



# Intersecting nonhomogeneous Cantor sets with their translations

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## ABSTRACT

A scheme is given to compute the Hausdorff dimensions for the intersection of a class of nonhomogeneous Cantor sets with their translations.

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## 1. Introduction

Let  $\beta \in (0, 1/2)$  and let

$$\psi_k(x) = \beta x + k(1 - \beta), \quad k = -1, 0, 1.$$

The middle- $(1 - 2\beta)$  Cantor set  $\Gamma_\beta \subseteq [0, 1]$  is defined as the unique invariant nonempty compact set under maps  $\psi_0$  and  $\psi_1$ :

$$\Gamma_\beta = \psi_0(\Gamma_\beta) \cup \psi_1(\Gamma_\beta). \quad (1)$$

One also call  $\Gamma_\beta$ , the self-similar set generated by the iterated function system (IFS)  $\{\psi_0, \psi_1\}$ . The set  $\Gamma_{1/3}$  is the classical middle third Cantor set. In the past two decades, intersection of Cantor sets has been the subject of several studies [1–12]. The context and motivation are numerous, but mainly come from the discipline of dynamical systems. A brief history of why intersection of Cantor sets is important in dynamical systems is described by Davis and Hu in [1]. The intersection  $\Gamma_{1/3} \cap (\Gamma_{1/3} + t)$  (or more general,  $\Gamma_\beta \cap (\Gamma_\beta + t)$ ) has been extensively studied by lots of authors.

When  $0 < \beta < 1/3$ ,  $\Gamma_\beta - \Gamma_\beta$  is the self-similar set generated by the IFS  $\{\psi_0, \psi_1, \psi_{-1}\}$  and satisfies the open set condition (see (3) in Section 2) so that for each  $t \in \Gamma_\beta - \Gamma_\beta$  the set  $\Gamma_\beta \cap (\Gamma_\beta + t)$  is just a generalized Moran set (see (5) in Section 2). Thus, the Hausdorff, packing and box-counting dimensions of  $\Gamma_\beta \cap (\Gamma_\beta + t)$  can then be determined. When  $\beta = 1/3$ , it can be dealt with in the same way though a bit of difficulty occurs. However, when  $1/3 < \beta < 1/2$ , the set  $\Gamma_\beta \cap (\Gamma_\beta + t)$  presents very complicated geometrical structure (see (4) in Section 2). A natural question is the following.

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**Question 1.** For  $\beta \in (1/3, 1/2)$  what are the Hausdorff, packing and box-counting dimensions of  $\Gamma_\beta \cap (\Gamma_\beta + t)$ ?

Note that the middle- $(1 - 2\beta)$  Cantor set  $\Gamma_\beta$  is a *homogeneous Cantor set* since it can be generated by an IFS with uniform contractive ratio (the maps  $\psi_0$  and  $\psi_1$  have the same contractive ratio  $\beta$  in (1)). Hence, points in sets  $\Gamma_\beta - \Gamma_\beta$  and  $\Gamma_\beta \cap (\Gamma_\beta + t)$  have an excellent algebraic representation.

Instead of  $\Gamma_\beta$ , we consider a class of nonhomogeneous Cantor sets. Let

$$F_1(x) = \beta x \quad \text{and} \quad F_2(x) = \beta^\tau x + 1 - \beta^\tau, \quad x \in \mathbb{R}, \tag{2}$$

where  $\beta \in (0, 1)$  and  $\tau \in \mathbb{N}$  are such that  $\beta + \beta^\tau < 1$ . It determines a unique nonempty compact set  $\Gamma_{\beta,\tau} \subseteq [0, 1]$  satisfying

$$\Gamma_{\beta,\tau} = F_1(\Gamma_{\beta,\tau}) \cup F_2(\Gamma_{\beta,\tau}).$$

Obviously,  $\Gamma_{\beta,1} = \Gamma_\beta$  and we will use  $\Gamma_{\beta,1}$  instead of  $\Gamma_\beta$  below. When  $\tau \neq 1$ , the set  $\Gamma_{\beta,\tau}$  is a *nonhomogeneous Cantor set* so that we have no analogous representations as shown in (3) and (4) for the homogeneous case. Again the similar question arises:

**Question 2.** What are the Hausdorff, packing and box-counting dimensions of  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$ ?

We mainly focus on the [Question 2](#) since it reduces to [Question 1](#) when  $\tau = 1$ . We will analyze the structure of  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  by making use of a geometric approach which were employed to investigate the self-similar sets with overlaps, e.g. in [13–18], just list a few. The main idea is to investigate for which  $t$  the set  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is of finite type so that it presents a graph-directed structure and then its Hausdorff, packing and box-counting dimensions can be calculated. For example, when  $\beta^{-1}$  is a Pisot number we have (see [Proposition 4.3](#))

$$\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t) \text{ is of finite type for each } t \in (\Gamma_{\beta,\tau} - \Gamma_{\beta,\tau}) \cap \mathbb{Z}[\beta].$$

This allows us to get (see [Examples 4.5](#) and [4.7](#))

(A)  $\dim_H(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + \beta - \beta^3)) = \dim_B(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + \beta - \beta^3)) = \frac{\log 2}{\log(\sqrt{3}+1)}$  where  $\beta = (\sqrt{3} - 1)/2 \in (1/3, 1/2)$  (homogeneous case, see [Question 1](#)).

(B)  $\dim_H(\Gamma_{\frac{1}{2},2} \cap (\Gamma_{\frac{1}{2},2} + \frac{3}{16})) = \dim_B(\Gamma_{\frac{1}{2},2} \cap (\Gamma_{\frac{1}{2},2} + \frac{3}{16})) = \frac{\log(3+\sqrt{5})-\log 2}{\log 4}$  (nonhomogeneous case, see [Question 2](#)).

In addition,  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  may present a graph-directed structure for some  $t$  even if  $\beta^{-1}$  is not a Pisot number. For instance, in [Example 4.6](#) we show that for any  $\beta \in (0, (3 - \sqrt{5})/2)$  (note that  $(3 - \sqrt{5})/2 > 1/3$ ) and any  $\ell \in \mathbb{N}$ , sets  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + (1 - \beta)\beta^\ell)$  and  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + (1 - \beta)(1 + \beta^\ell))$  present graph-directed structures and they all have the same Hausdorff and box dimensions  $\frac{\log 2}{-\log \beta}$ .

For [Question 1](#), an alternative way is to investigate for which  $\beta$  and  $t$  the set  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  presents a structure of the generalized Moran set so that their Hausdorff, packing and box-counting dimensions can be determined. For example, we obtain (see [Theorem 3.5](#) and [Corollary 3.6](#)).

(C) Let  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$ . Then  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set if  $\#N_{\Gamma_{\beta,1},k}(F_J([0, 1])) \leq 1$  for any  $k \in \mathbb{N} \cup \{0\}$  and any  $J \in \Sigma_2^k$ .

(D) Let  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$ . If for some  $m \in \mathbb{N}$  there exist no new type for members from  $\bigcup_{k>m} \mathcal{C}_{\Gamma_{\beta,1},k}$  and  $\#N_{\Gamma_{\beta,1},k}(F_J([0, 1])) \leq 1$  for any  $J \in \Sigma_2^k$  with  $k \leq m$ . Then  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set.

This paper is arranged as follows. A geometrical description for the structure of  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is arranged in [Section 2](#). Following the idea in [17], the concept of type is defined for some component sets of  $\Gamma_{\beta,\tau}$  by means of an equivalent relation and a sufficient and necessary condition for  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  to be of finite type is then established in [Section 3](#) (see [Theorem 3.3](#)). The final section is mainly devoted to a scheme for computing the Hausdorff dimensions of  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  (see [Theorem 4.4](#)). Some examples are included. This paper is mainly enlightened by [17].

## 2. Geometrical structure of $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$

In this section we first explain how to determine the Hausdorff, packing and box-counting dimensions of  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  with  $\beta \in (0, 1/3]$  and what difficulties we encounter for the case  $\beta \in (1/3, 1/2)$  and further for the nonhomogeneous case. Then we introduce some notations and analyze the geometrical structure of  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$ .

