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Self-similar structure on intersection of Cartesian product of Cantor triadic sets with their translations

Yuanyuan Yao · Wenxia Li

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Abstract Let C be the classical Cantor triadic set. For $\alpha, \beta \in [-1, 1]$, a sufficient and necessary condition for $(C \times C) \cap (C \times C + (\alpha, \beta))$ to be self-similar is obtained.

Keywords Cantor triadic set · Cartesian product · Intersection · Self-similar structure

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1 Introduction

By a *constrictive similitude* in $\mathbb{R}^d$ we mean a map $S : \mathbb{R}^d \to \mathbb{R}^d$ such that for some $0 < r < 1$, $|S(x) - S(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^d$. A constrictive similitude S in $\mathbb{R}^d$ has the form

$$S(x) = rAx + b \quad \text{with} \quad x, b \in \mathbb{R}^d$$

where A is a $d \times d$ orthogonal matrix. Thus, when $d = 1$ we have $A = (1)$ or (-1), and when $d = 2$

Y. Yao
Department of Mathematics, East China University of Science and Technology,
Shanghai 200237, China
e-mail: yaoyuanyuan@ecust.edu.cn

W. Li (✉)
Department of Mathematics, East China Normal University,
Shanghai 200062, China
e-mail: wxli@math.ecnu.edu.cn

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi) \tag{1}$$

which correspond to, respectively, a rotation through the angle θ and a reflection about the line through the origin making the angle $\frac{\theta}{2}$ with the x-axis.

A nonempty compact set $E \subset \mathbb{R}^d$ is called a *self-similar set* if there are constrictive similitudes $S_1, \ldots, S_N (N \geq 2)$ of $\mathbb{R}^d$ such that

$$E = \bigcup_{i=1}^{N} S_i(E). \tag{2}$$

The family of $\{S_i\}_{i=1}^N$ is called a generating *iterated function system* (IFS) of E. If the sets of the union in (2) are pairwise disjoint, we say that the IFS $\{S_i\}_{i=1}^N$ satisfies the *strong separation condition* (SSC).

We need some notations. For a finite set Ω of integers let $\Omega^{\mathbb{N}} = \{(i_k)_{k=1}^{\infty} : i_k \in \Omega\}$. Let $\Omega^* = \bigcup_{k=0}^{\infty} \Omega^k$, where $\Omega^k = \{(i_j)_{j=1}^k : i_j \in \Omega \text{ for all } 1 \leq j \leq k\}$ for $k \in \mathbb{N}$ and $\Omega^0 = \emptyset$. We sometimes write a finite sequence $(i_j)_{j=1}^k$ as $i_1 \ldots i_k$, an infinite sequence $(i_j)_{j=1}^{\infty}$ as $i_1 i_2 \ldots$. For a finite sequence $\mathbf{i} \in \Omega^k$ $(k \in \mathbb{N})$, $\bar{\mathbf{i}} := \mathbf{i}\mathbf{i} \ldots \in \Omega^{\mathbb{N}}$ denotes the infinite repeating of $\mathbf{i}$. A sequence $\mathbf{i} \in \Omega^{\mathbf{N}}$ is said *strong p-periodic* (or simply, *strong periodic*) if there exist two finite sequences $\mathbf{u}, \mathbf{v} \in \Omega^p$ for some $p \in \mathbb{N}$ such that $\mathbf{i} = \mathbf{u}\bar{\mathbf{v}}$ and $\mathbf{u} \preceq \mathbf{v}$, where $\mathbf{u} \preceq \mathbf{v}$ means $u_n \leq v_n$, $1 \leq n \leq p$ for $\mathbf{u} = u_1 \ldots u_p$ and $\mathbf{v} = v_1 \ldots v_p$. For $(i_\ell)_{\ell=1}^k, (j_\ell)_{\ell=1}^k \in \Omega^k$ we write $(i_\ell)_{\ell=1}^k + (j_\ell)_{\ell=1}^k = (i_\ell + j_\ell)_{\ell=1}^k$. In addition, a sequence $\mathbf{i} = (i_k)_{k=1}^{\infty} \in \Omega^{\mathbf{N}}$ is said eventually periodic if there exist two integers d, m such that $i_{k+d} = i_k$ for all $k \geq m$, and the integer d is called a period of $\mathbf{i}$. Thus a strong periodic sequence is eventually periodic.

