

COLORFUL PATTERNS WITH DISCRETE PLANAR SYMMETRIES FROM DYNAMICAL SYSTEMS

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Abstract

Automatic generation of colored patterns with discrete planar symmetries is considered from a dynamical system's point of view. Invariant mappings with such symmetries are constructed to serve as the density functions for the generation of colorful images.

Keywords: Cyclic and Dihedral Groups; Frieze and Crystallographic Groups; Invariant Mapping.

1. INTRODUCTION

Dynamical systems are mathematical models containing the rules describing the way that some quantity undergoes a change through time and have been applied in many branches of science, engineering and mathematics. Graphical presentations

of the phase portraits are simultaneously complex and artistically appealing.^{1,2} Particularly, dynamical systems with symmetry have attracted special attention. For instance, Chossat and Golubitsky³ showed that the symmetry increasing bifurcation in the discrete dynamics of symmetric mappings

resulted from the collision of conjugate chaotic attractors. Field and Golubitsky⁴ constructed the families of functions which can be used to create aesthetic chaotic attractors with the symmetries of the cyclic, dihedral groups and some wallpaper groups. Carter *et al.*⁵ illustrated chaotic attractors with symmetries from each of the frieze and planar crystallographic groups. Jones *et al.*⁶ generated chaotic attractors with cyclic and dihedral symmetries by applying a group summation technique. Brisson *et al.*⁷ investigated the generation of colorful chaotic attractors with spherical symmetries in three-dimensional space. Dumont *et al.*⁸ established a general method to create attractors with crystallographic symmetries in R^n space. Attractors with symmetry of the n -cube⁹ and tetrahedron¹⁰ have also been studied. The key idea of the above papers for the automatic generation of symmetrical patterns was based on the construction of *equivariant* mapping $f: R^2 \rightarrow R^2$, which satisfies the following restriction^{4,5}

$$f \circ \gamma = \gamma \circ f$$

for all elements γ of a symmetry group. The scheme of coloring the chaotic attractors is based on the frequency of visit at the pixels. Here, the attractors are supposed to be SRB(Sinai-Ruelle-Bowen)-attractors, then it is possible to approximate the SRB measure by a process of iteration. Thus SRB-attractors lend themselves readily to computation of a density function that approximates the SRB measure.^{11,12}

In this paper we generate artistic patterns with discrete planar symmetry groups from dynamical systems by the construction of *invariant* mapping $F: R^2 \rightarrow R$ satisfying¹³

$$F \circ \gamma = F$$

for all elements γ of the symmetry group. And these invariant mappings are used to determine the density functions.

2. DISCRETE PLANAR SYMMETRY GROUPS

The discrete symmetry groups in the plane are well known to consist of the identity, two infinite families of rotational groups (cyclic and dihedral), seven frieze groups that include one independent translation each and 17 crystallographic groups that include two independent translations each.¹⁴

2.1. Planar Cyclic and Dihedral Groups

Cyclic and dihedral groups are subgroups of permutation groups. The cyclic group, denoted by C_n , is generated by n -fold rotations about a single point; and the dihedral group, denoted by D_n , contains those rotations and a reflection through a line passing through the point of rotation. It is well known that a mapping with C_n symmetry should be equivariant with respect to $S_n(x, y) = (\cos(\frac{2\pi}{n})x - \sin(\frac{2\pi}{n})y, \sin(\frac{2\pi}{n})x + \cos(\frac{2\pi}{n})y)$; similarly, a mapping with D_n symmetry should be equivariant with respect to $M(x, y) = (x, -y)$ as well as S_n .

The theory of which polynomial maps have cyclic and dihedral symmetries is given in Field and Golubitsky.⁴ A truncated form for these maps can be written in complex coordinates as

$$f(z) = (\lambda + \alpha z \bar{z} + \beta Re(z^n) + \omega i)z + \gamma \bar{z}^{n-1}, \quad (1)$$

where $\lambda, \alpha, \beta, \gamma, \omega$ are real numbers, and $z = x + iy$ is the complex variable corresponding to the point (x, y) in the plane. This function $f(z)$ is equivariant with respect to C_n and when $\omega = 0$ then it is equivariant with respect to D_n .