It is easy to see that

$$\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t) \neq \emptyset \quad \text{if and only if } t \in \Gamma_{\beta,1} - \Gamma_{\beta,1},$$

and the compact set  $\Gamma_{\beta,1} - \Gamma_{\beta,1}$  is generated by the IFS  $\{\psi_0, \psi_1, \psi_{-1}\}$ , i.e.,

$$\Gamma_{\beta,1} - \Gamma_{\beta,1} = \psi_{-1}(\Gamma_{\beta,1} - \Gamma_{\beta,1}) \cup \psi_0(\Gamma_{\beta,1} - \Gamma_{\beta,1}) \cup \psi_1(\Gamma_{\beta,1} - \Gamma_{\beta,1}). \tag{3}$$

Thus we have

$$\Gamma_{\beta,1} = \left\{ \sum_{k=1}^{\infty} j_k \beta^{k-1} (1 - \beta) : (j_k)_{k=1}^{\infty} \in \{0, 1\}^{\mathbb{N}} \right\},$$

and

$$\Gamma_{\beta,1} - \Gamma_{\beta,1} = \left\{ \sum_{k=1}^{\infty} j_k \beta^{k-1} (1 - \beta) : (j_k)_{k=1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}} \right\}.$$

One can check that for a  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$  (cf. [8,9])

$$\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t) = \bigcup_{(t_k)_{k=1}^{\infty}} \left\{ \sum_{i=1}^{\infty} x_i \beta^{i-1} (1 - \beta) : x_i \in \{0, 1\} \cap \{t_i, t_i + 1\} \right\} \tag{4}$$

where the union is taken over all  $(t_k)_{k=1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}}$  satisfying  $t = \sum_{k=1}^{\infty} t_k \beta^{k-1} (1 - \beta)$ . When  $\beta \in (0, 1/3)$ , each  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$  has a unique  $(t_k)_{k=1}^{\infty}$  such that  $t = \sum_{k=1}^{\infty} t_k \beta^{k-1} (1 - \beta)$  and so

$$\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t) = \left\{ \sum_{i=1}^{\infty} x_i \beta^{i-1} (1 - \beta) : x_i \in \{0, 1\} \cap \{t_i, t_i + 1\} \right\}. \tag{5}$$

When  $\beta = 1/3$ ,  $\Gamma_{1/3,1} \cap (\Gamma_{1/3,1} + t)$  is either of form (5) or a finite set.

The set on the right side of (5) is called a *generalized Moran set* and its Hausdorff, packing and box-counting dimensions can be determined in terms of  $(t_k)_{k=1}^{\infty}$  (cf. [8,9]).

When  $\beta \in (1/3, 1/2)$ , a  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$  may have a uncountable number of  $(t_k)_{k=1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}}$  satisfying  $t = \sum_{k=1}^{\infty} t_k \beta^{k-1} (1 - \beta)$ , which make it hard to determine the Hausdorff, packing and box-counting dimensions of  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$ . Furthermore, the above approaches fail to be effective for the nonhomogeneous case.

Let  $\Sigma_2 = \{1, 2\}$  and  $\Sigma_2^* = \bigcup_{n \geq 0} \Sigma_2^n$  be the set of all finite words, where  $\Sigma_2^n$  is the set of all words of length  $n$  and  $\Sigma_2^0$  contains just the empty word  $\emptyset$ . For  $J \in \Sigma_2^n$  let  $|J| = n$  denote the length of  $J$ . For  $I \in \Sigma_2^m$  and  $J \in \Sigma_2^n$ , let  $IJ \in \Sigma_2^{m+n}$  be the concatenation of  $I$  and  $J$ . Let  $\Sigma_2^{\mathbb{N}} = \{J = (j_1, j_2, \dots) : j_i \in \Sigma_2\}$ . For a  $J = (j_1, j_2, \dots) \in \Sigma_2^{\mathbb{N}}$  and  $k \in \mathbb{N}$  let  $J|k = (j_1, \dots, j_k) \in \Sigma_2^k$ . Let  $\beta_1 = \beta$  and  $\beta_2 = \beta^\tau$ . For a  $J = (j_1, \dots, j_k) \in \Sigma_2^k$  denote (recall  $F_1$  and  $F_2$  are defined in (2))

$$F_J = F_{j_1} \circ F_{j_2} \circ \dots \circ F_{j_k}, \quad \beta_J = \beta_{j_1} \beta_{j_2} \dots \beta_{j_k},$$

where we adopt the convention that  $\beta_\emptyset = 1$  and  $F_\emptyset$  is the identity map on  $\mathbb{R}$ . Thus

$$\Gamma_{\beta,\tau} = \bigcap_{k=0}^{\infty} \bigcup_{J \in \Sigma_2^k} F_J([0, 1]).$$

We establish a bijection  $\Pi : \Sigma_2^{\mathbb{N}} \rightarrow \Gamma_{\beta,\tau}$ : for  $J = (j_1, j_2, \dots) \in \Sigma_2^{\mathbb{N}}$

$$\Pi(J) = \lim_{k \rightarrow \infty} F_{J|k}(0) = \lim_{k \rightarrow \infty} \left( b_{j_1} + \sum_{\ell=2}^k b_{j_\ell} \prod_{s=1}^{\ell-1} \beta_{j_s} \right) = b_{j_1} + \sum_{\ell=2}^{\infty} b_{j_\ell} \prod_{s=1}^{\ell-1} \beta_{j_s},$$

where  $b_1 = 0$ ,  $b_2 = 1 - \beta^\tau$ ,  $\beta_1 = \beta$  and  $\beta_2 = \beta^\tau$ . For each  $k \in \mathbb{N}$  let

$$\mathcal{M}_k = \{J = (j_1, \dots, j_m) \in \Sigma_2^* | \beta^{\tau k} \beta_{j_m} < \beta_J \leq \beta^{\tau k}\},$$

and let  $\mathcal{M}_0$  consist of only the empty word  $\emptyset$ . Then  $F_J([0, 1]), J \in \mathcal{M}_k$  are mutually disjoint and have comparable lengths, i.e.,  $\beta < |F_J([0, 1])|/|F_I([0, 1])| < \beta^{-\tau}$  for any  $I, J \in \mathcal{M}_k$ . Moreover,

$$\Gamma_{\beta,\tau} = \bigcap_{k=1}^{\infty} \bigcup_{J \in \mathcal{M}_k} F_J([0, 1]).$$

Let  $S_1(x) = \beta x + t(1 - \beta)$ ,  $S_2(x) = \beta^\tau x + (t + 1)(1 - \beta^\tau)$  and  $S_J = S_{j_1} \circ S_{j_2} \circ \dots \circ S_{j_k}$  for  $J = (j_1, \dots, j_k) \in \Sigma_2^k$  with  $S_\emptyset$  being the identity map. Then

$$\Gamma_{\beta,\tau} + t = S_1(\Gamma_{\beta,\tau} + t) \cup S_2(\Gamma_{\beta,\tau} + t) = \bigcap_{k=1}^{\infty} \bigcup_{J \in \mathcal{M}_k} S_J([t, 1 + t]).$$

The sets  $F_J([0, 1]), S_J([t, 1 + t])$  for  $J \in \mathcal{M}_k$  are called the *k-level component sets* of  $\Gamma_{\beta,\tau}$  and  $\Gamma_{\beta,\tau} + t$ , respectively (so  $F_\emptyset([0, 1]) = [0, 1]$  and  $S_\emptyset([t, 1 + t]) = [t, 1 + t]$  are the 0-level component sets).

For a *k-level component set*, say  $F_J([0, 1])$  with  $J \in \mathcal{M}_k, k \in \mathbb{N} \cup \{0\}$ , of  $\Gamma_{\beta,\tau}$  its *neighborhood*  $N_{\Gamma_{\beta,\tau},k}(F_J([0, 1]))$  is defined as the collection of *k-level component sets* of  $\Gamma_{\beta,\tau} + t$  which it intersects, i.e.,

$$N_{\Gamma_{\beta,\tau},k}(F_J([0, 1])) = \{S_I([t, 1 + t]) : I \in \mathcal{M}_k, F_J([0, 1]) \cap S_I([t, 1 + t]) \neq \emptyset\}. \tag{6}$$

Let

$$\Lambda_k = \{J \in \mathcal{M}_k : N_{\Gamma_{\beta,\tau},k}(F_J([0, 1])) \neq \emptyset\}.$$

We remark that for some  $F_J([0, 1])$  with  $J \in \mathcal{M}_k$ ,  $N_{\Gamma_{\beta,\tau,k}}(F_J([0, 1]))$  may be empty. If it is the case they do not intersect any  $k$ -level component sets of  $\Gamma_{\beta,\tau} + t$  and so have no contribution to  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$ . One can readily check that

$$\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t) = \bigcap_{k=1}^{\infty} \bigcup_{J \in \Lambda_k} F_J([0, 1]). \tag{7}$$