The following map π on $\Omega^{\mathbb{N}}$ will be used. For $(x_i)_{i=1}^{\infty} \in \Omega^{\mathbb{N}}$ let

$$\pi((x_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{x_i}{3^i}.$$

For $x \in \pi(\Omega^{\mathbb{N}})$ and $(x_i)_{i=1}^{\infty} \in \Omega^{\mathbb{N}}$, the $(x_i)_{i=1}^{\infty}$ is called a $\Omega-$*code* of x if $\pi((x_i)_{i=1}^{\infty})=x$.

The *classical Cantor triadic set* C is generated by the IFS $\{f_0(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3}\}$. For $\mathbf{i} = (i_k)_{k=1}^n \in \{0, 2\}^n$ denote $f_{\mathbf{i}}(x) = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(x)$. Then

$$C = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \{0,2\}^n} f_{\mathbf{i}}([0, 1]).$$

The left and right endpoints of $f_{\mathbf{i}}([0, 1])$ belong to C and they are termed as the left and right *endpoints* of C.

Clearly, $\pi(\{0, 2\}^{\mathbb{N}}) = C$. In fact, $\pi : \{0, 2\}^{\mathbb{N}} \to C$ is bijective, i.e., each $x \in C$ has a unique $\{0, 2\}$-code. In addition, we have $\pi(\{0, 1, 2\}^{\mathbb{N}}) = [0, 1]$ and $\pi(\{-2, 0, 2\}^{\mathbb{N}}) = [-1, 1]$. Each point of [0,1], except countable many ones, has a unique $\{0, 1, 2\}$-code and each exceptional point has just two $\{0, 1, 2\}$-codes of form $I\ell\overline{2},\ I(\ell+1)\overline{0}$ where $I \in \{0, 1, 2\}^*$ and $\ell \in \{0, 1\}$. Each endpoint of C, except points 0 and 1, has two $\{0, 1, 2\}$-codes. Analogically, each point of [-1,1], except countable many ones, has a unique $\{-2, 0, 2\}$-code and each exceptional point has just two $\{-2, 0, 2\}$-codes of form $I\ell\overline{2},\ I(\ell+2)\overline{-2}$ where $I \in \{-2, 0, 2\}^*$ and $\ell \in \{-2, 0\}$.

One can easily check that $C \cap (C+\alpha) \neq \emptyset$ if and only if $\alpha \in C - C = [-1, 1]$. In [1], Deng et al. obtained a sufficient and necessary condition for $C \cap (C+\alpha)$ to be a self-similar set.

Theorem A ([1, Theorem 1.1]) *Let $\alpha \in [-1, 1]$. Then $C \cap (C+\alpha)$ is a self-similar set if and only if α has a unique $\{-2, 0, 2\}$-code $(\alpha_k)_{k=1}^\infty \in \{-2, 0, 2\}^{\mathbb{N}}$ and $(2-|\alpha_k|)_{k=1}^\infty$ is strong periodic. Furthermore, if $C \cap (C+\alpha)$ is a self-similar set with more than one point, then there exists an IFS satisfying the strong separation condition.*

The following theorem describes the form of a generating IFS of C_α, defined below in (3), if it is self-similar (one can refer to [2] for detailed discussion on generating IFSs of a self-similar set in $\mathbb{R}$).

Theorem B ([1, Theorem 1.2]) *Suppose that $\alpha \in [-1, 1]$ has a unique $\{-2, 0, 2\}$-code $(\alpha_k)_{k=1}^\infty$. Let C_α be defined as in (3). If $(2-|\alpha_k|)_{k=1}^\infty \neq \overline{0}$ is strong p-periodic, then any generating IFS $\{f_i(x) = r_i x + b_i, r_i \in (0, 1)\}_{i=1}^N$ of C_α satisfies that $r_i = 3^{-q_i}$ for some positive integer q_i and*

$$b_i = \sum_{k=1}^{p+q_i} \frac{b_{ik}}{3^k},\ i = 1, \dots, N,$$

where all $b_{ik} = 0$ or 2. Moreover, each q_i is a period of $(2-|\alpha_k|)_{k=1}^\infty$.

We remark that the above results also hold for more general homogeneous symmetric Cantor sets in $\mathbb{R}$ (see [3,5]).