2.2. Planar Frieze and Crystallographic Groups

We consider four basic types of symmetry in the plane: translations, rotations, reflections, and glide reflections. Each frieze (or strip) group, of which there are seven, contains one direction of translation combined with zero or more of the other symmetries. The crystallographic groups are characterized by translations in two independent directions, which give rise to a lattice. In addition, the crystallographic groups may contain rotations of order two, three, four, and six. These different possible orders of rotation, together with reflections, and glide reflections, yield a total of seventeen crystallographic groups.¹⁵ A detailed discussion of the families of functions equivariant with respect to the frieze and crystallographic groups is well developed in Carter *et al.*⁵ Here, we only discuss functions related two of the planar symmetry groups.

Our first example is symmetry group $p1a1$. It contains a glide reflection in addition to one translation. To generate a chaotic attractor with $p1a1$ symmetry, a mapping should be equivariant with respect to symmetries: $\delta_0(x, y) = (x + 2\pi, y)$ and $\delta_2(x, y) = (x + \pi, -y)$. We take a mapping of the

plane $f_{\mathbf{P}}: R^2 \rightarrow R^2$ of the form

$$\begin{aligned} f_{\mathbf{P}} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ 0 \end{pmatrix} + \langle 1, \cos(x), \cos(2x), \sin(x), \sin(2x) \rangle \\ &\cdot \mathbf{P} \cdot \langle 1, y, y^2 \rangle \bmod \begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix}, \end{aligned} \quad (2)$$

where $\mathbf{P} = (p_{ijk})$ is one random $5 \times 2 \times 3$ array of real parameters. And it can be restricted to be equivariant with respect to the desired symmetries.

Crystallographic groups involve similar functions, but require equivariance with respect to two translations. For example, the group $p4g$ contains four-fold rotational symmetry in addition to two independent translations, and it also contains glide reflections. To create a chaotic attractor with $p4g$ symmetry, functions of the following form have equivariance with respect to symmetries $\delta_0(x, y), \delta_3(x, y) = (x, y + \pi), \delta_4(x, y) = (-y, x), \delta_2(x, y)$:

$$\begin{aligned} f_{\mathbf{P}} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} + \langle 1, \cos(x), \cos(2x), \sin(x), \sin(2x) \rangle \cdot \mathbf{P} \\ &\cdot \langle 1, \cos(y), \cos(2y), \sin(y), \sin(2y) \rangle \bmod \begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix}, \end{aligned} \quad (3)$$

where $\mathbf{P} = (p_{ijk})$ is one $5 \times 2 \times 5$ array of real parameters.

A detailed discussion of the images representative of chaotic attractors with each of the seven frieze and 17 crystallographic symmetries appears in Carter *et al.*⁵ In this paper, we use those functions in Carter *et al.*⁵ to create artistic images with discrete planar symmetries from dynamics.

3. CONSTRUCTION OF INVARIANT FUNCTIONS FOR CYCLIC AND DIHEDRAL SYMMETRY GROUPS

We determine an invariant function used for generating patterns with C_n (or D_n) symmetry. Note that if a function is invariant with respect to the generators of a group, then it is invariant with respect to the group itself. So, we only need to construct a function invariant with respect to S_n (or plus M).

Proposition 1. *If $F(x, y)$ is a linear combination of $g(x, y)$, where $g(x, y) = \sum_{i=0}^n h(S_n^i(x, y))$, $h(x, y): R^2 \rightarrow R$ is any real function, then $F(x, y)$ is invariant for C_n group symmetry. Furthermore, if $h(x, y)$ satisfies $h(x, y) = h(x, -y)$, then $F(x, y)$ is invariant for D_n group symmetry.*

Proof. The desired results can be easily obtained by direct calculation. \square

From Proposition 1, we take $F(x, y) = \sum_{i=0}^4 h(S_5^i(x, y))$, where $h(x, y) = \cos^2(x) \sin^3(y)$ to create the image shown in Fig. 1. And in Fig. 2, we take $F(x, y) = \sum_{i=0}^{15} h(S_{16}^i(x, y))$ where $h(x, y) = \sin^3(x) \cos^2(y)$. Remarkably, the image shown in Fig. 3, which was cut off from Fig. 2 and enlarged, appears to have fascinatingly subtle structure.

Proposition 2. *If $F(x, y) = h \circ g(x, y)$, where $g(x, y) = x^2 + y^2$, and $h(x, y): R \rightarrow R$ is any real function, then $F(x, y)$ is invariant for C_n and D_n .*

Proof. By direct calculation, $g(S_n(x, y)) = g(x, y)$ and $g(M(x, y)) = g(x, y)$, meaning $F(x, y)$ invariant for both C_n and D_n . \square

From Proposition 2, we take $F(x, y) = h(g(x, y))$, where $h(x) = 12 \cos(x) + \sin^3(x)$ to produce the image shown in Fig. 4.