Thus,  $\Lambda_k$  with  $k \in \mathbb{N} \cup \{0\}$  geometrically characterize  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$ . Let

$$\mathcal{C}_{\Gamma_{\beta,\tau,k}} = \{F_J([0, 1]) : J \in \Lambda_k\} \quad \text{and} \quad \mathcal{C}_{\Gamma_{\beta,\tau}} = \left\{ F_J([0, 1]) : J \in \bigcup_{k \geq 0} \Lambda_k \right\}.$$

A graph with vertex set  $\mathcal{C}_{\Gamma_{\beta,\tau}}$  is constructed as follows. For  $I \in \Lambda_k$  and  $J \in \Lambda_{k+1}$ , if there exists an  $\ell \in \Sigma_2^*$  such that  $J = I\ell$ , then we connect a directed edge  $\ell$  from  $F_I([0, 1])$  to  $F_J([0, 1])$  and denote this by  $F_I([0, 1]) \xrightarrow{\ell} F_J([0, 1])$ . In this case,  $F_I([0, 1])$  is called the *parent* of  $F_J([0, 1])$  and  $F_J([0, 1])$  an *offspring* of  $F_I([0, 1])$ . Clearly, each  $k$ -level component set in  $\mathcal{C}_{\Gamma_{\beta,\tau,k}}$  ( $k \geq 1$ ) of  $\Gamma_{\beta,\tau}$  has only one parent, but may have no offsprings. The *reduced graph* is then constructed by removing those vertexes having no offsprings and the edges going to them. The reduced graph determines a subset  $\Lambda$  of  $\Sigma_2^{\mathbb{N}}$ :

$$\Lambda = \{J = (j_1, j_2, \dots) \in \Sigma_2^{\mathbb{N}} : \text{there exist } n(J, 1) < n(J, 2) < \dots \text{ such that } (j_1, \dots, j_{n(J,k)}) \in \Lambda_k\}. \tag{8}$$

Then  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t) = \Pi(\Lambda)$  by (7). In addition, the concept of parent and offspring can be defined in the same way for the component sets in  $\{S_I([t, t + 1]) : I \in \bigcup_{k \geq 0} \mathcal{M}_k\}$  of  $\Gamma_{\beta,\tau} + t$ . Note that each component set of  $\Gamma_{\beta,\tau} + t$  has only one parent, but at least two offsprings.

### 3. Equivalence relation and type

We now establish an equivalence relation for members in  $\mathcal{C}_{\Gamma_{\beta,\tau}}$ . For  $\eta_1 \in \mathcal{C}_{\Gamma_{\beta,\tau,k_1}}$  and  $\eta_2 \in \mathcal{C}_{\Gamma_{\beta,\tau,k_2}}$ ,  $\eta_1$  and  $\eta_2$  is said to be *equivalent*, denoted by  $\eta_1 \sim \eta_2$ , if there exists a similarity map of form  $\xi(x) = \beta^{\tau(k_2-k_1)}x + c$  with  $c \in \mathbb{R}$  such that  $\xi(\eta_1) = \eta_2$  and

$$\{u : u \in N_{\Gamma_{\beta,\tau,k_2}}(\eta_2)\} = \{\xi(v) : v \in N_{\Gamma_{\beta,\tau,k_1}}(\eta_1)\}.$$

It is easy to check that  $\sim$  is indeed an equivalence relation. We shall use  $[\eta]$  to denote the equivalence class, and call it the *type* of  $\eta$ .

The following lemma describes some properties related to the offsprings, neighborhood of a pair of equivalent component sets.

**Lemma 3.1.** For  $I \in \Lambda_{k_1}$  and  $J \in \Lambda_{k_2}$  let  $N_{\Gamma_{\beta,\tau,k_1}}(F_I([0, 1])) = \{S_{I_1}([t, 1 + t]), \dots, S_{I_n}([t, 1 + t])\}$  and  $N_{\Gamma_{\beta,\tau,k_2}}(F_J([0, 1])) = \{S_{J_1}([t, 1 + t]), \dots, S_{J_m}([t, 1 + t])\}$ . Suppose  $F_I([0, 1]) \sim F_J([0, 1])$ . Then

- (I)  $n = m$ ,  $I\ell \in \Lambda_{k_1+1}$  if and only if  $J\ell \in \Lambda_{k_2+1}$ ;
- (II) Let  $F_{I_{p_i}}([0, 1])$ ,  $1 \leq i \leq s$  be all offsprings of  $F_I([0, 1])$  in the order from left to right,  $F_{J_{q_j}}([0, 1])$ ,  $1 \leq j \leq t$  be all offsprings of  $F_J([0, 1])$  in the order from left to right. Then  $s = t$  and  $[F_{I_{p_i}}([0, 1])] = [F_{J_{q_j}}([0, 1])]$  for  $i = 1, \dots, s$ .

**Proof.** Let  $\xi(x) = \beta^{\tau(k_2-k_1)}x + c$  be such that  $F_J([0, 1]) = \xi(F_I([0, 1]))$ , equivalently  $\xi \circ F_I = F_J$ .

(I)  $n = m$  just derives from the definition of equivalence. Without loss of generality, we assume that  $S_{J_i}([t, 1 + t]) = \xi(S_{I_i}([t, 1 + t]))$ ,  $1 \leq i \leq m$  (equivalently  $\xi \circ S_{I_i} = S_{J_i}$ ). We first prove  $I\ell \in \mathcal{M}_{k_1+1}$  if and only if  $J\ell \in \mathcal{M}_{k_2+1}$ . Recall that

$$\mathcal{M}_k = \{P = (p_1, \dots, p_m) \in \Sigma_2^* | \beta^{\tau k} p_m < \beta_P \leq \beta^{\tau k}\},$$

with  $\beta_1 = \beta$ ,  $\beta_2 = \beta^{\tau}$ . Suppose  $I\ell \in \mathcal{M}_{k_1+1}$  with  $\ell = (\ell_1, \dots, \ell_s)$ . Then  $(\beta^{\tau})^{k_1+1} \beta_{\ell_s} < \beta_I \beta_{\ell} \leq (\beta^{\tau})^{k_1+1}$ . By the equivalence of  $F_I([0, 1])$  and  $F_J([0, 1])$  we have  $\beta_J = (\beta^{\tau})^{k_2-k_1} \beta_I$ , which implies that  $(\beta^{\tau})^{k_2+1} \beta_{\ell_s} < \beta_J \beta_{\ell} \leq (\beta^{\tau})^{k_2+1}$ , i.e.,  $J\ell \in \mathcal{M}_{k_2+1}$ . The converse is given in the same way.

It remains to check that  $I\ell \in \Lambda_{k_1+1}$  if and only if  $J\ell \in \Lambda_{k_2+1}$ . Recall that

$$\Lambda_k = \{P \in \mathcal{M}_k : N_{\Gamma_{\beta,\tau,k}}(F_P([0, 1])) \neq \emptyset, k \geq 0\}.$$

So one only needs to show that  $N_{\Gamma_{\beta,\tau,k_1+1}}(F_{I\ell}([0, 1])) \neq \emptyset$  if and only if  $N_{\Gamma_{\beta,\tau,k_2+1}}(F_{J\ell}([0, 1])) \neq \emptyset$ . By symmetry, we merely show the “only if” part below. Suppose  $N_{\Gamma_{\beta,\tau,k_1+1}}(F_{I\ell}([0, 1])) \neq \emptyset$ . Then there exist some  $1 \leq i \leq m$  and  $\ell^* \in \Sigma_2^*$  such that

$$I_i \ell^* \in \mathcal{M}_{k_1+1} \quad \text{and} \quad F_{I_i \ell^*}([0, 1]) \cap S_{I_i \ell^*}([t, t + 1]) \neq \emptyset.$$

Since  $\xi \circ F_I = F_J$  and  $\xi \circ S_{I_i} = S_{J_i}$ , we have  $\xi(F_{I_i \ell^*}([0, 1])) = F_{J_i \ell^*}([0, 1])$  and  $\xi(S_{I_i \ell^*}([t, t + 1])) = S_{J_i \ell^*}([t, t + 1])$

$$J_i \ell^* \in \mathcal{M}_{k_2+1} \quad \text{and} \quad F_{J_i \ell^*}([0, 1]) \cap S_{J_i \ell^*}([t, t + 1]) \neq \emptyset,$$

leading to  $N_{\Gamma_{\beta,\tau,k_2+1}}(F_{J\ell}([0, 1])) \neq \emptyset$ .