Now let us consider the intersection of $C \times C$ with its translation. Clearly, $(C \times C) \cap (C \times C + (\alpha, \beta)) = (C \cap (C+\alpha)) \times (C \cap (C+\beta))$. So $C \times C \cap (C \times C + (\alpha, \beta)) \neq \emptyset$ if and only if $\alpha, \beta \in [-1, 1]$. A natural question is that for $\alpha, \beta \in [-1, 1]$ when $C \times C \cap (C \times C + (\alpha, \beta))$ is a self-similar set. In the present paper we prove that (see also Theorem 3.1).

Theorem 1.1 *Let $\alpha, \beta \in [-1, 1]$. Then $(C \times C) \cap (C \times C + (\alpha, \beta))$ is a self-similar set if and only if both α and β have unique $\{-2, 0, 2\}$-codes $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty \in \{-2, 0, 2\}^{\mathbb{N}}$ and both $(2-|\alpha_k|)_{k=1}^\infty$ and $(2-|\beta_k|)_{k=1}^\infty$ are strong periodic. Furthermore, if $(C \times C) \cap (C \times C + (\alpha, \beta))$ is a self-similar set with more than one point, then there exists an IFS satisfying the strong separation condition.*

In fact, we obtain a better result, see Corollary 3.2. Below let us recall some basic properties of $C \cap (C+\alpha)$.

When $\alpha \in [-1, 1]$ has two $\{-2, 0, 2\}$-codes, say $(\alpha_k)_{k=1}^\infty, (\alpha_k^*)_{k=1}^\infty \in \{-2, 0, 2\}^{\mathbb{N}}$,

$$C \cap (C+\alpha) = \pi\left(\prod_{k=1}^\infty (\{0, 2\} \cap (\{0, 2\} + \alpha_k))\right) \cup \pi\left(\prod_{k=1}^\infty (\{0, 2\} \bigcap (\{0, 2\} + \alpha_k^*))\right).$$

Note that the two codes of α are of form $I\ell\overline{2}$ and $I(\ell+2)\overline{-2}$. Thus $C\cap(C+\alpha)$ is a finite set of more than two points and so is not self-similar (In fact, any finite set E of more than two points in $\mathbb{R}^d$ is not self-similar since $S(E)\nsubseteq E$ for any constrictive similitude S). In other words, if $C\cap(C+\alpha)$ is self-similar, then α has a unique $\{-2,0,2\}$-code.

Suppose that $\alpha\in[-1,1]$ has a unique $\{-2,0,2\}$-code $(\alpha_k)_{k=1}^{\infty}\in\{-2,0,2\}^{\mathbb{N}}$. Then

$$C\cap(C+\alpha)=\pi\left(\prod_{k=1}^{\infty}(\{0,2\}\cap(\{0,2\}+\alpha_k))\right).$$

It is easy to see that the unique $\{0,2\}$-code of $\min\{x:x\in C\cap(C+\alpha)\}$ is $((\alpha_k+|\alpha_k|)/2)_{k=1}^{\infty}$, and the unique $\{0,2\}$-code of $\max\{x:x\in C\cap(C+\alpha)\}$ is $(2+(\alpha_k-|\alpha_k|)/2)_{k=1}^{\infty}$. Denote $\gamma_\alpha=\min\{x:x\in C\cap(C+\alpha)\}$ and $C_\alpha=C\cap(C+\alpha)-\gamma_\alpha$. Then,

$$\begin{aligned}C_\alpha&=C\cap(C+\alpha)-\gamma_\alpha=\pi\left(\{(x_k)_{k=1}^{\infty}\in\{0,2\}^{\mathbb{N}}:x_k\le 2-|\alpha_k|\}\right)\\&=\left\{\sum_{k=1}^{\infty}\frac{x_k}{3^k}:x_k\in\{0,2-|\alpha_k|\}\text{ for }k\in\mathbb{N}\right\}.\end{aligned}\tag{3}$$

Thus, when α has a unique $\{-2,0,2\}$-code $C\cap(C+\alpha)$ degenerates to a singleton if and only if $(2-|\alpha_k|)_{k=1}^{\infty}=\overline{0}$. And $C\cap(C+\alpha)$ contains infinitely many points if and only if $(2-|\alpha_k|)_{k=1}^{\infty}$ contains infinitely many 2s. The following properties of C_α are clear:

$$0\in C_\alpha\subseteq C\quad\text{and}\quad -C_\alpha=C_\alpha+2\gamma_\alpha-\alpha-1.\tag{4}$$