For D_4 symmetry group, we have following result.

Proposition 3. *A polynomial function $F: R^2 \rightarrow R$ is invariant with respect to D_4 group if and only*



Fig. 1 Pattern with C_5 symmetry for $\lambda = -2.08, \alpha = 1.0, \beta = -0.33, \gamma = 0.007, \omega = 0.3$, and $r = 0.85$.



Fig. 2 Pattern with D_{16} symmetry for $\lambda = 2.39, \alpha = -2.5, \beta = -0.1, \gamma = 0.9, \omega = 0$, and $r = 0.5$.

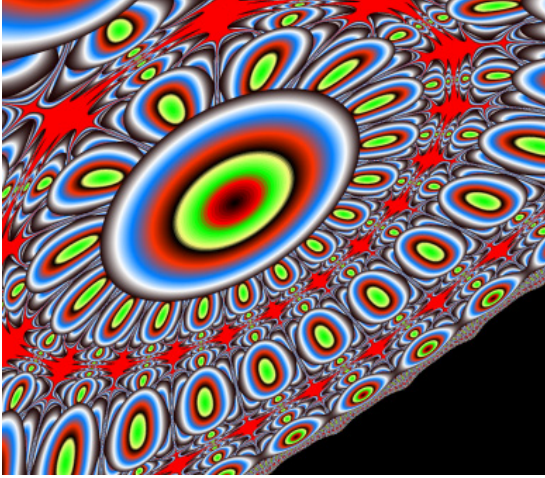


Fig. 3 A part cut off from Fig. 2 and enlarged.

if it can be written as a linear combination of the functions $h_{i,j}$, where

$$h_{i,j}(x,y) = x^i y^j + x^j y^i \quad (4)$$

and i, j are even numbers.

Proof. Note that $S_4(x,y) = \begin{pmatrix} x & -y \\ -y & x \end{pmatrix}$. Let $F(x,y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j$, then $F(x,y) = F(S_4(x,y)) = F(M(x,y))$ means that

$$\begin{aligned} \sum_{i,j \geq 0} a_{i,j} x^i y^j &= \sum_{i,j \geq 0} (-1)^j a_{i,j} x^i y^j \\ &= \sum_{i,j \geq 0} (-1)^i a_{i,j} y^i x^j. \end{aligned} \quad (5)$$

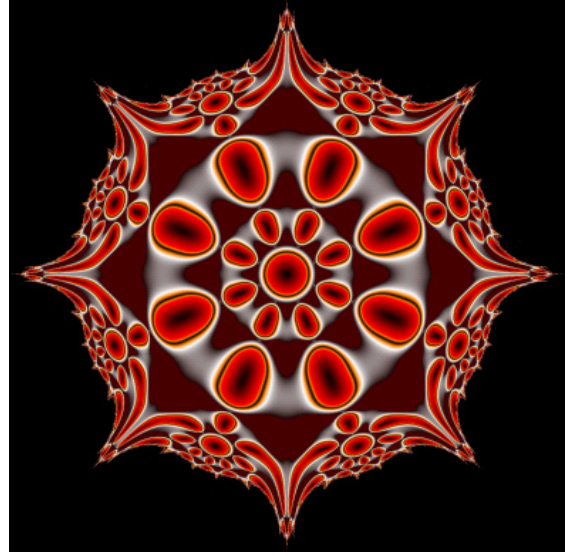


Fig. 4 Pattern with D_8 symmetry for $\lambda = -1.75, \alpha = 1.805, \beta = 0.01, \gamma = 1.0, \omega = 0$, and $r = 0.52$.

Comparing the coefficients of $x^i y^j$ and $x^j y^i$ in Eq. (5), then

$$\begin{cases} a_{i,j} = (-1)^j a_{i,j} = (-1)^j a_{j,i} \\ a_{j,i} = (-1)^i a_{j,i} = (-1)^i a_{i,j} \end{cases} \quad (6)$$

By Eq. (6), we obtain that i, j should be even numbers and $a_{i,j} = a_{j,i}$. So any polynomial function invariant with respect to D_4 group can be written as a linear combination of the function $h_{i,j}$.

