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The Hausdorff Dimension of Sets Related to the General Sierpinski Carpets

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Abstract In this paper we study a class of subsets of the general Sierpinski carpets for which the allowed two digits in the expansions occur with proportional frequency. We calculate the Hausdorff and box dimensions of these subsets and give necessary and sufficient conditions for the corresponding Hausdorff measure to be positive finite.

Keywords general Sierpinski carpets, Hausdorff dimension, Hausdorff measure

MR(2000) Subject Classification 28A80, 28A78

1 Introduction

Let T be the expanding endomorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the matrix $\operatorname{diag}(n, m)$ where $2 \le m < n$ are integers. The simplest invariant sets for T have the form

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k : \ d_k \in D \text{ for all } k \ge 1 \right\},$$

where $D \subseteq I \times J$ is a set of digits with $I = \{0, 1, \dots, n-1\}$ and $J = \{0, 1, \dots, m-1\}$. Alternatively, define a map $K_T : (I \times J)^{\mathbb{N}} \to \mathbb{T}^2$ by

$$K_T(x) = \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k x_k, \quad x = (x_k)_{k=1}^{\infty} \in (I \times J)^{\mathbb{N}}.$$

Then $K(T, D) = K_T(D^{\mathbb{N}})$. So each element of $K(T, D)$ can be represented as an expansion in base $\operatorname{diag}(n^{-1}, m^{-1})$ with digits in D. The set $K(T, D)$, called the general Sierpinski carpet, was first studied by McMullen [1] and Bedford [2], independently, to determine its Hausdorff and box-counting dimensions. From then on, some further problems related to the Sierpinski carpet $K(T, D)$ are proposed and considered by lots of authors. Peres [3–4] studied its packing

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and Hausdorff measures. Kenyon and Peres [5–6] extended the results of McMullen [1] and Bedford [2] to the compact subsets of the 2-torus corresponding to shifts of finite type or sofic shifts and to the Sierpinski sponges. King [7] determined the singularity spectrum for general Sierpinski carpets. Olsen [8] extended King's results to $\mathbb{R}^d$ by analyzing the multifractal structure of self-affine invariant measures supported by the Sierpinski sponges.

Let σ denote the projection of $\mathbb{R}^2$ onto its second coordinate. Let

$$B = \sigma(D) \text{ and } n_b = \#\{d \in D : \sigma(d) = b\} \text{ for each } b \in B,$$

where and throughout this paper we use $\#A$ to denote the cardinality of a finite set A. D is said to have *uniform horizontal fibres* if $n_b = n_b'$ for all $b, b' \in B$. Let

$$\alpha = \log_n m \text{ and } \theta = \frac{\alpha - 1}{\alpha}.$$

For any $x = (x_j)_{j=1}^{\infty} \in D^{\mathbb{N}}$ and $d \in D$, set

$$N_k(x, d) = \#\{1 \le j \le k : x_j = d\}.$$

Whenever there exists the limit

$$\zeta(x, d) = \lim_{k\to\infty} \frac{N_k(x, d)}{k}, \tag{1}$$

it is called the *frequency* of the digit d in the coding x. When we write the symbol $\zeta(x, d)$ we are already assuming the existence of the limit in (1). As we know, lots of interesting results have been established for the study of certain subsets of self-similar sets, e.g., the so-called multifractal analysis. Some detailed description on this topic and recent developments are included in [9]. Unfortunately, less analogous results have been revealed for the general self-affine sets. However, some results for a typical subset of the general Sierpinski carpet, a special class of self-affine sets, were achieved by Nielsen [10].

For a probability vector $\mathbf{p} = (p_d)_{d\in D}$ on D, i.e., $\sum_{d\in D} p_d = 1$ with each $p_d \in (0, 1)$, let

$$\Xi_{\mathbf{p}} = \{x = (x_j)_{j=1}^{\infty} \in D^{\mathbb{N}} : \zeta(x, d) = p_d,\ d \in D\}, \tag{2}$$

where $\zeta(x, d)$ is defined by (1). A probability vector $\mathbf{p} = (p_d)_{d\in D}$ is said to be *uniformly distributed* on D if $p_d = \frac{1}{\#D}$ for all $d \in D$. For any Borel subset E of $\mathbb{R}^2$, let $\dim_H E$ and $\dim_B E$, respectively, denote its Hausdorff and box dimensions, and $\mathscr{H}^\gamma(E)$ denote its γ-dimensional Hausdorff measure. For a probability vector $\mathbf{p} = (p_d)_{d\in D}$ on D we denote $q_b = \sum_{d\in D, \sigma(d)=b} p_d$ for $b \in B = \sigma(D)$. Then $(q_b)_{b\in B}$ is a probability measure on B. Nielsen in [10] obtained that for each probability vector $\mathbf{p} = (p_d)_{d\in D}$ (see [10, Theorems 1 and 3])

[R1] $\dim_H K_T(\Xi_{\mathbf{p}}) = -\alpha \sum_{d\in D} p_d \log_m p_d - (1 - \alpha) \sum_{b\in B} q_b \log_m q_b$;

[R2] $\dim_B K_T(\Xi_{\mathbf{p}}) = \dim_B K(T, D) = (1 - \alpha) \log_m \#B + \alpha \log_m \#D$;

[R3] Denote $\gamma = \dim_H K_T(\Xi_{\mathbf{p}})$.