The latter property in (4) implies that

$$-rC_\alpha+r(\alpha+1-2\gamma_\alpha)+b=rC_\alpha+b.\tag{5}$$

Note that $C_\alpha\times C_\beta$ is a translation of $(C\times C)\cap(C\times C+(\alpha,\beta))$. We shall consider the self-similarity of $C_\alpha\times C_\beta$ instead of $(C\times C)\cap(C\times C+(\alpha,\beta))$.

This paper is arranged as follows. In the next section, we give a new condition for C_α to be self-similar. In the last section, we give the proof of Theorem 1.1 (see Theorem 3.1). We also describe the form of generating IFSs of $C_\alpha\times C_\beta$ when it is self-similar (see Proposition 3.4).

2 A new condition for C_α to be self-similar

In this section, we give a new condition for C_α to be self-similar, i.e., C_α is self-similar if and only if there exists a constrictive similitude S such that $S(C_\alpha)\subseteq C_\alpha$.

Lemma 2.1 *Suppose that* $\alpha\in[-1,1]$ *has a unique* $\{-2,0,2\}$*-code* $(\alpha_k)_{k=1}^{\infty}$ *and* $(2-|\alpha_k|)_{k=1}^{\infty}\neq\overline{0}$. *Let* $0<r<1$ *and* $b\in\mathbb{R}$. *If* $rC_\alpha+b\subseteq C_\alpha$, *then* $0<r<\frac{1}{2}$.

Proof By (4) and the assumption $rC_\alpha + b \subseteq C_\alpha$ we have

$$\begin{aligned} rC_\alpha + (r-1)(2\gamma_\alpha - \alpha - 1) - b &= r(C_\alpha + (2\gamma_\alpha - \alpha - 1)) - (2\gamma_\alpha - \alpha - 1) - b \\ &= -rC_\alpha - (2\gamma_\alpha - \alpha - 1) - b \subseteq -C_\alpha - (2\gamma_\alpha - \alpha - 1) = C_\alpha. \end{aligned}$$

Thus, the self-similar set F generated by the IFS $\{rx+b, rx+(r-1)(2\gamma_\alpha-\alpha-1)-b\}$ is a subset of C_α. When $b \neq (r-1)(2\gamma_\alpha - \alpha - 1) - b$ this implies $0 < r < \frac{1}{2}$ since F is an interval if $r \geq \frac{1}{2}$.

In the following we show that the case $b = (r-1)(2\gamma_\alpha - \alpha - 1) - b$ doesn't happen.

Suppose $b = (r-1)(2\gamma_\alpha - \alpha - 1) - b$. Then by (4)

$$-C_\alpha = C_\alpha - \frac{2b}{1-r}$$

and so by (3)

$$\frac{2b}{1-r} = \max C_\alpha = \sum_{k=1}^{\infty} \frac{2 - |\alpha_k|}{3^k}.$$

Thus

$$\frac{b}{1-r} = \sum_{k=1}^{\infty} \frac{t_k}{3^k} \notin C_\alpha$$

since by the assumption $t_k = \frac{2-|\alpha_k|}{2} \in \{0, 1\}$ contains infinitely many 1s. However, the point $\frac{b}{1-r}$, as the fixed point of the map $g(x) = rx + b$, should lie in C_α. □

We will need the following

Lemma 2.2 (see Lemma 2.6 in [1], Lemma 3.1 in [3]) *Let Ω be a finite set of integers. Let $(x_k)_{k=1}^{\infty} \in \Omega^{\mathbb{N}}$. If there exists a positive integer q such that $x_{k+q} \geq x_k$ for all $k \in \mathbb{N}$, then $(x_k)_{k=1}^{\infty}$ is strong periodic and q is a period of $(x_k)_{k=1}^{\infty}$.*

Proof Put $\tau_i = \max\{x_{i+kq} : k \in \mathbb{N} \cup \{0\}\}$, $1 \leq i \leq q$. Let

$$m = \max_{1 \leq i \leq q} k_i \quad \text{where} \quad k_i = \min\{k \in \mathbb{N} \cup \{0\} : x_{i+kq} = \tau_i\}.$$

Then $(x_k)_{k=1}^{\infty} = \overline{x_1 \dots x_q}$ if $m = 0$, or $(x_k)_{k=1}^{\infty} = x_1 \dots x_{mq} \overline{x_{1+mq} \dots x_{q+mq}}$ if $m \geq 1$. Thus, the desired results follow.