(II) Since  $I\ell \in \Lambda_{k_1+1}$  if and only if  $J\ell \in \Lambda_{k_2+1}$  by (I), we have  $s = t$  and  $P_i = Q_i$  for  $1 \leq i \leq s$ . For simplicity, it suffices to show  $[F_{P_1}([0, 1])] = [F_{Q_1}([0, 1])]$ . Note that

$$N_{\Gamma_{\beta,\tau,k_1+1}}(F_{P_1}([0, 1])) = \{S_{I_i\ell}([t, t + 1]) : F_{P_1}([0, 1]) \cap S_{I_i\ell}([t, t + 1]) \neq \emptyset, 1 \leq i \leq n, I_i\ell \in \mathcal{M}_{k_1+1}\},$$

and

$$N_{\Gamma_{\beta,\tau,k_2+1}}(F_{Q_1}([0, 1])) = \{S_{J_i\ell}([t, t + 1]) : F_{Q_1}([0, 1]) \cap S_{J_i\ell}([t, t + 1]) \neq \emptyset, 1 \leq i \leq n, J_i\ell \in \mathcal{M}_{k_2+1}\}.$$

Therefore, the desired result follows from the fact  $\xi \circ F_I = F_J$  and  $\xi \circ S_{I_i} = S_{J_i}$ .  $\square$

**Definition 3.2.**  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is said to be of *finite type* if and only if  $\{\omega : \omega \in \mathcal{C}_{\Gamma_{\beta,\tau}}\}$  is a finite set.

**Theorem 3.3.** Let  $t \in \Gamma_{\beta,\tau} - \Gamma_{\beta,\tau}$ . For a component set  $\eta$  of  $\Gamma_{\beta,\tau}$  or  $\Gamma_{\beta,\tau} + t$  we denote by  $\hat{\eta}$  its left endpoint. For a  $k \in \mathbb{N} \cup \{0\}$  and an  $\eta \in \mathcal{C}_{\Gamma_{\beta,\tau,k}}$  let

$$R_{\Gamma_{\beta,\tau,k}}(\eta) = \{\beta^{-\tau k}(\hat{v} - \hat{\eta}) : v \in N_{\Gamma_{\beta,\tau,k}}(\eta)\} \quad \text{and} \quad R_{\Gamma_{\beta,\tau}}(\eta) = \{R_{\Gamma_{\beta,\tau,k}}(\eta) : \eta \in \mathcal{C}_{\Gamma_{\beta,\tau,k}}, k \in \mathbb{N} \cup \{0\}\}.$$

Then  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is of *finite type* if and only if  $R_{\Gamma_{\beta,\tau}}$  is a finite set.

**Proof.** Let us first recall that for  $\eta_1 \in \mathcal{C}_{\Gamma_{\beta,\tau,k_1}}$  and  $\eta_2 \in \mathcal{C}_{\Gamma_{\beta,\tau,k_2}}$ ,  $\eta_1 \sim \eta_2$  if and only if there exists a linear map of form  $\xi(x) = \beta^{\tau(k_2-k_1)}x + c$  with  $c \in \mathbb{R}$  such that

$$\xi(\eta_1) = \eta_2 \quad \text{and} \quad \{u : u \in N_{\Gamma_{\beta,\tau,k_2}}(\eta_2)\} = \{\xi(v) : v \in N_{\Gamma_{\beta,\tau,k_1}}(\eta_1)\}.$$

By Lemma 3.1(I) one can assume  $N_{\Gamma_{\beta,\tau,k_2}} = \{u_1 = \eta_2, u_2, \dots, u_n\}$  and  $N_{\Gamma_{\beta,\tau,k_1}}(\eta_1) = \{v_1 = \eta_1, v_2, \dots, v_n\}$  which satisfies  $\xi(v_i) = u_i, i = 1, 2, \dots, n$ . Thus  $\eta_1 \sim \eta_2$  if and only if for all  $i = 1, 2, \dots, n$

$$\beta^{-\tau k_1}|v_i| = \beta^{-\tau k_2}|u_i| \quad \text{and} \quad \beta^{-\tau k_1}(\hat{v}_i - \hat{\eta}_1) = \beta^{-\tau k_2}(\hat{u}_i - \hat{\eta}_2). \tag{9}$$

This means that  $R_{\Gamma_{\beta,\tau,k_1}}(\eta_1) = R_{\Gamma_{\beta,\tau,k_2}}(\eta_2)$  when  $\eta_1 \sim \eta_2$ . So the necessity is true. On the other hand, for a  $k$ -level component set  $\eta$  of  $\Gamma_{\beta,\tau}$  or  $\Gamma_{\beta,\tau} + t$  we have  $\beta^{-\tau k}|\eta| \in \{\beta^\ell : \ell = 0, 1, \dots, \tau - 1\}$ , which implies the sufficiency.  $\square$

**Remark 3.4.** For the case  $\tau = 1$ , each  $k$ -level component set of  $\Gamma_{\beta,1}$  is uniquely determined by its left endpoint. Thus, for  $\eta_1 \in \mathcal{C}_{\Gamma_{\beta,1,k_1}}$  and  $\eta_2 \in \mathcal{C}_{\Gamma_{\beta,1,k_2}}$ ,  $\eta_1 \sim \eta_2$  if and only if

$$\{\beta^{-k_1}(\hat{v} - \hat{\eta}_1) : v \in N_{\Gamma_{\beta,1,k_1}}(\eta_1)\} = \{\beta^{-k_2}(\hat{u} - \hat{\eta}_2) : u \in N_{\Gamma_{\beta,1,k_2}}(\eta_2)\},$$

where  $\hat{\eta}$  denotes the left endpoint of the component set  $\eta$  of  $\Gamma_{\beta,1}$  or  $\Gamma_{\beta,1} + t$ .

When  $\tau = 1$ , let us recall that  $\Gamma_{\beta,1} = \Gamma_\beta$  and, correspondingly,  $\mathcal{M}_k = \Sigma_2^k$ . As shown in (5),  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set when  $\beta \in (0, 1/3)$ . The following theorem shows that  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  sometimes may be a generalized Moran set without the restriction on  $\beta$ .

**Theorem 3.5.** Let  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$ . Then  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set if  $\#N_{\Gamma_{\beta,1,k}}(F_J([0, 1])) \leq 1$  for any  $k \in \mathbb{N} \cup \{0\}$  and any  $J \in \Sigma_2^k$ .

**Proof.** We begin with  $k = 1$ . Then there are two possible cases.

Case 1.  $\#N_{\Gamma_{\beta,1,1}}(F_1([0, 1])) = \#N_{\Gamma_{\beta,1,1}}(F_2([0, 1])) = 1$ . In this case,  $N_{\Gamma_{\beta,1,1}}(F_1([0, 1])) = S_1([t, t + 1])$  and  $N_{\Gamma_{\beta,1,1}}(F_2([0, 1])) = S_2([t, t + 1])$ , implying  $F_1([0, 1]) \sim F_2([0, 1])$ .

Case 2. Only one  $j \in \Sigma_2^1$  is such that  $\#N_{\Gamma_{\beta,1,1}}(F_j([0, 1])) = 1$ . Surely  $F_j([0, 1]) \sim F_j([0, 1])$  in this simple case.

Thus, for both cases described above we have that all  $F_I([0, 1]), I \in \Lambda_1$  are equivalent. Now suppose that all  $F_I([0, 1]), I \in \Lambda_k$  are equivalent. From Lemma 3.1(II) it follows that either each  $F_I([0, 1])$  with  $I \in \Lambda_k$  has only one offspring and they are equivalent, or each  $F_I([0, 1])$  with  $I \in \Lambda_k$  has two offsprings and all these offsprings are equivalent ( $F_{I_1}([0, 1]) \sim F_{I_2}([0, 1])$ ) derives from the same argument as that in case 1;  $F_{I_i}([0, 1]) \sim F_{I_j}([0, 1])$  for  $i = 1, 2$  follows from Lemma 3.1(II). Therefore, for each  $k \in \mathbb{N}$ ,  $F_I([0, 1]), I \in \Lambda_k$  are equivalent, i.e., they are of same type. Hence,  $\Lambda_k$  is of form  $\prod_{i=1}^k D_i$  with  $\emptyset \neq D_i \subseteq \{1, 2\}$ , i.e.,  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t) \neq \emptyset$  is a generalized Moran set.  $\square$

**Corollary 3.6.** Let  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$ . If for some  $m \in \mathbb{N}$  there exist no new type for members from  $\bigcup_{k>m} \mathcal{C}_{\Gamma_{\beta,1,k}}$  and  $\#N_{\Gamma_{\beta,1,k}}(F_J([0, 1])) \leq 1$  for any  $J \in \Sigma_2^k$  with  $k \leq m$ . Then  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set.