(a) If $\mathbf{p}$ is uniformly distributed on D and if D has uniform horizontal fibers then

$$0 < \mathscr{H}^\gamma(K_T(\Xi_{\mathbf{p}})) < +\infty;$$

(b) If $\mathbf{p}$ is not uniformly distributed on D or if D does not have uniform horizontal fibers then

$$\mathscr{H}^\gamma(K_T(\Xi_{\mathbf{p}})) = +\infty.$$

In the present paper, we like to investigate another class of subsets of the general Sierpinski carpet. For any fixed two distinct digits $d_s,\ d_t \in D$ and $\beta > 0$ we consider the set

$$\Omega(d_s, d_t, \beta) = \left\{x = (x_i)_{i=1}^{\infty} \in D^{\mathbf{N}} : \zeta(x, d_s) = \beta\zeta(x, d_t) > 0\right\}. \tag{3}$$

Then $\Omega(d_s, d_t, \beta)$ is the subset of $D^{\mathbb{N}}$ such that for each $x \in \Omega(d_s, d_t, \beta)$, the frequency of d_s is β times the frequency of d_t. And so $K_T(\Omega(d_s, d_t, \beta))$ is the subset of the $K(T, D)$ whose elements have their codings with a prescribed proportional frequencies for two prescribed digits. Clearly, $K_T(\Omega(d_s, d_t, \beta))$ is T-invariant, dense in $K(T, D)$ but not compact in general. Let

$$\Sigma = \left\{\mathbf{p} = (p_d)_{d\in D} : p_d \in (0, 1),\ \sum_{d\in D} p_d = 1 \text{ and } p_{d_s} = \beta p_{d_t}\right\}. \tag{4}$$

To avoid triviality, we assume that $\#D \geq 3$ since Σ is a singleton when $\#D = 2$ (and so $\Omega(d_s, d_t, \beta) = \Xi_{\mathbf{p}}$ with $\mathbf{p} = ((1 + \beta)^{-1}, \beta(1 + \beta)^{-1})$). It is easy to see that

$$K_T(\Omega(d_s, d_t, \beta)) \supset \bigcup_{\mathbf{p}\in\Sigma} K_T(\Xi_{\mathbf{p}}). \tag{5}$$

We emphasize that the inclusion is proper since $K_T(\Omega(d_s, d_t, \beta))$ contains points for which $\zeta(x, d), d \in D \setminus \{d_s, d_t\}$ are not well-defined. Thus, it directly follows from [R2] that

$$\dim_B K_T(\Omega(d_s, d_t, \beta)) = \dim_B K(T, D) = (1 - \alpha)\log_m \#B + \alpha \log_m \#D.$$

We define a function on Σ by

$$\begin{aligned} f(\mathbf{p}) &= -\alpha \sum_{d\in D} p_d \log_m p_d - (1 - \alpha) \sum_{d\in D} p_d \log_m q_{\sigma(d)} \\ &= -\alpha \sum_{d\in D} p_d \log_m p_d - (1 - \alpha) \sum_{b\in B} q_b \log_m q_b. \end{aligned} \tag{6}$$

Thus we have

$$\dim_H K_T(\Omega(d_s, d_t, \beta)) \geq \sup_{\mathbf{p}\in\Sigma} \dim_H K_T(\Xi_{\mathbf{p}}) = \sup_{\mathbf{p}\in\Sigma} f(\mathbf{p}),$$

by (5), (R1) and (6). Note that the function $f(\mathbf{p})$ can be continuously extended to $\mathrm{cl}(\Sigma)$ (the closure of Σ) by interpreting $0 \log_m 0$ as 0. Then $f(\mathbf{p})$ can obtain its maximum $f_{\max}$ on $\mathrm{cl}(\Sigma)$. Indeed, the maximum $f_{\max}$ can not be reached on the boundary of $\mathrm{cl}(\Sigma)$, and there exists a unique point $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ such that $f(\mathbf{p}^*) = f_{\max} = \max_{\mathbf{p}\in\mathrm{cl}(\Sigma)} f(\mathbf{p}) = \max_{\mathbf{p}\in\Sigma} f(\mathbf{p})$. This fact is shown in the next section as Proposition 2.4. Throughout this paper, the notation $\mathbf{p}^* = (p_d^*)_{d\in D}$ is always assumed to be the unique maximum point of $f(\mathbf{p})$ and $q_b^* = \sum_{d\in D, \sigma(d)=b} p_d^*$ for $b \in B$ whenever they occur. More precisely, as we can see in Proposition 2.4, $\mathbf{p}^* = (p_d^*)_{d\in D}$ is determined by (9) and so

$$f(\mathbf{p}^*) = \alpha\left(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1 - (\beta + 1)p_{d_t}^*)\right).$$

Therefore, we can obtain a lower bound for the Hausdorff dimension of $K_T(\Omega(d_s, d_t, \beta))$, i.e.,

$$\dim_H K_T(\Omega(d_s, d_t, \beta))) \geq \alpha\left(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1 - (\beta + 1)p_{d_t}^*)\right).$$

However, our main result shows that the opposite inequality also holds. In this paper, we obtain the following results.

Theorem 1.1 *Let $f(\mathbf{p})$ be given by (6). Let $\mathbf{p}^* = (p_d^*)_{d\in D} \in \mathrm{cl}(\Sigma)$ be such that $f(\mathbf{p}^*) = \max_{\mathbf{p}\in\mathrm{cl}(\Sigma)} f(\mathbf{p})$. Then*

$$\dim_H K_T(\Omega(d_s,d_t,\beta)) = f(\mathbf{p}^*) = \alpha\biggl(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1-(\beta+1)p_{d_t}^*)\biggr),$$

where $q_b^* = \sum_{d\in D,\sigma(d)=b} p_d^*$ for $b\in B$.