Lemma 2.3 *Suppose that $\alpha \in [-1, 1]$ has a unique $\{-2, 0, 2\} - code$ $(\alpha_k)_{k=1}^{\infty}$. Let $0 < r < 1$ and $b \in \mathbb{R}$. If $rC_\alpha + b \subseteq C_\alpha$, then $(2 - |\alpha_k|)_{k=1}^{\infty}$ is strong periodic.*

Proof If $C_\alpha = \{0\}$, then $(2-|\alpha_k|)_{k=1}^\infty = \overline{0}$ is strong periodic. Suppose $C_\alpha \neq \{0\}$. Then the condition that $rC_\alpha + b \subseteq C_\alpha$ ensures that the set $\mathcal{A} = \{t \in \mathbb{N} : \alpha_t = 0\}$ is infinite. Label $\mathcal{A}$ as $\{t_k : k \in \mathbb{N}\}$. By Lemma 2.1 we have $2r \in (0, 1)$. Let $(p_k)_{k=1}^\infty \in \{0, 1, 2\}^{\mathbb{N}}$ be a $\{0, 1, 2\}$-code of $2r$, i.e., $2r = \sum_{k=1}^\infty \frac{p_k}{3^k}$. Let $s = \min\{k \in \mathbb{N} : p_k \neq 0\}$. By Lemma 2.2 it suffices to show

$$t_k + s \in \mathcal{A} \quad \text{for all } k \in \mathbb{N}.$$

Otherwise, there exists an $i \in \mathbb{N}$ such that $t_i + s \notin \mathcal{A}$. Note that $\frac{2}{3^{t_i}} \in C_\alpha$ and $2 - |\alpha_{t_i+s}| = 0$. Hence we have

$$\frac{2r}{3^{t_i}} + b = \sum_{k=1}^{t_i+s-1} \frac{b_k}{3^k} + \frac{p_s}{3^{t_i+s}} + \sum_{k=t_i+s+1}^{\infty} \frac{p_{k-t_i} + b_k}{3^k},$$

where $(b_k)_{k=1}^\infty$ is the $\{0, 2\}$-code of b (note that $b \in C_\alpha$). Put

$$\xi = \frac{p_s}{3^{t_i+s}} + \sum_{k=t_i+s+1}^{\infty} \frac{p_{k-t_i} + b_k}{3^k}.$$

Then

$$\frac{1}{3^{t_i+s}} \leq \xi \leq \frac{4}{3^{t_i+s}}.$$

Case A $\xi = \frac{1}{3^{t_i+s}}$. In this case, we have

$$2r = \frac{1}{3^s} \quad \text{and} \quad b = \sum_{k=1}^{t_i+s-1} \frac{b_k}{3^k}.$$

By taking $t_n, t_m \in \mathcal{A}$ with $t_n > t_m > t_i$ we have $\frac{2}{3^{t_m}} + \frac{2}{3^{t_n}} \in C_\alpha$ but

$$r\left(\frac{2}{3^{t_m}} + \frac{2}{3^{t_n}}\right) + b = \sum_{k=1}^{t_i+s-1} \frac{b_k}{3^k} + \frac{1}{3^{s+t_m}} + \frac{1}{3^{s+t_n}} \notin C_\alpha,$$

a contradiction.

Case B $\xi = \frac{4}{3^{t_i+s}}$. In this case, $(b_k)_{k=1}^\infty$ contains only finitely many 0s. So b is a right endpoint. So $rx + b \notin C_\alpha$ for $x \in C_\alpha \setminus \{0\}$ close enough to 0, a contradiction.