**Proof.** We claim that  $\#N_{\Gamma_{\beta,1,k}}(F_J([0, 1])) \leq 1$  still hold for any  $J \in \Sigma_2^k$  with  $k > m$  and so the desired result follows from Theorem 3.5. In fact, if  $\#N_{\Gamma_{\beta,1,k}}(F_J([0, 1])) > 1$  for some  $J \in \Sigma_2^k$  and  $k > m$ , then the type of  $F_J([0, 1])$  is new.  $\square$

As a direct application, one can see that when  $\beta \in (0, 1/3)$ ,  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set for all  $t \in \Gamma_{\beta,1} - \Gamma_{\beta,1}$  since  $\#N_{\Gamma_{\beta,1,k}}(F_J([0, 1])) \leq 1$  always hold (also see (5)). When  $\beta = 1/3$ , for each  $t \in \Gamma_{1/3,1} - \Gamma_{1/3,1}$  there exists an  $m \in \mathbb{N}$

such that  $\#N_{\Gamma_{1/3,1,k}}(F_J([0, 1])) \leq 1$  for any  $J \in \Sigma_2^k$  with  $k > m$ . Thus, for each  $t \in \Gamma_{1/3,1} - \Gamma_{1/3,1}, \Gamma_{1/3,1} \cap (\Gamma_{1/3,1} + t)$  is a finite union of generalized Moran sets. As will be shown in Example 4.6, for each  $\beta \in (1/3, (3 - \sqrt{5})/2), \Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  are generalized Moran sets for some  $t$ .

**4. Finite type conditions and graph-directed structure**

4.1. Finite type conditions

In this subsection a condition is given such that  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is of finite type. The following well-known result about Pisot numbers can be found in [19].

**Lemma 4.1.** *Suppose that  $\beta^{-1}$  is a Pisot number. Let  $M, N$  be two positive constants and let*

$$\Omega_{M,N} = \left\{ \sum_{i=0}^k a_i \beta^{-i} : |a_i| < M, a_i \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\} \right\} \cap [-N, N].$$

Then  $\Omega_{M,N}$  is a finite set.

Let  $\mathbb{Z}[\beta] = \left\{ \sum_{i=0}^m a_i \beta^i : a_i \in \mathbb{Z}, m \in \mathbb{N} \cup \{0\} \right\}$ . The following derives directly from Lemma 4.1.

**Corollary 4.2.** *Suppose that  $\beta^{-1}$  is a Pisot number and that  $N$  is positive constant. Let  $D \subseteq r\mathbb{Z}[\beta]$  be a finite set with  $r \in \mathbb{R}$ . Then*

$$\mathfrak{B}_{D,N} = \left\{ \sum_{i=0}^k e_i \beta^{-i} : e_i \in D, k \in \mathbb{N} \cup \{0\} \right\} \cap [-N, N]$$

is a finite set.

**Proof.** Let

$$D = \left\{ r \sum_{j=0}^{p_i} a_{ij} \beta^j : a_{ij} \in \mathbb{Z}, p_i \in \mathbb{N} \cup \{0\}, i = 1, \dots, \ell \right\}.$$

Let  $s = \max_{1 \leq i \leq \ell} p_i$ . Since  $h = \max\{|a_{ij}| : 1 \leq i \leq \ell, 0 \leq j \leq p_i\} < \infty$  and

$$\mathfrak{B}_{D,N} = \beta^s r \left( \left\{ \sum_{i=0}^k \frac{e_i}{r} \beta^{-i-s} : e_i \in D, k \in \mathbb{N} \cup \{0\} \right\} \cap [-N\beta^{-s}r^{-1}, N\beta^{-s}r^{-1}] \right),$$

the desired result follows from Lemma 4.1.  $\square$

**Proposition 4.3.** *Suppose that  $\beta^{-1}$  is a Pisot number. Then  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is of finite type for each  $t \in (\Gamma_{\beta,\tau} - \Gamma_{\beta,\tau}) \cap \mathbb{Z}[\beta]$ .*

**Proof.** Let  $R_{\Gamma_{\beta,\tau}}$  be as in Theorem 3.3. By Theorem 3.3 it suffices to show  $R_{\Gamma_{\beta,\tau}}$  is a finite set. For an  $\eta = F_J([0, 1])$  with  $J = (j_1, \dots, j_m) \in \Lambda_k$  and a  $\delta = S_I([t, t + 1]) \in N_{\Gamma_{\beta,\tau,k}}(\eta)$  with  $I = (i_1, \dots, i_n) \in \mathcal{M}_k$ , we have (recall  $\beta_1 = \beta$  and  $\beta_2 = \beta^\tau$ )

$$\hat{\eta} = F_J(0) = b_{j_1} + \sum_{\ell=2}^m b_{j_\ell} \prod_{s=1}^{\ell-1} \beta_{j_s}$$

and

$$\hat{\delta} = S_I(t) = c_{i_1} + \sum_{\ell=2}^n c_{i_\ell} \prod_{s=1}^{\ell-1} \beta_{i_s} + \beta_I t$$

where  $b_1 = 0, b_2 = 1 - \beta^\tau, c_1 = t(1 - \beta)$  and  $c_2 = (t + 1)(1 - \beta^\tau)$ . Thus

$$\beta^{-\tau k}(\hat{\delta} - \hat{\eta}) = (c_{i_1} - b_{j_1})\beta^{-\tau k} + t\beta_1\beta^{-\tau k} + \sum_{\ell=2}^n c_{i_\ell}\beta^{-\tau k} \prod_{s=1}^{\ell-1} \beta_{i_s} - \sum_{\ell=2}^m b_{j_\ell}\beta^{-\tau k} \prod_{s=1}^{\ell-1} \beta_{j_s}.$$

Note that  $J = (j_1, \dots, j_m) \in \Lambda_k$  and  $I = (i_1, \dots, i_n) \in \mathcal{M}_k$ , which implies that both  $\beta^{-\tau k} \prod_{s=1}^{\ell-1} \beta_{i_s}$  and  $\beta^{-\tau k} \prod_{s=1}^{\ell-1} \beta_{j_s}$  in above formula are of form  $\beta^{-p}$  with  $1 \leq p \leq \tau k - 1, \beta_1\beta^{-\tau k} \in \{1, \beta, \dots, \beta^{\tau-1}\}$ . Since  $|\beta^{-\tau k}(\hat{\delta} - \hat{\eta})| \leq 1$ , we have

$$\beta^{-\tau k}(\hat{\delta} - \hat{\eta}) \in \left\{ \sum_{i=0}^\ell e_i \beta^{-i} : e_i \in D, \ell \in \mathbb{N} \cup \{0\} \right\} \cap [-1, 1]$$

where  $D \subseteq \mathbb{Z}[\beta]$  is a finite set independent of  $k$ . Therefore,  $R_{\Gamma_{\beta,\tau}}$  is a finite set by Corollary 4.2.  $\square$

4.2. Graph-directed structure and examples

For a fixed  $t \in \Gamma_{\beta,\tau} - \Gamma_{\beta,\tau}$ , suppose that  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is of finite type. Let  $\mathcal{V} = \{\omega_1, \omega_2, \dots, \omega_m\}$  be the set of all types for the vertexes of the reduced graph where  $\omega_1 = [F_\emptyset([0, 1])]$  (recall that  $F_\emptyset$  is the identity map on  $\mathbb{R}$ ). For each pair  $1 \leq i, j \leq m$  the set  $E_{i,j}$  is defined as follows. Take a vertex of type  $\omega_i$  from the reduced graph, say  $F_I([0, 1]) \in \mathcal{C}_{\Gamma_{\beta,\tau},k}$  with  $[F_I([0, 1])] = \omega_i$ . Suppose that all offsprings of  $F_I([0, 1])$  in the reduced graph are  $F_{J_\ell}([0, 1]), \ell = 1, \dots, p$ , i.e.,  $F_{J_\ell}([0, 1]) \in \mathcal{C}_{\Gamma_{\beta,\tau},k+1}$  and  $F_I([0, 1]) \xrightarrow{J_\ell} F_{J_\ell}([0, 1])$ . Let

$$E_{i,j} = \{J_\ell : [F_{J_\ell}([0, 1])] = \omega_j, \ell = 1, \dots, p\}.$$

Obviously,  $E_{i,j}$  may be empty, but  $\bigcup_{j=1}^m E_{i,j} \neq \emptyset$  for each  $1 \leq i \leq m$  from the definition of the reduced graph. By Lemma 3.1 the set  $E_{i,j}$  is independent of the choice of  $F_I([0, 1])$ . By means of  $E_{i,j}, 1 \leq i, j \leq m$  a directed graph IFS  $\{\phi_{i,j,J} : 1 \leq i, j \leq m, E_{i,j} \neq \emptyset \text{ and } J \in E_{i,j}\}$  is defined by

$$\phi_{i,j,J}(x) = F_J(x) = \beta_j x + F_J(0) \quad \text{for } 1 \leq i, j \leq m \text{ and } J \in E_{i,j}$$

when  $E_{i,j} \neq \emptyset$ . This directed graph IFS satisfies the strong separation condition since for each  $1 \leq i \leq m$

$$[0, 1] \supset \bigcup_{1 \leq j \leq m, E_{i,j} \neq \emptyset} \bigcup_{J \in E_{i,j}} \phi_{i,j,J}([0, 1]) \quad \text{with disjoint union on the right hand,}$$

and it determines unique compact sets  $T_i, 1 \leq i \leq m$  such that (cf. [20,21])

$$T_i = \bigcup_{1 \leq j \leq m, E_{i,j} \neq \emptyset} \bigcup_{J \in E_{i,j}} \phi_{i,j,J}(T_j), \quad i = 1, 2, \dots, m. \tag{10}$$