Conversely, direct computation shows F is invariant with respect to D_4 . \square

From Proposition 3, we construct $F(x,y) = 15(x^4 y^6 + x^6 + y^4) + 2(x^2 + y^2) + 12(x^4 y^2 + x^2 y^4)$ to generate the image shown as in Fig. 5.

4. CONSTRUCTION OF INVARIANT FUNCTIONS FOR FRIEZE AND CRYSTALLOGRAPHIC GROUPS

Since the frieze and crystallographic patterns contain one direction of translation and two non-parallel directions of translations respectively, we only need to determine the invariant functions on a period parallelogram. Namely we only need to construct the functions invariant with respect to the rest of the generators but $\delta(x,y) = (x+T, y)$ and $\delta'(x,y) = (x, y+T')$, where T, T' are the translation periods.

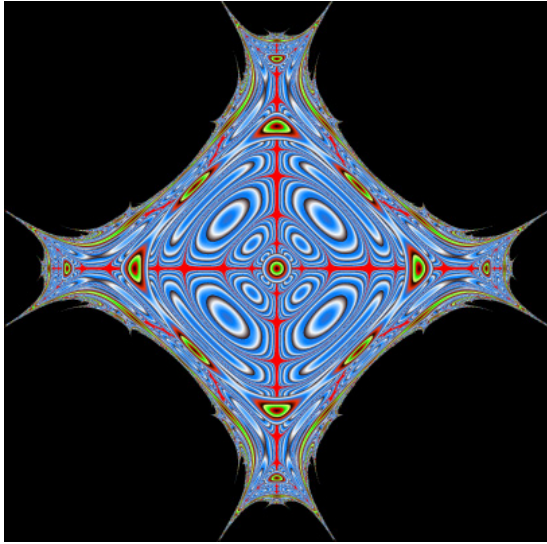


Fig. 5 Pattern with D_4 symmetry for $\lambda = -2.18, \alpha = 10, \beta = 12, \gamma = 1.0, \omega = 0$, and $r = 0.22$.

4.1. Invariant Functions for Frieze Groups

We first determine an invariant function used for creating images with $p112$ symmetry. The $p112$ symmetry group contains two-fold rotations (180°), so we require the function invariant with respect to $\delta_2(x, y) = (-x, -y)$, i.e., $F(\delta_2(x, y)) = F(-x, -y) = F(x, y)$. In Fig. 6, we take $F(x, y) = |\cos(2x) + 0.52y^2 + \sin(xy)|$.

The frieze group $pm11$ has lines of symmetry perpendicular to the axis of translation. We construct functions by requiring invariance with respect to $\delta_6(x, y) = (-x, y)$. Similar to that of $p112$, F must be an even function for x . In Fig. 7, we take $F(x, y) = x^2 \sin^2(y) + 0.3 \cos^2(xy) + |\cos(x)| \sin^2(5y)$.

The symmetry group $p1m1$ is characterized by a line of reflection parallel to the direction of translation. That is, we require invariance with respect to $\delta_7(x, y) = (x, -y)$, i.e., F is an even function for

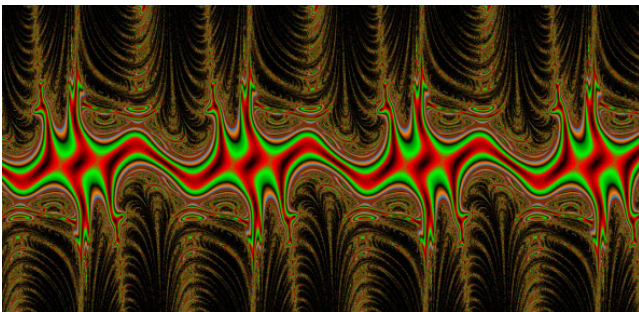


Fig. 6 Pattern with $p112$ symmetry for $r = 2.5\pi$.

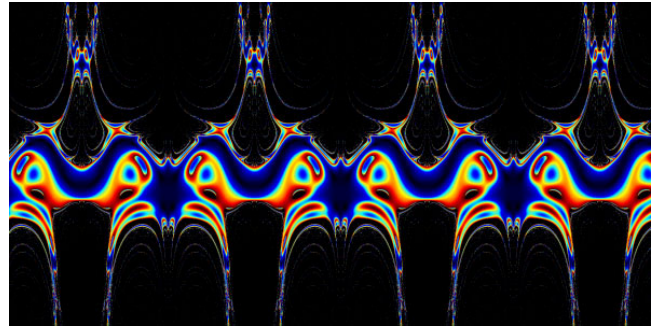


Fig. 7 Pattern with $pm11$ symmetry for $r = 2.0\pi^2 - 4$.