Remark As shown in Proposition 2.4, $\mathbf{p}^* = (p_d^*)_{d\in D}$ is determined by (9). By means of the third equality in (9), an alternative expression for $\dim_H K_T(\Omega(d_s,d_t,\beta))$ is given by

$$\begin{aligned}\dim_H K_T(\Omega(d_s,d_t,\beta)) = {}& -\frac{\alpha\beta}{1+\beta}\log_m \beta - \alpha\log_m p_{d_t}^* \\ & - \frac{(1-\alpha)\beta}{1+\beta}\log_m q_{\sigma(d_s)}^* - \frac{(1-\alpha)}{1+\beta}\log_m q_{\sigma(d_t)}^*. \end{aligned} \tag{7}$$

Generally, it is difficult to give an explicit expression for the maximum point $\mathbf{p}^* = (p_d^*)_{d\in D}$. However, this is possible for some special cases.

Corollary 1.2 *Suppose $\sigma(D\setminus\{d_s,d_t\})\cap\sigma(\{d_s,d_t\}) = \emptyset$.*

(I) *If $\sigma(d_s)=\sigma(d_t)$, then*

$$\dim_H K_T(\Omega(d_s,d_t,\beta)) = -\frac{\alpha\beta}{1+\beta}\log_m\beta + \log_m\Biggl(\beta^{\frac{\alpha\beta}{1+\beta}} \sum_{d\in D\setminus\{d_s,d_t\}} n_{\sigma(d)}^{\alpha-1} + (1+\beta)^\alpha\Biggr);$$

(II) *If $\sigma(d_s)\neq\sigma(d_t)$, then*

$$\dim_H K_T(\Omega(d_s,d_t,\beta)) = -\frac{\beta}{1+\beta}\log_m\beta + \log_m\Biggl(\beta^{\frac{\beta}{1+\beta}} \sum_{d\in D\setminus\{d_s,d_t\}} n_{\sigma(d)}^{\alpha-1} + \beta + 1\Biggr).$$

Proof The first equality in (9) gives that

$$\frac{p_d^*}{p_{d'}^*} = \left(\frac{q_{\sigma(d)}^*}{q_{\sigma(d')}^*}\right)^\theta \quad \text{for } d,d'\in D\setminus\{d_s,d_t\},$$

implying that $p_d^* = p_{d'}^*$ whenever $d,d'\in D\setminus\{d_s,d_t\}$ lie on the same horizontal fibre, i.e., $\sigma(d)=\sigma(d')$. Under the condition that $\sigma(D\setminus\{d_s,d_t\})\cap\sigma(\{d_s,d_t\})=\emptyset$, we have $q_{\sigma(d)}^* = n_{\sigma(d)}p_d^*$ for any $d\in D\setminus\{d_s,d_t\}$. Thus,

$$\frac{p_d^*}{p_{d'}^*} = \left(\frac{n_{\sigma(d)}}{n_{\sigma(d')}}\right)^{\alpha-1} \quad \text{for } d,d'\in D\setminus\{d_s,d_t\}.$$

Therefore, for any $d\in D\setminus\{d_s,d_t\}$, we have

$$p_d^* = \frac{n_{\sigma(d)}^{\alpha-1}}{\sum_{d\in D\setminus\{d_s,d_t\}} n_{\sigma(d)}^{\alpha-1}}(1-(\beta+1)p_{d_t}^*),$$

and

$$q_{\sigma(d)}^* = \frac{n_{\sigma(d)}^{\alpha}}{\sum_{d\in D\setminus\{d_s,d_t\}} n_{\sigma(d)}^{\alpha-1}}(1-(\beta+1)p_{d_t}^*).$$

By the third equality of (9), we obtain

$$p^*_{d_t} = \frac{(1+\beta)^{\alpha-1}}{\beta^{\frac{\alpha\beta}{1+\beta}} \sum_{d\in D\setminus\{d_s,d_t\}} n_{\sigma(d)}^{\alpha-1} + (1+\beta)^{\alpha}} \quad \text{when} \quad \sigma(d_s) = \sigma(d_t),$$

and

$$p^*_{d_t} = \frac{1}{\beta^{\frac{\beta}{1+\beta}} \sum_{d\in D\setminus\{d_s,d_t\}} n_{\sigma(d)}^{\alpha-1} + 1 + \beta} \quad \text{when} \quad \sigma(d_s) \neq \sigma(d_t).$$

Hence, (I) and (II) are then established by (7). □

Remark Under the condition of Corollary 1.2, we have that if, furthermore, all $n_{\sigma(d)}, d \in D \setminus \{d_s, d_t\}$ are equal, then

$$\dim_H K_T(\Omega(d_s, d_t, \beta)) = \dim_P K_T(\Omega(d_s, d_t, \beta)).$$

This can be established since one can get that $\lim_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) = -\gamma$ in the proof of Theorem 1.1 where we only obtain that $\limsup_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \geq -\gamma$ for the general case.