For  $s \geq 0$  let  $A_s = (A_s(i, j))_{m \times m}$  be the incidence matrix where

$$A_s(i, j) = \begin{cases} \sum_{J \in E_{i,j}} \beta_j^s & \text{if } E_{i,j} \neq \emptyset \\ 0 & \text{if } E_{i,j} = \emptyset. \end{cases}$$

Let  $\lambda_s$  be the spectral radius of  $A_s$ . Then

$$\dim_H \bigcup_{i=1}^m T_i = \dim_B \bigcup_{i=1}^m T_i = s,$$

where  $s$  is the unique non-negative real number such that  $\lambda_s = 1$  (cf. [20,21]).

**Theorem 4.4.** Let  $t \in \Gamma_{\beta,\tau} - \Gamma_{\beta,\tau}$  be such that  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  is of finite type. Let the compact subsets  $T_i, 1 \leq i \leq m$  be defined as in (10). Let  $s \geq 0$  be such that the spectral radius of the incidence matrix  $A_s = (A_s(i, j))_{m \times m}$  equals to 1. Then

$$\dim_H(\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)) = \dim_B(\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)) = s.$$

**Proof.** From the construction of the reduced graph it follows that  $\dim_H T_1 = \dim_B T_1 = s$  since the root vertex  $[0, 1] = F_\emptyset([0, 1])$  is of type  $\omega_1$  and so  $T_1$  contains a copy of  $T_i$  for all  $2 \leq i \leq m$ . However, we have  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t) = \Pi(\Lambda) = T_1$  by (8).  $\square$

In the following, we consider some examples for which the sets  $\Gamma_{\beta,\tau} \cap (\Gamma_{\beta,\tau} + t)$  are of finite type and then compute their Hausdorff dimensions by Theorem 4.4.

**Example 4.5.** Take  $\beta = \frac{\sqrt{3}-1}{2} \in (1/3, 1/2), \tau = 1$  and  $t = \beta - \beta^3$ . Then  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is of finite type by Proposition 4.3. and so we have  $\dim_H(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)) = \dim_B(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)) = \frac{\log 2}{\log(\sqrt{3}+1)}$  by Theorem 4.4.

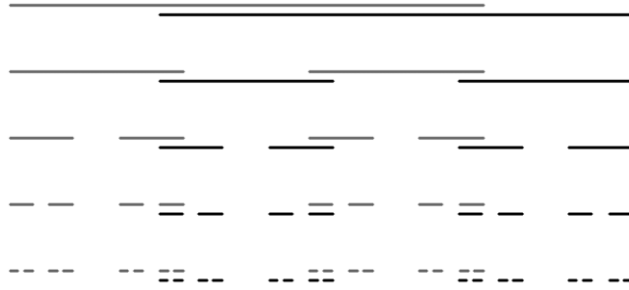
We can find out all types of  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  by means of Remark 3.4. First we have  $R_{\Gamma_{\beta,1},0}([0, 1]) = t$  which gives the type  $\omega_1 = [[0, 1]]$ . For  $k = 1$  we have (Readers can learn information from Fig. 1)

$$N_{\Gamma_{\beta,1},1}([0, \beta]) = \{[t, t + \beta]\}, \quad N_{\Gamma_{\beta,1},1}([1 - \beta, 1]) = \{[t, t + \beta], [t + 1 - \beta, t + 1]\},$$

and so

$$R_{\Gamma_{\beta,1},1}([0, \beta]) = \beta^{-1}t, \quad R_{\Gamma_{\beta,1},1}([1 - \beta, 1]) = \{\beta^{-1}(t - 1 + \beta), \beta^{-1}t\}.$$

Then one gets two new types  $\omega_2 = [[0, \beta]]$  and  $\omega_3 = [[1 - \beta, 1]]$  following  $\omega_1$ , which is represented symbolically as  $\omega_1 \rightarrow \omega_2 + \omega_3$ . For  $k = 2, [0, \beta]$  has unique offspring  $[\beta - \beta^2, \beta] \in \mathcal{C}_{\Gamma_{\beta,1},2}$  for which  $N_{\Gamma_{\beta,1},2}([\beta - \beta^2, \beta]) = \{[t, t + \beta^2]\}$



**Fig. 1.** The first five levels of component sets of  $\Gamma_{\beta,1}$  and  $\Gamma_{\beta,1} + t$  with  $\beta = \frac{\sqrt{3}-1}{2}$  and  $t = \beta - \beta^3$ . Gray and dark intervals denote the component sets of  $\Gamma_{\beta,1}$  and  $\Gamma_{\beta,1} + t$ , respectively.

and  $R_{\Gamma_{\beta,1,2}}([\beta - \beta^2, \beta]) = \{\beta^{-2}(t - \beta + \beta^2)\}$ . We denote  $\omega_4 = [[\beta - \beta^2, \beta]]$  and so  $\omega_2 \rightarrow \omega_4$ . Using the same argument, one can check that

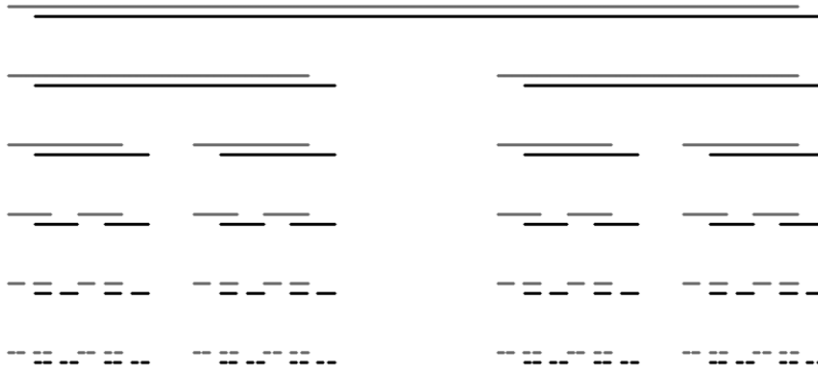
$$\omega_3 \rightarrow \omega_4 + \omega_5, \quad \omega_4 \rightarrow \omega_6, \quad \omega_5 \rightarrow \omega_6 \quad \text{and} \quad \omega_6 \rightarrow \omega_6 + \omega_6.$$

Thus these give all types and then the incidence matrix is

$$A_s = \begin{pmatrix} 0 & \beta^s & \beta^s & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^s & 0 & 0 \\ 0 & 0 & 0 & \beta^s & \beta^s & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta^s \\ 0 & 0 & 0 & 0 & 0 & \beta^s \\ 0 & 0 & 0 & 0 & 0 & 2\beta^s \end{pmatrix}.$$

The spectral radius of  $A_s$  is  $2\beta^s$  and so the desired result follows.

**Example 4.6.** Let  $\tau = 1$ . Then for any  $\beta \in (0, \frac{3-\sqrt{5}}{2})$  and any  $t \in \{(1 - \beta)\beta^\ell, (1 - \beta)(1 + \beta^\ell) : \ell \in \mathbb{N}\}$  the set  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is of finite type and is a generalized Moran set. Moreover,  $\dim_H(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)) = \dim_B(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)) = \frac{\log 2}{-\log \beta}$  by Theorem 4.4.



**Fig. 2.** The first six levels of component sets of  $\Gamma_{\beta,1}$  and  $\Gamma_{\beta,1} + t$  for  $\beta = 0.38$  and  $t = (1 - \beta)\beta^3$ . Gray and dark intervals denote the component sets of  $\Gamma_{\beta,1}$  and  $\Gamma_{\beta,1} + t$ , respectively.

