \sum_{i \leqslant k-1} \frac{\tau_i}{q^i} + \frac{\beta}{q^k} + \sum_{i=1}^{\infty} \frac{\beta_i}{q^{i+k}} \tag{16}$$

(where $\{0, 1, q\}^0$ *denotes the empty set).*

We verify this claim as follows.

Case I: $x \in I_1$. Let $k \in \mathbb{N}$ be such that $q^k x \in [1/q, 1) = I_2$ (this k exists and is unique).

(I_1) If $q^k x \in (1/q, 1)$ then $q^k x$ lies in the interiors of both $\phi_0(I)$ and $\phi_1(I)$. Thus there exist distinct $(\alpha_i), (\beta_i) \in \{0, 1, q\}^{\mathbb{N}} \backslash \{0^\infty, q^\infty\}$ such that

$$q^k x = \frac{0}{q} + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{i+1}} = \frac{1}{q} + \sum_{i=1}^{\infty} \frac{\beta_i}{q^{i+1}},$$

so (16) is true for x.

(I_2) Suppose that $q^k x = 1/q$. Then $q^{k+1} x = 1$ lies in the interiors of both $\phi_0(I)$ and $\phi_1(I)$. We only need to repeat the above argument.

Case II: $x \in I_2 = [1/q, 1)$. Then we have $x = 1/q$ or $x \in (1/q, 1)$. We only need to repeat the arguments in Case I. The same idea works for $x \in I_3 \cup I_4$.

Case III: $x \in I_5$. Let $k \in \mathbb{N}$ and $(\varepsilon_i) \in \{0, 1, q\}^{\mathbb{N}}$ with $\varepsilon_1 \neq q$ such that

$$x = \sum_{i \leqslant k} \frac{q}{q^i} + \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^{i+k}}.$$

If $(\varepsilon_i) \neq 0^\infty$, then the claim is true for $y := \sum_{i=1}^{\infty} \varepsilon_i / q^i \in I \backslash I_5$ and hence for x. If $(\varepsilon_i) = 0^\infty$, then

$$\begin{aligned} x &= \sum_{i<k} \frac{q}{q^i} + \frac{1}{q^{k-1}} \left(\sum_{i \leqslant \ell-1} \frac{\tau_i}{q^i} + \frac{\alpha}{q^\ell} + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{i+\ell}} \right) \\ &= \sum_{i<k} \frac{q}{q^i} + \sum_{i \leqslant \ell-1} \frac{\tau_i}{q^{i+k-1}} + \frac{\alpha}{q^{k+\ell-1}} + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{i+k+\ell-1}} \\ &= \sum_{i<k} \frac{q}{q^i} + \frac{1}{q^{k-1}} \left(\sum_{i \leqslant \ell-1} \frac{\tau_i}{q^i} + \frac{\beta}{q^\ell} + \sum_{i=1}^{\infty} \frac{\beta_i}{q^{i+\ell}} \right) \\ &= \sum_{i<k} \frac{q}{q^i} + \sum_{i \leqslant \ell-1} \frac{\tau_i}{q^{i+k-1}} + \frac{\beta}{q^{k+\ell-1}} + \sum_{i=1}^{\infty} \frac{\beta_i}{q^{i+k+\ell-1}}, \end{aligned}$$

where $(\tau_i) \in \{0, 1, q\}^{\ell-1}$, the distinct $\alpha, \beta \in \{0, 1, q\}$ and the distinct $(\alpha_i), (\beta_i) \in \{0, 1, q\}^{\mathbb{N}} \backslash \{0^\infty, q^\infty\}$ are such that

$$1 = \sum_{i \leqslant \ell-1} \frac{\tau_i}{q^i} + \frac{\alpha}{q^\ell} + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{i+\ell}} = \sum_{i \leqslant \ell-1} \frac{\tau_i}{q^i} + \frac{\beta}{q^\ell} + \sum_{i=1}^{\infty} \frac{\beta_i}{q^{i+\ell}}.$$

The proof of the above claim is thus complete. □

THEOREM 4.2.