As to the corresponding Hausdorff measure, we have the following theorem:

Theorem 1.3 *Denote* $\gamma = \dim_H K_T(\Omega(d_s, d_t, \beta))$. *We have*

(I) *If* $\beta \neq 1$, *then* $\mathscr{H}^\gamma(K_T(\Omega(d_s, d_t, \beta))) = +\infty$;

(II) *If* $\beta = 1$ *and* D *has uniform horizontal fibres then* $0 < \mathscr{H}^\gamma(K_T(\Omega(d_s, d_t, \beta))) < +\infty$;

(III) *If* $\beta = 1$ *and* D *does not have uniform horizontal fibres then* $\mathscr{H}^\gamma(K_T(\Omega(d_s, d_t, \beta))) = +\infty$.

The rest of this paper is organized as follows. In Section 2, some basic facts and known results needed in the proof of our theorems are described. Proofs of Theorems 1.1 and 1.3 are arranged in Section 3.

2 Preliminaries

As in [1, 3–4, 10], a class of approximate squares are used to calculate the various dimensions of the general Sierpinski carpets and its subsets. For each $x = (x_j)_{j=1}^\infty \in (I \times J)^{\mathbb{N}}$ and each positive integer k, let

$$\begin{aligned} Q_k(x) = \big\{K_T(y) : y = (y_j)_{j=1}^\infty \in (I\times J)^{\mathbb{N}},\ y_j = x_j \ \text{ for } 1 \leq j \leq [\alpha k] \\ \text{and } \sigma(y_j) = \sigma(x_j) \text{ for } [\alpha k] + 1 \leq j \leq k\big\}, \end{aligned}$$

where, as usual, $[a]$ with $a \in \mathbb{R}$ denotes the greatest integer function. The sets $Q_k(x)$ are approximate squares in $[0,1]^2$, whose sides have length $n^{-[\alpha k]}$ and m^{-k}. Note that the ratio of the sides of $Q_k(x)$ is at most n, and their diameters $\mathrm{diam} Q_k(x)$ satisfy

$$\sqrt{2} m^{-k} \leq \mathrm{diam} Q_k(x) \leq \sqrt{2} n m^{-k}.$$

So in the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares $Q_k(x)$. The following lemma appears in [10] in which the approximate square $Q_k(x)$ behaves as an analogue as the ball does in the classical density theorems. It is just a reformulation of the Rogers–Taylor density theorem as stated by Peres in Section 2 of [4].

Lemma 2.1 [10, Lemma 4] *Suppose that δ is a positive number, that μ is a finite Borel measure in $[0,1]^2$, and that E is a subset of $(I\times J)^{\mathbb{N}}$ such that $K_T(E)$ is a Borel subset of $[0,1]^2$, and $\mu(K_T(E))>0$. Put*

$$A(x)=\limsup_{k\to\infty}(k\delta+\log_m\mu(Q_k(x)))$$

for each point $x\in E$.

1) *If $A(x)=-\infty$ for all $x\in E$, then $\mathscr{H}^{\delta}(K_T(E))=+\infty$;*

2) *If $A(x)=+\infty$ for all $x\in E$, then $\mathscr{H}^{\delta}(K_T(E))=0$;*

3) *If there are numbers a and b such that $a\le A(x)\le b$ for all $x\in E$, then $0<\mathscr{H}^{\delta}(K_T(E))<+\infty$.*

The Borel measures on $[0,1]^2$ to which the above lemmas will be applied are constructed as follows. Let $\mathbf{p}=(p_d)_{d\in D}$ be a probability vector on D, i.e., $\sum_{d\in D}p_d=1$ with each $p_d\in(0,1)$. Then $\mathbf{p}$ determines a unique infinite product Borel probability measure, denoted by $\mu_{\mathbf{p}}$, on $D^{\mathbb{N}}$. For any finite sequence $(x_\ell)_{\ell=1}^k\in D^k$,

$$\mu_{\mathbf{p}}([(x_\ell)_{\ell=1}^k])=\prod_{j=1}^k p_{x_j},$$

where $[(x_\ell)_{\ell=1}^k]:=\{y=(y_j)_{j=1}^\infty\in D^{\mathbb{N}}:y_j=x_j \text{ for } 1\le j\le k\}$ is a cylinder set of $D^{\mathbb{N}}$ with base $(x_\ell)_{\ell=1}^k$. Let $\widetilde{\mu}_{\mathbf{p}}$ be the Borel probability measure on $K_T(D^{\mathbb{N}})$ which is the image measure of $\mu_{\mathbf{p}}$ under K_T, i.e., $\widetilde{\mu}_{\mathbf{p}}(A)=\mu_{\mathbf{p}}(K_T^{-1}A)$ for Borel set $A\subseteq\mathbb{R}^2$. From the definition of $Q_k(x)$ it follows that for any $x=(x_j)_{j=1}^\infty\in D^{\mathbb{N}}$ (cf. formula (4) in [10], also formula (4.4) in [11]),

$$\widetilde{\mu}_{\mathbf{p}}(Q_k(x))=\prod_{j=1}^{[\alpha k]}p_{x_j}\cdot\prod_{j=[\alpha k]+1}^{k}q_{\sigma(x_j)}. \tag{8}$$

Then the Kolmogrov Strong Law of Large Numbers shows that $\widetilde{\mu}_{\mathbf{p}}(K_T(\Xi_{\mathbf{p}}))=1$. We give the proof for completeness.