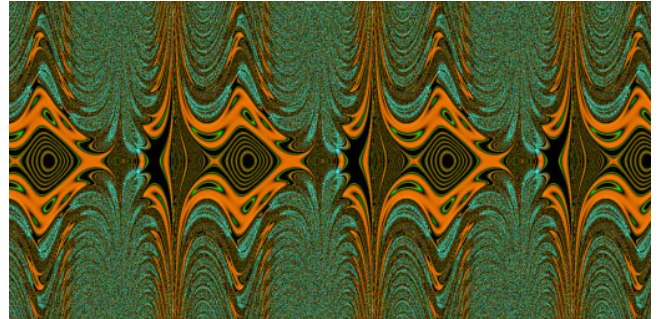


Fig. 8 Pattern with $p1m1$ symmetry for $r = 4.5\pi - 3$.

y . Figure 8 is an image representative of this group and $F(x, y) = |xy^2 + 3 \sin(x^3) + \cos(5y) + 0.2y^2 + 10 \sin^2 2x \cos^2(2y)|$.

A glide reflection is a translation accompanied by a reflection across the axis of translation, which is characterized by δ_2 . The $p1a1$ frieze pattern contains the glide reflection. We require that our function be invariant with respect to δ_2 . Figure 9 is an image representative of this group and $F(x, y) = 12|\sin^5(x) \sin(3y) + 25 \cos(2x) \cos^3(y) + 2 \cos(x) \sin^5(2y)|$.

The $pma2$ frieze pattern is characterized by lines of reflection perpendicular to the axis of translation, glide reflections, and two-fold rotations. Obviously,

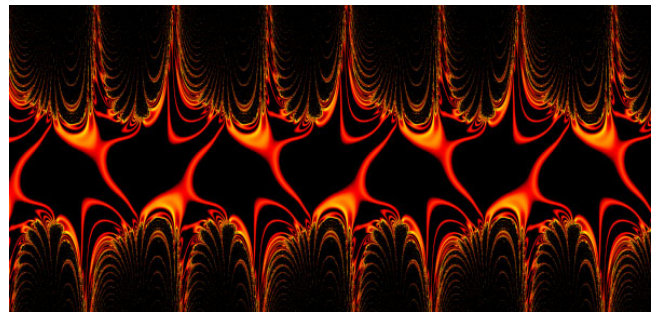


Fig. 9 Pattern with $p1a1$ symmetry for $r = 1.5\pi$.

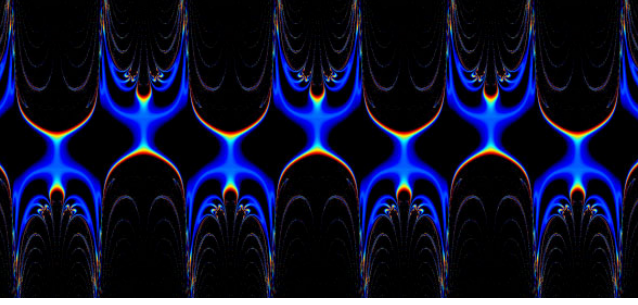


Fig. 10 Pattern with $pma2$ symmetry for $r = 5.5$.

it is a $p1a1$ frieze pattern with added half turns and mirror reflections. We determine that our function be invariant with respect to δ_2 and δ_6 . In Fig. 10, we take $F(x, y) = 22|\cos^5(2x)\cos(3y) + \cos(4x)\cos^3(y) + \cos(2x)\sin^5(2y^2)|$.

The final frieze group $pmm2$ contains reflections both parallel and perpendicular to be axis of translation, and a 180° rotation. This time the invariant function F is restricted by the combinations of the generators of $p1m1$ and $pm11$. Obviously, F is an even function for x, y . In Fig. 11, we take $F(x, y) = 10\cos^4(x)\sin^2(3y^2) + 25\sin^2(x)\cos^2(y) + 12|\cos(2x)\sin^5(2y^2) + \cos(x)|$.