Case C $\frac{1}{3^{t_i+s}} < \xi < \frac{4}{3^{t_i+s}}$. In this case we have

$$\xi = \frac{1}{3^{t_i+s}} + \frac{a_0}{3^{t_i+s}} + \frac{a_1}{3^{t_i+s+1}} + \cdots$$

where each $a_k \in \{0, 1, 2\}$ and $(a_k)_{k=0}^{\infty} \notin \{\overline{0}, \overline{2}\}$. Hence

$$\frac{2r}{3^{t_i}} + b = \sum_{k=1}^{t_i+s-1} \frac{b_k}{3^k} + \frac{1+a_0}{3^{t_i+s}} + \sum_{k=1}^{\infty} \frac{a_k}{3^{t_i+s+k}}.$$

When $a_0 = 0$, then $\sum_{k=1}^{\infty} \frac{a_k}{3^{t_i+s+k}} > 0$. Thus $\frac{2r}{3^{t_i}} + b \notin C_\alpha$ by the fact $t_i + s \notin \mathcal{A}$.

When $a_0 = 1$, we also have $\frac{2r}{3^{t_i}} + b \notin C_\alpha$ by the fact $t_i + s \notin \mathcal{A}$.

When $a_0 = 2$, $\frac{2r}{3^{t_i}} + b \in C_\alpha$ forces (a $\{0, 1, 2\}$-code of a point of C_α either contains no digit 1 or contains just one digit 1 of the form $I1\overline{0}$ or $I1\overline{2}$, $I \in \{0, 2\}^*$)

$$\frac{2r}{3^{t_i}} + b = \sum_{k=1}^{q-1} \frac{b_k}{3^k} + \frac{1}{3^q} = \sum_{k=1}^{q-1} \frac{b_k}{3^k} + \sum_{k=q+1}^{\infty} \frac{2}{3^k}$$

with $q \leq t_i + s - 1$. This is impossible by the fact $t_i + s \notin \mathcal{A}$. □

Comparing with Theorem A, the following theorem gives a new equivalent condition for C_α to be self-similar.

Theorem 2.4 *Suppose that* $\alpha \in [-1, 1]$ *has a unique* $\{-2, 0, 2\}$*-code* $(\alpha_k)_{k=1}^{\infty}$. *Then the following two statements are equivalent:*

(I) C_α *is self-similar;*
(II) *there exists a constrictive similitude* S *such that* $S(C_\alpha) \subseteq C_\alpha$.

Proof (I) ⇒ (II) is clear. (II) ⇒ (I) can be deduced by Lemma 2.3 and Theorem A. □

Combining with Theorems A and B, the following corollary is direct.

Corollary 2.5 *Suppose that* $\alpha \in [-1, 1]$ *has a unique* $\{-2, 0, 2\}$*-code and that* $C_\alpha \neq \{0\}$. *Let* $0 < r < 1$ *and* $b \in \mathbb{R}$. *If* $rC_\alpha + b \subseteq C_\alpha$, *then* $r = \frac{1}{3^q}$ *with* $q \in \mathbb{N}$ *and the* $\{0, 2\}$*-code of* b *contains finitely many* 2*s.*

Proof Let $(\alpha_k)_{k=1}^{\infty}$ be the unique $\{-2, 0, 2\}$-code of α. The assumption $C_\alpha \neq \{0\}$ means that $(2 - |\alpha_k|)_{k=1}^{\infty} \neq \overline{0}$. Let $S(x) = rx + b$. Then C_α is self-similar by Theorem 2.4. From Theorem A it follows that $(2 - |\alpha_k|)_{k=1}^{\infty}$ is strong periodic. Let $\{f_i(x) = r_i x + b_i, r_i \in (0, 1)\}_{i=1}^{N}$ be a generating IFS of C_α. Then $\{S(x) = rx + b, f_i(x) = r_i x + b_i, r_i \in (0, 1)\}_{i=1}^{N}$ is also a generating IFS of C_α. Thus we have $r = \frac{1}{3^q}$ with $q \in \mathbb{N}$ and the $\{0, 2\}$-code of b contains finitely many 2s by Theorem B.