Lemma 2.2 *Let $\mathbf{p}=(p_d)_{d\in D}$ be a probability vector. Let $\widetilde{\mu}_{\mathbf{p}}$ and $\Xi_{\mathbf{p}}$ be given by* (8) *and* (2). *Then $\widetilde{\mu}_{\mathbf{p}}(K_T(\Xi_{\mathbf{p}}))=1$.*

Proof For $d\in D$, let

$$\Gamma_d=\{x=(x_j)_{j=1}^\infty\in D^{\mathbb{N}}:\zeta(x,d)=p_d\}.$$

Then $\Xi_{\mathbf{p}}=\bigcap_{d\in D}\Gamma_d$. So it suffices to show that $\mu_{\mathbf{p}}(\Gamma_d)=1$. Consider a sequence of random variables $\{X_j\}_{j=1}^\infty$ on the probability space $(D^{\mathbb{N}},\mathscr{F},\mu_{\mathbf{p}})$ ($\mathscr{F}$ is the Borel σ-algebra) by letting

$$X_j(x)=\begin{cases}1 & x_j=d,\\ 0 & x_j\ne d,\end{cases}$$

for each $x=(x_j)_{j=1}^\infty\in D^{\mathbb{N}}$. Then $X_1,X_2,\dots$ are independent and identically distributed random variables with $\mu_{\mathbf{p}}(X_1=1)=p_d$ and $\mu_{\mathbf{p}}(X_1=0)=1-p_d$. By Kolmogrov Strong Law of Large Numbers, we have that for $\mu_{\mathbf{p}}$-a.e. $x=(x_j)_{j=1}^\infty\in D^{\mathbb{N}}$,

$$\zeta(x,d)=\lim_{k\to\infty}\frac{\#\{1\le j\le k:x_j=d\}}{k}=\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^k X_j(x)=E(X_1)=p_d,$$

implying $\mu_{\mathbf{p}}(\Gamma_d) = 1$. □

Note that $\Omega(d_s, d_t, \beta) \supset \Xi_{\mathbf{p}}$ for each $\mathbf{p} = (p_d)_{d\in D} \in \Sigma$. The following corollary follows immediately from Lemma 2.2.

Corollary 2.3 *Let* $\Omega(d_s, d_t, \beta)$ *and* Σ *be defined as* (3) *and* (4). *Then* $\widetilde{\mu}_{\mathbf{p}}(K_T(\Omega(d_s, d_t, \beta))) = 1$ *for any* $\mathbf{p} = (p_d)_{d\in D} \in \Sigma$.

Our next target is to maximize the expression (6) under the constraint $\mathbf{p} \in \Sigma$. We use the notation log to denote the natural logarithm.

Proposition 2.4 *Let* $f(\mathbf{p})$ *be defined by* (6). *There exists a unique probability vector* $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ *such that*

$$f(\mathbf{p}^*) = f_{\max} = \max_{\mathbf{p}\in \mathrm{cl}(\Sigma)} f(\mathbf{p}).$$

More precisely, $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ *is uniquely determined by*

$$\begin{cases} p_d^* = \dfrac{q_{\sigma(d)}^{*\theta}}{\sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta}}(1-(\beta+1)p_{d_t}^*), \quad d \in D\setminus\{d_s, d_t\}, \\ p_{d_s}^* = \beta p_{d_t}^*, \\ \alpha(\beta+1)\log(1-(\beta+1)p_{d_t}^*) - \alpha(\beta+1)\log \displaystyle\sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \alpha\beta\log\beta \\ \quad -\alpha(\beta+1)\log p_{d_t}^* - (1-\alpha)\beta\log q_{\sigma(d_s)}^* - (1-\alpha)\log q_{\sigma(d_t)}^* = 0, \end{cases} \tag{9}$$

where, as before, $q_b^* = \sum_{d\in D,\sigma(d)=b} p_d^*$ *for* $b \in B$.

Proof Clearly, $f(\mathbf{p})$ can obtain its maximum on $\mathrm{cl}(\Sigma)$ since $f(\mathbf{p})$ is continuous and $\mathrm{cl}(\Sigma)$ is compact. We first show that the maximum point is unique. Note that $f(\mathbf{p})$ is a strictly concave function in $\mathbf{p}$. In fact, the first summand of $f(\mathbf{p})$ is strictly concave and the second is concave. On the other hand, $\mathrm{cl}(\Sigma)$ is convex, the constraint inequalities are both convex and concave and its constraint equalities are all linear. By a well-known property of convex programming , there exists a unique $\mathbf{p}^* \in \mathrm{cl}(\Sigma)$ such that $f(\mathbf{p})$ attains its maximum at the point $\mathbf{p}^*$.

We then show that the maximum of $f(\mathbf{p})$ is obtained in Σ, equivalently, that $\mathbf{p}^* \in \Sigma$. Let

$$Z_1(\mathbf{p}) = -\alpha\sum_{d\in D} p_d \log_m p_d \quad \text{and} \quad Z_2(\mathbf{p}) = (\alpha-1)\sum_{b\in B} q_b \log_m q_b.$$