Now we construct invariant functions for crystallographic groups $pgg, pg, pmg, pmm, p31m$, and $p4g$. Other invariant functions for the rest of the crystallographic groups can be found in Lu *et al.*¹³

4.2. Invariant Function for pgg, pg

The group pgg contains perpendicular glide reflections. To construct an invariant function used for creating images with pgg symmetry, we only need to determine a function invariant with respect to δ_2, δ_3 , where $\delta_3(x, y) = (-x, y + \pi)$.

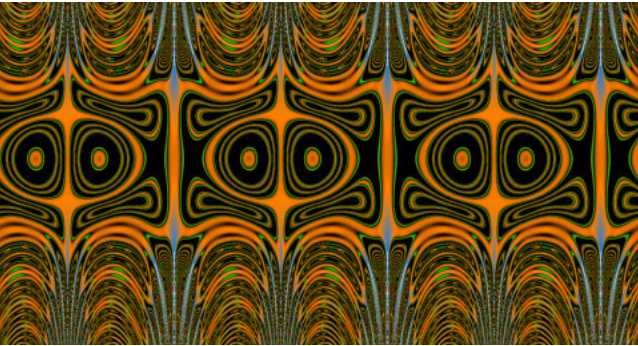


Fig. 11 Pattern with $pmm2$ symmetry for $r = 15.7$.

Proposition 4. Let $p_i(x), q_i(y), i = 1, 2, \dots$, be functions which satisfies one of following restrictions.

(I) if $p_i(x)$ and $q_i(y)$ both are odd functions, then

$$p_i(x + \pi) = -p_i(x), q_i(y + \pi) = -q_i(y);$$

(II) if $p_i(x)$ and $q_i(y)$ both are even functions, then

$$p_i(x + \pi) = p_i(x), q_i(y + \pi) = q_i(y);$$

(III) if $p_i(x)$ and $q_i(y)$ are odd and even functions, respectively, then

$$p_i(x + \pi) = p_i(x), q_i(y + \pi) = -q_i(y);$$

(IV) if $p_i(x)$ and $q_i(y)$ are even and odd functions, respectively, then

$$p_i(x + \pi) = -p_i(x), q_i(y + \pi) = q_i(y).$$

If $F(x, y)$ is a linear combination of $p_i(x)q_i(y)$, then F is invariant for pgg .

Proof. Let $F(z) = \sum_i a_i p_i(x)q_i(y)$, where $z = (x, y) \in R^2$, and $a_i \in R, i = 1, 2, \dots$, are a sequence of constants. One can readily verify that

$$p_i(x)q_i(y) = p_i(x + \pi)q_i(-y) = p_i(-x)q_i(y + \pi).$$

Then, $F(z) = F(\delta_2 z) = F(\delta_3 z)$, which shows that f is invariant for pgg . \square

In Fig. 12, we take $F(x, y) = 16|\sin^3(5x) \times \sin(3y) + 12\cos^5(x)\sin^3(2y)| + 15\sin^4(x)\cos^2(y)$.

Note that the group pg contains glide reflection in only one direction. We only need to determine a function invariant with respect to δ_2 . So, a function invariant for pgg constructed by Proposition 4 is also invariant for pg . For instance, in Fig. 13, we take $F(x, y) = 22|\sin^3(x)\sin(3y)| +$

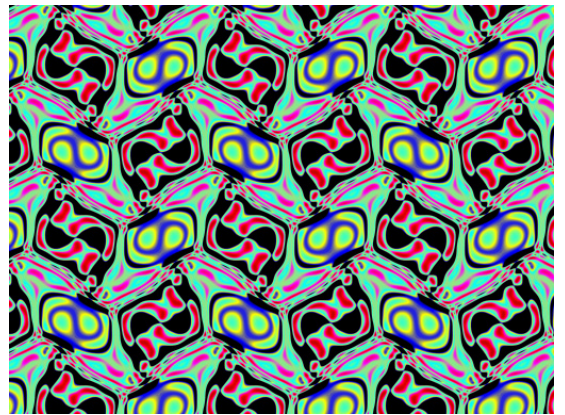


Fig. 12 Image with pgg symmetry for $r = 2.5\pi$.