It is easy to check that  $\{(1 - \beta)\beta^\ell, (1 - \beta)(1 + \beta^\ell) : \ell \in \mathbb{N}\} \subseteq (\Gamma_{\beta,1} - \Gamma_{\beta,1})$ . Let  $t = (1 - \beta)\beta^\ell$  with some  $\ell \in \mathbb{N}$ . For  $k \in \mathbb{N}$  we denote by  $1_k$  the member of  $\Sigma_2^k$  consisting of only the digit 1, i.e.,  $1_k = (1, \dots, 1) \in \Sigma_2^k$ . Then  $F_{1_k}([0, 1])$  and  $S_{1_k}([0, 1])$  are the most left  $k$ -level component sets of  $\Gamma_{\beta,1}$  and  $(\Gamma_{\beta,1} + t)$ , respectively. Note that for  $j = 0, 1, \dots, \ell - 1$

$$(1 - \beta)\beta^\ell < \beta^{\ell-j} \quad \text{and} \quad (1 - \beta)\beta^\ell + \beta^{\ell-j} < \beta^{\ell-j-1} - \beta^{\ell-j}, \tag{11}$$

where the second inequality follows from the condition  $\beta \in (0, \frac{3-\sqrt{5}}{2})$ . The first inequality in (11) shows that the left endpoint of  $S_{1_{\ell-j}}([0, 1])$  is behind the right endpoint of  $F_{1_{\ell-j}}([0, 1])$ , the second inequality in (11) shows that the right endpoint of  $S_{1_{\ell-j}}([0, 1])$  is behind the left endpoint of  $F_{1_{\ell-j-1}}([0, 1])$ . Considering into account the geometrical symmetry of  $\Gamma_{\beta,1}$  and  $\Gamma_{\beta,1} + t$ , this implies that  $\#N_{\Gamma_{\beta,1},i}(F_j([0, 1])) = 1$  for all  $J \in \Sigma_2^i$  with  $1 \leq i \leq \ell$ . Furthermore,

$F_{1_{\ell 2}}([0, 1]) = S_{1_{\ell+1}}([0, 1])$  since these two intervals have same length and common left endpoint  $t = (1 - \beta)\beta^\ell$ . Therefore,  $\#N_{\Gamma_{\beta,1,i}}(F_j([0, 1])) = 1$  are also true for  $J \in \Lambda_2^i$  with  $i \geq \ell + 1$ . So  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is a generalized Moran set by Theorem 3.5. On the other hand, from the proof of Theorem 3.5 it follows that for each  $1 \leq k \leq \ell$  all members of  $\mathcal{C}_{\Gamma_{\beta,1,k}}$  have the same type, say  $\omega_k$ , and all members of  $\bigcup_{k>\ell} \mathcal{C}_{\Gamma_{\beta,1,k}}$  share the same type, denoted by  $\omega_{\ell+1}$ . Then  $\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)$  is of finite type. By using  $\omega_1$  to denote the type of the 0-level component  $[0, 1]$  we have (see Fig. 2 for the case that  $\beta = 0.38$  and  $t = (1 - \beta)\beta^3$ )

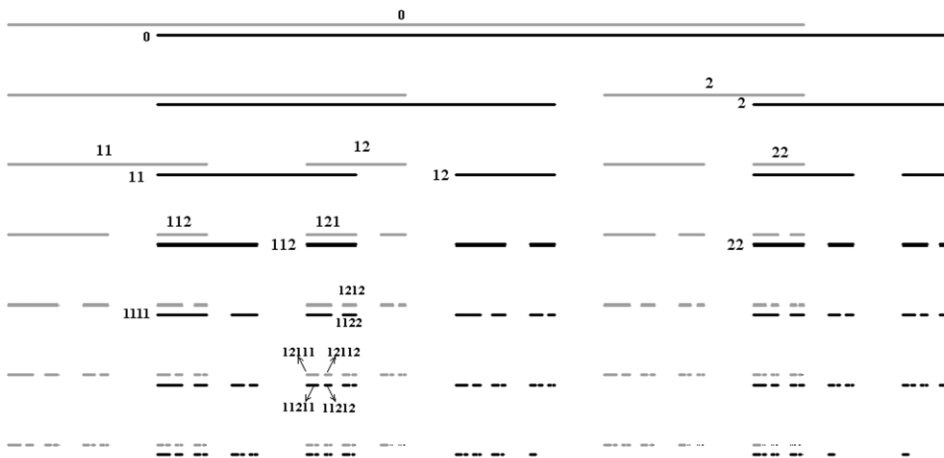
$$\omega_k \rightarrow \omega_{k+1} + \omega_{k+1}, \quad k = 1, \dots, \ell - 1; \quad \omega_{\ell+1} \rightarrow \omega_{\ell+2} \quad \text{and} \quad \omega_{\ell+2} \rightarrow \omega_{\ell+2} + \omega_{\ell+2},$$

which gives the incidence matrix

$$A_s = \begin{pmatrix} 0 & 2\beta^s & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 2\beta^s & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 2\beta^s & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 2\beta^s & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \beta^s \\ 0 & 0 & 0 & \dots & 0 & 0 & 2\beta^s \end{pmatrix}$$

and so  $\dim_H(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)) = \dim_B(\Gamma_{\beta,1} \cap (\Gamma_{\beta,1} + t)) = \frac{\log 2}{-\log \beta}$ . Finally, the case  $t = (1 - \beta)(1 + \beta^\ell)$  can be dealt with in the same way.

**Example 4.7.** Take  $\beta = \frac{1}{2}$ ,  $\tau = 2$  and  $t = \frac{3}{16}$ . Then  $\dim_H(\Gamma_{1/2,2} \cap (\Gamma_{1/2,2} + 3/16)) = \dim_B(\Gamma_{1/2,2} \cap (\Gamma_{1/2,2} + 3/16)) = \frac{\log(3+\sqrt{5})-\log 2}{\log 4}$ .



**Fig. 3.** The first six levels of component sets of  $\Gamma_{1/2,2}$  and  $\Gamma_{1/2,2} + 3/16$ . Gray and dark intervals denote the component sets of  $\Gamma_{1/2,2}$  and  $\Gamma_{1/2,2} + 3/16$ , respectively.

**Proof.** The set  $\Gamma_{1/2,2} \cap (\Gamma_{1/2,2} + 3/16) (\neq \emptyset)$  is of finite type by Proposition 4.3. We find all types of  $\Gamma_{1/2,2} \cap (\Gamma_{1/2,2} + 3/16)$  as follows. By  $\omega_1$  we denote the type of  $[0, 1] = F_\emptyset([0, 1])$ . Direct computation (also see the Fig. 3) gives that  $\Lambda_1 = \{11, 12, 2\}$  and

$$N_{\Gamma_{1/2,2,1}}(F_{11}([0, 1])) = \{S_{11}([3/16, 1 + 3/16])\}, \quad N_{\Gamma_{1/2,2,1}}(F_{12}([0, 1])) = \{S_{11}([3/16, 1 + 3/16])\}$$

and

$$N_{\Gamma_{1/2,2,1}}(F_2([0, 1])) = \{S_2([3/16, 1 + 3/16])\}.$$

Thus  $F_{11}([0, 1]) \sim F_2([0, 1])$  (simply obtained by (9)) and we denote by  $\omega_2$  the corresponding type. Let  $\omega_3$  denote the type of  $F_{12}([0, 1])$ . So  $\omega_1 \rightarrow \omega_2 + \omega_3 + \omega_2$ . Note that  $\Lambda_2 = \{112, 121, 22\}$  and

$$N_{\Gamma_{1/2,2,2}}(F_{112}([0, 1])) = \{S_{1111}([3/16, 1 + 3/16])\} \quad \text{and} \quad N_{\Gamma_{1/2,2,2}}(F_{121}([0, 1])) = \{S_{1112}([3/16, 1 + 3/16])\}.$$