Then $f(\mathbf{p}) = Z_1(\mathbf{p}) + Z_2(\mathbf{p})$. Suppose $\mathbf{p}^* = (p_d^*)_{d\in D} \in \mathrm{cl}(\Sigma)\setminus\Sigma$. Let $D_1 = \{d \in D : p_d^* = 0\}$ and $D_2 = D \setminus D_1$. Then both D_1 and D_2 are nonempty. Take $\widetilde{\mathbf{p}} = (\widetilde{p}_d)_{d\in D} \in \Sigma$. Let $\mathbf{p}_t = t\widetilde{\mathbf{p}} + (1-t)\mathbf{p}^* = (t\widetilde{p}_d + (1-t)p_d^*)_{d\in D}$, $t \in [0, 1]$. Then $\mathbf{p}_t \in \Sigma$ for $t \in (0, 1]$ and $\mathbf{p}_0 = \mathbf{p}^*$. Note that

$$\begin{aligned} Z_1'(\mathbf{p}_t) = \frac{d}{dt}Z_1(\mathbf{p}_t) &= -\alpha\frac{d}{dt}\Big(\sum_{d\in D}(t\widetilde{p}_d + (1-t)p_d^*)\log_m(t\widetilde{p}_d + (1-t)p_d^*)\Big) \\ &= -\alpha\sum_{d\in D}(\widetilde{p}_d - p_d^*)\log_m(t\widetilde{p}_d + (1-t)p_d^*) \\ &= -\alpha\Big(\sum_{d\in D_1}\widetilde{p}_d\log_m(t\widetilde{p}_d) + \sum_{d\in D_2}(\widetilde{p}_d - p_d^*)\log_m(t\widetilde{p}_d + (1-t)p_d^*)\Big). \end{aligned}$$

Thus we have $\lim_{t\to 0+} Z_1'(\mathbf{p}_t) = +\infty$. The same argument shows that $\lim_{t\to 0+} Z_2'(\mathbf{p}_t) = +\infty$ if $q_b^* = 0$ for some $b \in B$, or is equal to a finite real number. Therefore, $\lim_{t\to 0+} f'(\mathbf{p}_t) = +\infty$.

Note that $\lim_{t\to 0+} f(\mathbf{p}_t) = f(\mathbf{p}^*)$. Thus, $f(\mathbf{p}_t) > f(\mathbf{p}^*) = f_{\max}$ when t is small enough, leading to a contradiction. Now let

$$\begin{aligned}L(\mathbf{p}, \lambda_1, \lambda_2) = &- \alpha \sum_{d\in D} p_d \log_m p_d - (1-\alpha) \sum_{b\in B} q_b \log_m q_b \\ &+ \frac{\lambda_1}{\log m}(p_{d_s} - \beta p_{d_t}) + \frac{\lambda_2}{\log m}\Big(\sum_{d\in D} p_d - 1\Big).\end{aligned}$$

Since $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ is the unique point such that $f(\mathbf{p}^*) = \max_{\mathbf{p}\in\Sigma} f(\mathbf{p})$ and $f(\mathbf{p})$ is a strictly concave function in $\mathbf{p}$, $\mathbf{p}^*$ is uniquely solved by (method of Lagrange multipliers)

$$\begin{cases} \dfrac{\partial L}{\partial p_d} = 0, \ \ d \in D, \\ \dfrac{\partial L}{\partial \lambda_i} = 0, \ \ i = 1, 2, \end{cases}$$

i.e.,

$$\begin{cases} -\alpha(\log p_d + 1) - (1-\alpha)(\log q_{\sigma(d)} + 1) + \lambda_2 = 0, \ \ d \in D \setminus \{d_s, d_t\}, \\ -\alpha(\log p_{d_s} + 1) - (1-\alpha)(\log q_{\sigma(d_s)} + 1) + \lambda_1 + \lambda_2 = 0, \\ -\alpha(\log p_{d_t} + 1) - (1-\alpha)(\log q_{\sigma(d_t)} + 1) - \beta\lambda_1 + \lambda_2 = 0, \\ p_{d_s} - \beta p_{d_t} = 0, \\ \displaystyle\sum_{d\in D} p_d - 1 = 0. \end{cases}$$

So $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ satisfies

$$\begin{cases} p_d^* = \dfrac{q_{\sigma(d)}^{*\theta}}{\sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta}}(1 - (\beta+1)p_{d_t}^*), \ \ d \in D \setminus \{d_s, d_t\}, \\ p_{d_s}^* = \beta p_{d_t}^*, \\ \alpha(\beta+1)\log(1 - (\beta+1)p_{d_t}^*) - \alpha(\beta+1)\log \displaystyle\sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \alpha\beta\log\beta \\ \quad - \alpha(\beta+1)\log p_{d_t}^* - (1-\alpha)\beta\log q_{\sigma(d_s)}^* - (1-\alpha)\log q_{\sigma(d_t)}^* = 0. \end{cases}$$ □

By means of (9) we can rewrite $f(\mathbf{p}^*)$ as

$$f(\mathbf{p}^*) = \alpha\Big(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1 - (\beta+1)p_{d_t}^*)\Big).$$

In fact, from the first equality in (9) it follows that

$$\begin{aligned}\sum_{d\in D} p_d^* \log_m p_d^* &= \sum_{d\in D\setminus\{d_s,d_t\}} p_d^* \log_m p_d^* + \sum_{d\in\{d_s,d_t\}} p_d^* \log_m p_d^* \\ &= (1 - (1+\beta)p_{d_t}^*)\Big(\log_m(1 - (1+\beta)p_{d_t}^*) - \log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta}\Big) \\ &\quad + \sum_{b\in B} q_b^* \log_m q_b^{*\theta} - \sum_{d\in\{d_s,d_t\}} p_d^* \log_m q_{\sigma(d)}^{*\theta} + \sum_{d\in\{d_s,d_t\}} p_d^* \log_m p_d^*.\end{aligned}$$