(a) *$q-1$ has $2^{\aleph_0}$ q-expansions with respect to $\{0, 1, q\}$ when $1 < q < 2$.*

(b) *Let $2 < q \leqslant (3+\sqrt{5})/2$. If $(\varepsilon_i) \in \Pi^{-1}(q-1)$ satisfies*

$$(\varepsilon_i) \in \{1, q\}^{\mathbb{N}} \quad \textit{and} \quad 1^\infty < (\varepsilon_{n+i}) < (\varepsilon_i) \quad \textit{when } \varepsilon_n = 1,$$

then (ε_i) is the unique q-expansion of $q-1$.

(c) *Let $2 < q \leqslant (3+\sqrt{5})/2$. Suppose that $(\varepsilon_i)_{i=1}^\infty = q^{t_1}1^{t_2}q^{t_3}1^{t_4} \cdots \in \Pi^{-1}(q-1)$. Then $(\varepsilon_i)_{i=1}^\infty$ is the unique q-expansion of $q-1$ if it satisfies*

$$2 \leqslant t_1 \quad \textit{and} \quad 1 \leqslant t_{2k-1} \leqslant t_1 - 1 \textit{ for } k \geqslant 2 \tag{17}$$

or

$$1 \leqslant t_1,\ 1 \leqslant t_{2k-1} \leqslant t_1 \quad \textit{and} \quad t_2 < t_{2k} \textit{ for } k \geqslant 2.$$

(d) *There exist $2^{\aleph_0}$ many $q \in (2, (3+\sqrt{5})/2]$ such that $q-1$ has a unique q-expansion.*

Proof. Part (a) is clear from Theorem 4.1(c). Part (c) follows directly from (b).

Next, we prove (b). We first point out that $q-1$ has quasi-greedy q-expansions with respect to the digit sets $\{0, 1, q\}$ and $\{0, q-1, q\}$ when $2 < q \leqslant (3+\sqrt{5})/2$. By Lemma 3.1(a), we need to check the following (note that $(\widetilde{\mathcal{Q}}_i^1) = (q-1)^\infty$ by Lemma 3.2 (A)):

$$\begin{cases} 1^\infty < (\varepsilon_{n+i}) < (\mathcal{Q}_i^{q-1}) & \text{if } \varepsilon_n = 1, \\ (\varepsilon_{n+i}) > (q - \widetilde{\mathcal{Q}}_i^{q-1}) & \text{if } \varepsilon_n = q. \end{cases}$$

In fact, when $\varepsilon_n = 1$ we have

$$1^\infty < (\varepsilon_{n+i}) < (\varepsilon_i) \leqslant (\mathcal{Q}_i^{q-1}).$$

On the other hand, we have $\widetilde{\mathcal{Q}}_1^{q-1} = q$ since $q > 2$. This implies the second inequality. Thus (b) is true.

To prove (d), we let $\varepsilon = (\varepsilon_i)_{i=1}^\infty = q^{t_1}1^{t_2}q^{t_3}1^{t_4} \ldots$, with t_i satisfying (17). Let

$$M(q) = \frac{1}{q} + \sum_{i=2}^{\infty} \frac{\varepsilon_{i-1}}{q^i}, \quad q \in \left[2, \frac{3+\sqrt{5}}{2}\right].$$

Then $M(q)$ is continuous and strictly decreasing. Note that $M(2) > 1$ and $M((3+\sqrt{5})/2) < 1$. Therefore, there exists a unique $q_* \in (2, (3+\sqrt{5})/2)$ such that

$$M(q_*) = 1 \quad \text{and so} \quad \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q_*^i} = q_* - 1.$$

Now, $q_* - 1$ has a unique q_*-expansion by (c). Moreover, for different choices of $\varepsilon = (\varepsilon_i)_{i=1}^{\infty}$ the corresponding q_* are different by the uniqueness of their expansions. This finishes the proof of (d). □

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