3 Proofs of main theorems

It is clear that $C_\alpha \times C_\beta$ is not self-similar if either α or β has two $\{-2, 0, 2\}$-codes. For example, let α have two $\{-2, 0, 2\}$-codes. Then C_α is a finite set of more than two points. If C_β is a singleton, then $C_\alpha \times C_\beta$ is not self-similar. Suppose C_β is not a singleton. Take $P \in C_\alpha \times \{0\}$ and $\delta > 0$ such that $B_\delta(P) \cap (C_\alpha \times \{0\}) \subsetneq C_\alpha \times \{0\}$ where $B_\delta(P)$ denotes the ball of radius δ and center P. Thus $B_\delta(P) \cap (C_\alpha \times C_\beta)$ cannot contain any similar copy of $C_\alpha \times C_\beta$, implying $C_\alpha \times C_\beta$ is not self-similar.

Theorem 3.1 *Let $\alpha, \beta \in [-1, 1]$. Then $(C \times C) \cap (C \times C + (\alpha, \beta))$ is a self-similar set if and only if both α and β have unique $\{-2, 0, 2\}$-codes $(\alpha_k)_{k=1}^{\infty}, (\beta_k)_{k=1}^{\infty} \in \{-2, 0, 2\}^{\mathbb{N}}$ and both $(2 - |\alpha_k|)_{k=1}^{\infty}$ and $(2 - |\beta_k|)_{k=1}^{\infty}$ are strong periodic. Furthermore, if $(C \times C) \cap (C \times C + (\alpha, \beta))$ is a self-similar set with more than one point, then there exists an IFS which satisfies the strong separation condition.*

Proof The 'if' part can be directly verified. Let the sequences $(2 - |\alpha_k|)_{k=1}^{\infty}$ and $(2 - |\beta_k|)_{k=1}^{\infty}$ be strong p-periodic and strong q-periodic, respectively. Then there exist $I_\alpha, I_\beta, J_\alpha, J_\beta \in \{0, 2\}^{pq}$ such that $I_\alpha + J_\alpha, I_\beta + J_\beta \in \{0, 2\}^{pq}$ and

$$(2 - |\alpha_k|)_{k=1}^{\infty} = I_\alpha \overline{I_\alpha + J_\alpha} \quad \text{and} \quad (2 - |\beta_k|)_{k=1}^{\infty} = I_\beta \overline{I_\beta + J_\beta}.$$

One can check that the IFS $\{h_{\sigma,\tau}(x, y) = 3^{-pq}(x, y) + (\pi(\sigma\overline{0}), \pi(\tau\overline{0})) : \sigma \preceq I_\alpha J_\alpha, \tau \preceq I_\beta J_\beta\}$ generates $C_\alpha \times C_\beta$ and satisfies the SSC.

Now suppose $(C \times C) \cap (C \times C + (\alpha, \beta))$ is self-similar. Then $C_\alpha \times C_\beta$ is self-similar and both α and β have unique $\{-2, 0, 2\}$-codes. Let $S(x) = rAx + b$ be such that $S(C_\alpha \times C_\beta) \subseteq C_\alpha \times C_\beta$ where $r \in (0, 1)$, $b = (b_1, b_2)^T$ and A is of form (1). Without loss of generality, suppose $\cos\theta \neq 0$ (otherwise use $S \circ S$ instead of S). Thus one has

$$tC_\alpha + sC_\beta + b_1 \subseteq C_\alpha$$

where $|t| \in (0, 1)$. This leads to

$$tC_\alpha + b_1 \subseteq C_\alpha$$

and so $(2 - |\alpha_k|)_{k=1}^{\infty}$ is strong periodic by Theorems 2.4 and A. That $(2 - |\beta_k|)_{k=1}^{\infty}$ is strong periodic can be obtained in the same way. □

The above proof indeed proves the following

Corollary 3.2 *Let $\alpha, \beta \in [-1, 1]$. Then $(C \times C) \cap (C \times C + (\alpha, \beta))$ is a self-similar set if and only if there exists a constrictive similitude S such that $S((C \times C) \cap (C \times C + (\alpha, \beta))) \subseteq (C \times C) \cap (C \times C + (\alpha, \beta))$.*

In the following we shall explore the form of a generating IFS of $C_\alpha \times C_\beta$ when it is self-similar. To do it we need the following lemma.