Thus

$$\begin{aligned}
f(\mathbf{p}^*) &= -\alpha \sum_{d\in D} p_d^* \log_m p_d^* - (1-\alpha) \sum_{b\in B} q_b^* \log_m q_b^* \\
&= \alpha\bigg(\sum_{b\in B} q_b^* \log_m q_b^{*\theta} - \sum_{d\in D} p_d^* \log_m p_d^* \bigg) \\
&= \alpha\bigg\{ (1-(1+\beta)p_{d_t}^*) \bigg(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1-(1+\beta)p_{d_t}^*) \bigg) \\
&\quad + \sum_{d\in\{d_s,d_t\}} p_d^* \log_m q_{\sigma(d)}^{*\theta} - \sum_{d\in\{d_s,d_t\}} p_d^* \log_m p_d^* \bigg\} \\
&= \alpha\bigg(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1-(\beta+1)p_{d_t}^*) \bigg) \\
&\quad + \alpha\bigg\{ -(1+\beta)p_{d_t}^* \bigg(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1-(1+\beta)p_{d_t}^*) \bigg) \\
&\quad + \sum_{d\in\{d_s,d_t\}} p_d^* \log_m q_{\sigma(d)}^{*\theta} - \sum_{d\in\{d_s,d_t\}} p_d^* \log_m p_d^* \bigg\} \\
&= \alpha\bigg(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1-(\beta+1)p_{d_t}^*) \bigg).
\end{aligned}$$

The last equality above is obtained by the second and third equalities in (9). Its verification is left for readers.

3 Proofs

In this section, we give the proofs of Theorems 1.1 and 1.3. It will be done based on Lemma 2.1 and [R3].

Proof of Theorem 1.1 As discussed in Section 1, we only need to show $\dim_H K_T(\Omega(d_s,d_t,\beta)) \le f(\mathbf{p}^*)$.

For any $x=(x_j)_{j=1}^\infty \in \Omega(d_s,d_t,\beta)$, any $k\in\mathbb{N}$ and any $d\in D$, set

$$S_k(x) = \sum_{d\in D\setminus\{d_s,d_t\}} N_k(x,d) \log_m q_{\sigma(d)}^*. \tag{10}$$

For simplicity, denote

$$\gamma = f(\mathbf{p}^*) = \alpha\bigg(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*\theta} - \log_m(1-(\beta+1)p_{d_t}^*) \bigg).$$

As shown in (9), we have that for any $d\in D\setminus\{d_s,d_t\}$,

$$\log_m p_d^* = \theta \log_m q_{\sigma(d)}^* + \log_m(1-(\beta+1)p_{d_t}) - \log_m \sum_{d\in D\setminus\{d_s,d_t\}} q_{\sigma(d)}^{*^\theta}.$$

Thus we can rewrite (8) as

$$\begin{aligned}
&\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \\
&\quad = \sum_{j=1}^{[\alpha k]} \log_m p_{x_j}^* + \sum_{j=[\alpha k]+1}^{k} \log_m q_{\sigma(x_j)}^*
\end{aligned}$$

$$
\begin{aligned}
&= N_{[\alpha k]}(x,d_s)\log_m p^*_{d_s} + N_{[\alpha k]}(x,d_t)\log_m p^*_{d_t} + \sum_{d\in D\setminus\{d_s,d_t\}} N_{[\alpha k]}(x,d)\log_m p^*_d \\
&\quad + (N_k(x,d_s) - N_{[\alpha k]}(x,d_s))\log_m q^*_{\sigma(d_s)} + (N_k(x,d_t) - N_{[\alpha k]}(x,d_t))\log_m q^*_{\sigma(d_t)} \\
&\quad + \sum_{d\in D\setminus\{d_s,d_t\}} (N_k(x,d) - N_{[\alpha k]}(x,d))\log_m q^*_{\sigma(d)} \\
&= N_{[\alpha k]}(x,d_s)\log_m \beta p^*_{d_t} + N_{[\alpha k]}(x,d_t)\log_m p^*_{d_t} \\
&\quad + \sum_{d\in D\setminus\{d_s,d_t\}} N_{[\alpha k]}(x,d)\bigg(\theta\log_m q^*_{\sigma(d)} + \log_m(1-(\beta+1)p^*_{d_t}) - \log_m \sum_{d\in D\setminus\{d_s,d_t\}} {q^*_{\sigma(d)}}^{\theta}\bigg) \\
&\quad + (N_k(x,d_s) - N_{[\alpha k]}(x,d_s))\log_m q^*_{\sigma(d_s)} + (N_k(x,d_t) - N_{[\alpha k]}(x,d_t))\log_m q^*_{\sigma(d_t)} \\
&\quad + \sum_{d\in D\setminus\{d_s,d_t\}} (N_k(x,d) - N_{[\alpha k]}(x,d))\log_m q^*_{\sigma(d)} \\
&= N_{[\alpha k]}(x,d_s)\log_m \beta + (N_{[\alpha k]}(x,d_s) + N_{[\alpha k]}(x,d_t))\log_m p^*_{d_t} \\
&\quad + ([\alpha k] - N_{[\alpha k]}(x,d_t) - N_{[\alpha k]}(x,d_s))\log_m(1-(\beta+1)p^*_{d_t}) \\
&\quad - ([\alpha k] - N_{[\alpha k]}(x,d_t) - N_{[\alpha k]}(x,d_s))\log_m \sum_{d\in D\setminus\{d_s,d_t\}} {q^*_{\sigma(d)}}^{\theta} \\
&\quad + (N_k(x,d_s) - N_{[\alpha k]}(x,d_s))\log_m q^*_{\sigma(d_s)} + (N_k(x,d_t) - N_{[\alpha k]}(x,d_t))\log_m q^*_{\sigma(d_t)} \\
&\quad + S_k(x) - \frac{1}{\alpha}S_{[\alpha k]}(x).
\end{aligned}
$$