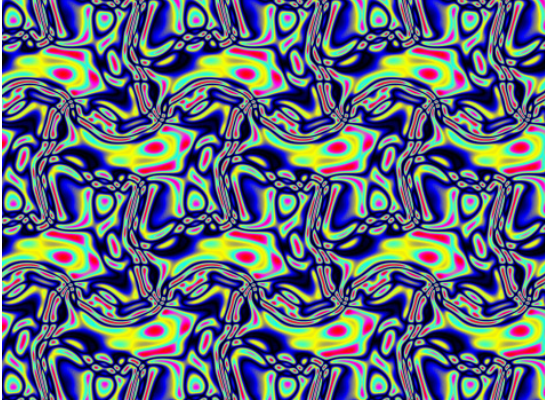


Fig. 13 Image with pg symmetry for $r = 3.5\pi^2 - 7$.

$$15 \cos^2(2x) \cos^2(y) + \sin^2(2x) \cos^4(y) + 2|\cos(x) \sin^5(2y)|.$$

4.3. Invariant Function for pmg , pmm

The symmetry group pmg contains lines of reflection perpendicular to lines of glide reflection. We only need to determine a function invariant with respect to δ_7, δ_3 .

Proposition 5. Let $p_i(x), q_i(y), i = 1, 2, \dots$, be functions satisfying following restrictions:

- (I) $q_i(y)$ is an even function; and
- (II) if $p_i(x)$ is an odd function, then $-q_i(y) = q_i(y + \pi)$; if $p_i(x)$ is an even function, then $q_i(y) = q_i(y + \pi)$.

If $F(x, y)$ is a linear combination of $p_i(x)q_i(y)$, then F is invariant for pmg .

Proof. Let $F(x, y) = \sum_i a_i p_i(x) q_i(y)$, where $a_i, i = 1, 2, \dots$ are a sequence of constants. By direct calculation,

$$p_i(x)q_i(y) = p_i(x)q_i(-y) = p_i(-x)q_i(y + \pi).$$

Then,

$$F(z) = F(\delta_3 z) = f(\delta_7 z), z = (x, y) \in R^2,$$

so F is invariant for pmg . \square

From Proposition 5, we take $F(x, y) = 16x^2|\cos(2y)| + 12x^2 \cos^2(y)$ to product the image shown in Fig. 14.

The symmetry group pmm has perpendicular reflections and two non-parallel translations. We need only to determine a function invariant with respect to δ_6, δ_7 .

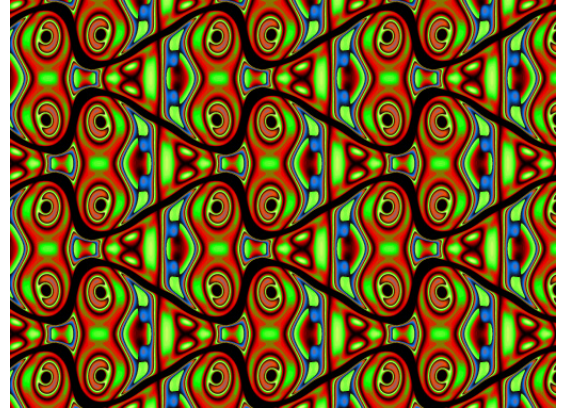


Fig. 14 Image with pmg symmetry for $r = 10.5$.

Proposition 6. The item $F(x, y)$ is a linear combination of $h_1(x)h_2(y)$ where both $h_1(x)$ and $h_2(y)$ are even functions, then F is invariant for pmm .

Proof. Direction calculation shows that F is invariant for pmm . \square

From Proposition 6, we can construct the invariant function for pmm . For example, in Fig. 15, we take $F = 16 \cos^2(2x)|\cos(y)| + 10x^2 + 12y^4 \sin^2(x)$.

4.4. Invariant Function for $p31m$, $p4g$

The group $p31m$ has third turns and mirrors in addition to translational symmetry. We only need to construct a function invariant with δ_7, δ_8 , where $\delta_8 = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y)$.

Proposition 7. If $F(x, y)$ is a linear combination of $g(x, y)$, where $g(x, y) = \sum_{i=0}^3 h(\delta_8^i(x, y))$, $h(x, y) : R^2 \rightarrow R$ is any function which satisfies $h(x, y) = h(x, -y)$, then F is invariant for $p31m$.

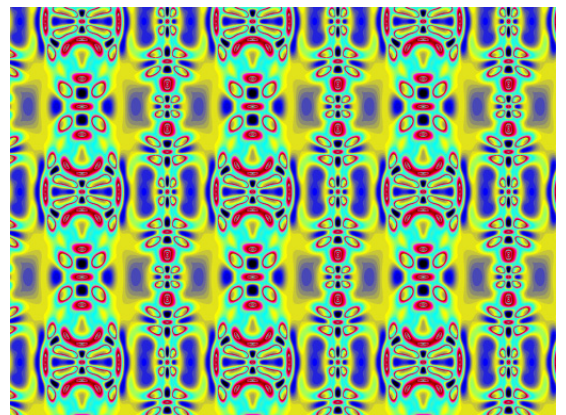


Fig. 15 Image with pmm symmetry for $r = 2.5\pi - 1$.