Hence,  $F_{112}([0, 1]) \sim F_{121}([0, 1])$  by (9) which corresponds to a new type  $\omega_4$ . As a result,  $\omega_2 \rightarrow \omega_4$  and  $\omega_3 \rightarrow \omega_4$ . Now we only need consider the offsprings of  $F_{121}([0, 1])$  and their corresponding types.  $F_{121}([0, 1])$  has three offsprings  $F_J([0, 1]), J \in \Lambda_3 = \{12111, 12112, 1212\}$  and

$$N_{\Gamma_{1/2,2,3}}(F_{12111}([0, 1])) = \{S_{112111}([3/16, 1 + 3/16])\}, \quad N_{\Gamma_{1/2,2,3}}(F_{1212}([0, 1])) = \{S_{1122}([3/16, 1 + 3/16])\}$$

and

$$N_{\Gamma_{1/2,2,3}}(F_{12112}([0, 1])) = \{S_{11212}([3/16, 1 + 3/16])\}.$$

So we have  $F_{12111}([0, 1]) \sim F_{1212}([0, 1]) \sim F_{121}([0, 1])$  but  $F_{12112}([0, 1])$  has a new type  $\omega_5$ . So  $\omega_4 \rightarrow \omega_4 + \omega_5 + \omega_4$ . Finally, the same argument yields  $\omega_5 \rightarrow \omega_4 + \omega_5$ . Therefore, the incidence matrix

$$A_s = \begin{pmatrix} 0 & 2\left(\frac{1}{4}\right)^s & \left(\frac{1}{8}\right)^s & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1}{4}\right)^s & 0 \\ 0 & 0 & 0 & \left(\frac{1}{2}\right)^s & 0 \\ 0 & 0 & 0 & 2\left(\frac{1}{4}\right)^s & \left(\frac{1}{8}\right)^s \\ 0 & 0 & 0 & \left(\frac{1}{2}\right)^s & \left(\frac{1}{4}\right)^s \end{pmatrix}$$

and  $\dim_H(\Gamma_{1/2,2} \cap (\Gamma_{1/2,2} + 3/16)) = \dim_B(\Gamma_{1/2,2} \cap (\Gamma_{1/2,2} + 3/16)) = \frac{\log(3+\sqrt{5})-\log 2}{\log 4}$  by Theorem 4.4.  $\square$

The scheme described in this paper can be employed to deal with a more general case. For instance, one can consider the self-similar sets  $E$  generated by the IFS  $\{h_k(x) = \beta^{\tau_k}x + b_k : \beta \in (0, 1), \tau_k \in \mathbb{N}, 1 \leq k \leq \ell\}$ . When  $b_k \in r\mathbb{Z}[\beta]$  and  $t \in r\mathbb{Z}[\beta] \cap (E - E)$  with some  $r \in \mathbb{R}$ ,  $E \cap (E + t)$  present a graph-directed structure. We give an example as follows.

**Example 4.8.** Let  $\beta$  be the positive root of  $3x^3 + 2x - 1 = 0$ . Let the compact set  $E \subset \mathbb{R}$  be generated by the IFS  $\{h_1, h_2, h_3\}$  where

$$h_1(x) = \beta x, \quad h_2(x) = \beta^2 x + \frac{2}{3}(1 - \beta^2) \quad \text{and} \quad h_3(x) = \beta^3 x + 1 - \beta^3.$$

Then  $\dim_H(E \cap (E + \frac{2}{3}(1 - \beta^2))) = \dim_B(E \cap (E + \frac{2}{3}(1 - \beta^2))) = \frac{\log \gamma}{-3 \log \beta}$  where

$$\gamma = \frac{7}{3} + \frac{34}{3(199 + 3\sqrt{33})^{\frac{1}{3}}} + \frac{(199 + 3\sqrt{33})^{\frac{1}{3}}}{3} \approx 6.22226.$$

**Proof.** For simplicity, we denote  $t = \frac{2}{3}(1 - \beta^2)$ . First let  $E + t = \bigcup_{k=1}^3 S_k(E + t)$ . The  $k$ -level component sets of  $E$  and  $E + t$  are determined by

$$\mathcal{M}_k = \left\{ J = (j_1, \dots, j_m) \in \bigcup_{n=0}^{\infty} \{1, 2, 3\}^n : \beta^{3k} \beta_{j_m} < \beta_j \leq \beta^{3k} \right\},$$

where we denote  $\beta_j = \beta_{j_1} \cdots \beta_{j_m}$  for  $J = (j_1, \dots, j_m)$  and  $\beta_j := \beta^j$  for  $j \in \{1, 2, 3\}$ . For  $I \in \mathcal{M}_{k_1}$  and  $J \in \mathcal{M}_{k_2}$  we say that  $h_I([0, 1])$  and  $h_J([0, 1])$  have the same type if there exists a similarity map of form  $\xi(x) = \beta^{3(k_2 - k_1)}x + c$  with  $c \in \mathbb{R}$  such that  $\xi(h_I([0, 1])) = h_J([0, 1])$  and

$$\{u : u \in N_{E,k_2}(h_J([0, 1]))\} = \{\xi(v) : v \in N_{E,k_1}(h_I([0, 1]))\},$$

where as in (6)  $N_{E,k}(h_J([0, 1]))$  is the neighborhood of  $h_J([0, 1])$ . Also by  $\Lambda_k$  we denote the set  $\{J \in \mathcal{M}_k : N_{E,k}(h_J([0, 1])) \neq \emptyset\}$ . Now we are ready to decide all types of  $E \cap (E + t)$ . By  $\omega_1$  we denote the type of  $[0, 1] = h_{\emptyset}([0, 1])$ . Direct computation (also see the Fig. 4) yields that  $\Lambda_1 = \{21, 22, 23, 3\}$  and

$$\begin{aligned} N_{E,1}(h_{21}([0, 1])) &= \{S_{111}([t, 1 + t])\}, & N_{E,1}(h_{22}([0, 1])) &= \{S_{112}([t, 1 + t])\}, \\ N_{E,1}(h_{23}([0, 1])) &= \{S_{113}([t, 1 + t])\}, & N_{E,1}(h_3([0, 1])) &= \{S_{13}([t, 1 + t])\}. \end{aligned}$$

Let  $\omega_i, i = 2, 3, 4, 5$  denote the types of  $h_{2k}([0, 1]), k = 1, 2, 3$  and  $h_3([0, 1])$ , respectively. We obtain

$$\omega_1 \rightarrow \omega_2 + \omega_3 + \omega_4 + \omega_5.$$

Note that  $h_{2k}([0, 1]) = S_{11k}([t, 1 + t])$  for  $k = 1, 2, 3$ . Therefore any  $J = (2, k, \dots, j_m)$  satisfying  $\beta^6 \beta_{j_m} < \beta_j \leq \beta^6$  belongs to  $\Lambda_2$ . It is easy to check that

$$\{21111, 21112, 21113, 2112, 2113, 2121, 2122, 2123, 213\} \subset \Lambda_2$$

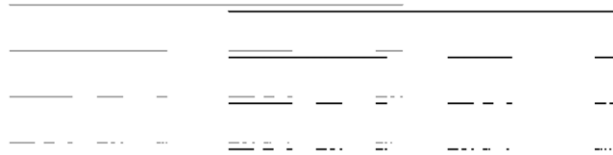


Fig. 4. The first four levels of component sets of  $E$  and  $E + t$ . Gray and dark intervals denote the component sets of  $E$  and  $E + t$ , respectively.

and then we have

$$\omega_2 \rightarrow 4\omega_2 + 3\omega_3 + 2\omega_4.$$

Similarly, we can obtain

$$\omega_3 \rightarrow 2\omega_2 + 2\omega_3 + \omega_4, \quad \omega_4 \rightarrow \omega_2 + \omega_3 + \omega_4.$$

Now we only need consider the offsprings of  $h_3([0, 1])$  and their corresponding types. Since  $\beta$  is the positive root of  $3x^3 + 2x - 1 = 0$ ,  $h_3([0, 1])$  and  $S_{13}([t, 1+t])$  have the same left endpoints. Moreover we have  $h_{31}([0, 1]) = S_{13}([t, 1+t])$ . Consequently,  $\{31111, 3112, 3113, 312, 313\} \subset \Lambda_2$  and

$$\omega_5 \rightarrow 2\omega_2 + 2\omega_3 + \omega_4.$$

Therefore, the incidence matrix is

$$A_s = \begin{pmatrix} 0 & \beta^{3s} & \beta^{4s} & \beta^{5s} & \beta^{3s} \\ 0 & 4\beta^{3s} & 3\beta^{4s} & 2\beta^{5s} & 0 \\ 0 & 2\beta^{2s} & 2\beta^{3s} & \beta^{4s} & 0 \\ 0 & \beta^{1s} & \beta^{2s} & \beta^{3s} & 0 \\ 0 & 2\beta^{3s} & 2\beta^{4s} & \beta^{5s} & 0 \end{pmatrix},$$

whose spectral radius is

$$\left( \frac{7}{3} + \frac{34}{3(199 + 3\sqrt{33})^{\frac{1}{3}}} + \frac{(199 + 3\sqrt{33})^{\frac{1}{3}}}{3} \right) \beta^{3s} =: r_s.$$

Thus  $\dim_H(E \cap (E + t)) = \dim_B(E \cap (E + t)) = s$  where  $s$  satisfies  $r_s = 1$ .  $\square$

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