Lemma 3.3 ([4, Theorem 5]) *Let $E \subseteq \mathbb{R}^2$ be a self-similar set generated by the IFS $\{S_i(x) = r_i A_{\theta_i} x + b_i : r_i \in (0, 1), b_i \in \mathbb{R}^2, 1 \leq i \leq n\}$ where each A_{θ_i} is of the form (1), corresponding to θ_i. Suppose that the group G generated by $\{A_{\theta_i} : 1 \leq i \leq n\}$ is such that the set $\{\theta : A_\theta \in G\}$ is dense in $[0, 2\pi)$. Then*

$$\dim_H P_\xi(E) = \min\{\dim_H E, 1\} \text{ for all } \xi \in [0, \pi),$$

where P_ξ denotes the orthogonal projection onto the line through the origin and making angle ξ with the x-axis, dim_H denotes Hausdorff dimension.

When $C_\alpha \times C_\beta$ degenerates to the singleton $\{(0,0)\}$, any IFS $\{S_i(x) = r_i A_i x : r_i \in (0,1), 1 \le i \le n\}$ will generate $C_\alpha \times C_\beta$. For general case we have

Proposition 3.4 *Let $\alpha, \beta \in [-1,1]$ and let $C_\alpha \times C_\beta$ be not a singleton. Suppose that $C_\alpha \times C_\beta$ is a self-similar set generated by the IFS $\{S_i(x) = r_i A_i x + b_i : r_i \in (0,1), b_i \in \mathbb{R}^2, 1 \le i \le n\}$ where each A_i is of the form (1). Then*

$$A_i \in \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}, \quad 1 \le i \le n.$$

Proof One can check that by (5)

$$\begin{aligned} & r \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} C_\alpha \\ C_\beta \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_\alpha \\ C_\beta \end{pmatrix} + \begin{pmatrix} c_1 - 2r\gamma_\beta \sin\theta + r\beta\sin\theta + r\sin\theta \\ c_2 + 2r\gamma_\beta\cos\theta - r\beta\cos\theta - r\cos\theta \end{pmatrix} \end{aligned}$$

where $\gamma_\beta = \min\{x : x \in C \cap (C+\beta)\}$. Thus, without loss of generality we assume

$$A_i = \begin{pmatrix} \cos 2\pi\theta_i & -\sin 2\pi\theta_i \\ \sin 2\pi\theta_i & \cos 2\pi\theta_i \end{pmatrix}, \quad 1 \le i \le n.$$

We claim all θ_i are rational. Otherwise, the set of rotation angles in the group generated by $\{A_i : 1 \le i \le n\}$ is dense in $[0, 2\pi)$. From Lemma 3.3 it follows that

$$\dim_H C_\alpha = \dim_H C_\beta = \min\{\dim_H(C_\alpha \times C_\beta), 1\},$$

implying $\dim_H C_\alpha = \dim_H C_\beta = \dim_H(C_\alpha \times C_\beta) = 0$. However, we have $\dim_H(C_\alpha \times C_\beta) > 0$.

Since $S_i(C_\alpha \times C_\beta) \subseteq C_\alpha \times C_\beta$ we have that for some $d_i, e_i \in \mathbb{R}$

$$|r_i \cos 2\pi\theta_i| C_\alpha + d_i \subseteq C_\alpha \quad \text{and} \quad r_i^2 \sin^2 2\pi\theta_i C_\alpha + e_i \subseteq C_\alpha.$$

We claim that $\cos 2\pi\theta_i \in \{0, \pm 1\}$, i.e., either $\cos 2\pi\theta_i = 0$ or $\sin 2\pi\theta_i = 0$. Otherwise, by Corollary 2.5 there are $p, q \in \mathbb{N}$ such that

$$r_i |\cos 2\pi\theta_i| = 3^{-p} \quad \text{and} \quad r_i^2 \sin^2 2\pi\theta_i = 3^{-q}.$$

Thus $r_i^2 = 3^{-2p} + 3^{-q}$. Now take $m \in \mathbb{N}$ such that $m\theta_i \in \mathbb{Z}$ and consider the similitude $S_i^m(x)$, the mth iteration of S_i. By the same argument we have $r_i^m = 3^{-\ell}$ with $\ell \in \mathbb{N}$. Therefore, we have

$$3^{-2\ell} = (3^{-2p} + 3^{-q})^m$$

which is impossible. □

We remark that A_i may take the form $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$, e.g., $C_{\frac{2}{3}} \times C_0$ can be generated by the IFS

$$\left\{ \frac{1}{3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{3i+j}{9} \end{pmatrix}, ij \in \{0, 2\}^2 \right\}.$$

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