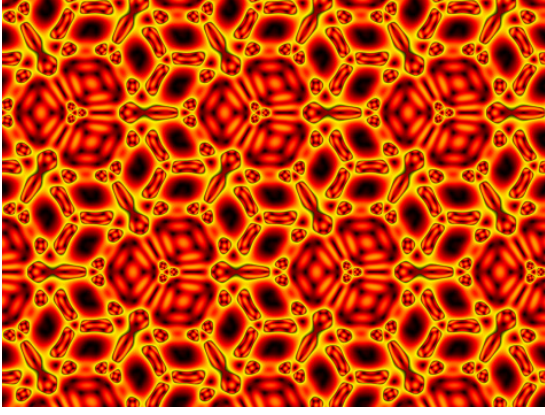


Fig. 16 Image with $p31m$ symmetry for $r = 5.5$.

Proof. Direct computation shows that F is invariant for $p31m$. \square

In Fig. 16, we take $F(x, y) = \sum_{i=0}^2 (0.0123h_1(\delta_8^i(x, y)) + h_2(\delta_8^i(x, y)))$, where $h_1(x, y) = x^2y^2$, $h_2(x, y) = |x|y^2$.

As discussed in Sec. 2.2, the symmetry group $p4g$ contains the four-fold rotational symmetry as well as glide reflections. We require the function invariant with respect to δ_4, δ_2 .

Proposition 8. Let $h(x, y): R^2 \rightarrow R$ be a linear combination of $h_1(x)h_2(y)$, where $h_1(x)$ is a periodic function with least period π , and $h_2(x)$ is an even function. If the term $F(x, y)$ is a linear combination of $g(x, y)$, where $g(x, y) = \sum_{i=0}^3 h(\delta_4^i(x, y))$, then F is invariant for $p4g$.

Proof. Direct calculation shows that F is invariant for $p4g$. \square

In Fig. 17, we take $F(x, y) = \sum_{i=0}^3 h(\delta_4^i(x, y))$ where $h(x, y) = \cos^2(2x)\cos^4(2y)$.

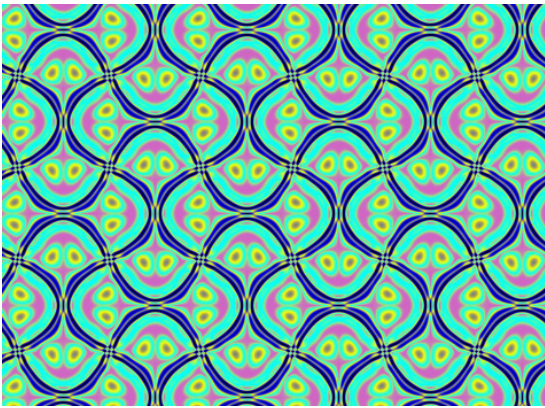


Fig. 17 Image with $p4g$ symmetry for $r = 2.5\pi^2 - 6$.

5. COLOR SCHEME

We consider the coloring of a point z_0 on the displaying scope U . Let X denote a crystallographic pattern, $\psi(X)$ be the symmetry group of X . For a given positive integer n and a positive real number r , take a mapping f equivariant with respect to $\psi(X)$ (e.g., Eqs. (1)–(3)) and determine the F values $\{F(f^k(z_0))\}_{k=1}^n$ of the orbit $\{f^k(z_0)\}_{k=1}^n$ of z_0 under f where f^k is the k th iteration of f . If $F(f^k(z_0)) < r$ for some $k \leq n$, the iteration exits and then we take $\hat{F} = \frac{F(f^k(z_0))}{r}$ as the density at z_0 , which is used to color the pixel z_0 . Otherwise, the pixel z_0 is colored by the background color. Obviously \hat{F} is a (symmetric) density function on X since $F(f^k(\delta(z_0))) = F(\delta f^k(z_0)) = F(f^k(z_0))$ for any $\delta \in \psi(X)$. Therefore, the δ symmetrically placed pixels have the same density and thus obtain the same color. Outstanding features of this color scheme are described in Lu *et al.*¹³

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