Therefore, for each $x = (x_i)_{i=1}^{\infty} \in \Omega(d_s,d_t,\beta)$ we have

$$
\begin{aligned}
&\limsup_{k\to\infty}\frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \\
&\quad = \alpha\zeta(x,d_s)\log_m\beta + (\alpha\zeta(x,d_s) + \alpha\zeta(x,d_t))\log_m p^*_{d_t} \\
&\qquad + \alpha(1-\zeta(x,d_t)-\zeta(x,d_s))\log_m(1-(\beta+1)p^*_{d_t}) \\
&\qquad - \alpha(1-\zeta(x,d_t)-\zeta(x,d_s))\log_m \sum_{d\in D\setminus\{d_s,d_t\}} {q^*_{\sigma(d)}}^{\theta} \\
&\qquad + (\zeta(x,d_s) - \alpha\zeta(x,d_s))\log_m q^*_{\sigma(d_s)} + (\zeta(x,d_t) - \alpha\zeta(x,d_t))\log_m q^*_{\sigma(d_t)} \\
&\qquad + \limsup_{k\to\infty}\left(\frac{S_k(x)}{k} - \frac{S_{[\alpha k]}(x)}{\alpha k}\right) \\
&\quad = -\alpha\bigg(\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q^{*\theta}_{\sigma(d)} - \log_m(1-(\beta+1)p^*_{d_t})\bigg) \\
&\qquad - \zeta(x,d_t)\bigg\{\alpha(\beta+1)\log_m(1-(\beta+1)p^*_{d_t}) \\
&\qquad - \alpha(\beta+1)\log_m \sum_{d\in D\setminus\{d_s,d_t\}} q^{*\theta}_{\sigma(d)} - \alpha\beta\log_m\beta - \alpha(\beta+1)\log_m p^*_{d_t} \\
&\qquad - (1-\alpha)\beta\log_m q^*_{\sigma(d_s)} - (1-\alpha)\log_m q^*_{\sigma(d_t)}\bigg\} \\
&\qquad + \limsup_{k\to\infty}\left(\frac{S_k(x)}{k} - \frac{S_{[\alpha k]}(x)}{\alpha k}\right) \\
&\quad = -\gamma + \limsup_{k\to\infty}\left(\frac{S_k(x)}{k} - \frac{S_{[\alpha k]}(x)}{\alpha k}\right),
\end{aligned}
$$

where the last equality is obtained by the third equality in (9). In the following, we show that for each point $x = (x_j)_{j=1}^{\infty} \in \Omega(d_s, d_t, \beta)$,

$$\limsup_{k\to\infty} \left(\frac{S_k(x)}{k} - \frac{S_{[\alpha k]}(x)}{\alpha k} \right) \geq 0. \tag{11}$$

Obviously, for every point $x = (x_j)_{j=1}^{\infty} \in \Omega(d_s, d_t, \beta)$ and any $k \in \mathbb{N}$, from (10) it follows that

$$\sup_k |S_{k+1}(x) - S_k(x)| \leq \max_{b\in B} |\log_m q_b^*| \,. \tag{12}$$

For a fixed $x = (x_j)_{j=1}^{\infty} \in \Omega(d_s, d_t, \beta)$, let $T(k) = S_k(x)$. We extend T to $[1, +\infty)$ by piecewise linear interpolation. Then T is a Lipschitz function by (12). Now define $g : [0, \infty) \to \mathbb{R}$ by

$$g(z) = \mathrm{e}^{-z} T(\mathrm{e}^z).$$

We claim that $g(z)$ is bounded and uniformly continuous on $[0, \infty)$. Indeed,

$$|g(z)| \leq |g(0)|\mathrm{e}^{-z} + |g(z) - g(0)\mathrm{e}^{-z}| \leq |T(1)| + \mathrm{e}^{-z}|T(\mathrm{e}^z) - T(1)| \leq |T(1)| + \mathrm{Lip}T,$$

and for any $\delta > 0$,

$$\begin{aligned} |g(z+\delta) - g(z)| &= |\mathrm{e}^{-(z+\delta)} T(\mathrm{e}^{z+\delta}) - \mathrm{e}^{-z} T(\mathrm{e}^z)| \\ &\leq \mathrm{e}^{-(z+\delta)} |T(\mathrm{e}^{z+\delta}) - T(\mathrm{e}^z)| + |g(z)|(1 - \mathrm{e}^{-\delta}) \\ &\leq (1 - \mathrm{e}^{-\delta})\mathrm{Lip}T + (1 - \mathrm{e}^{-\delta})(|T(1)| + \mathrm{Lip}T). \end{aligned}$$

Now for any $v > -\log\alpha$,

$$\begin{aligned} \left| \int_{-\log\alpha}^{v} (g(z) - g(z + \log\alpha)) dz \right| &= \left| \int_{-\log\alpha}^{v} g(z)dz - \int_{-\log\alpha}^{v} g(z + \log\alpha)dz \right| \\ &= \left| \int_{-\log\alpha}^{v} g(z)dz - \int_{0}^{v+\log\alpha} g(z)dz \right| \\ &= \left| \int_{0}^{-\log\alpha} g(z)dz + \int_{v}^{v+\log\alpha} g(z)dz \right| \\ &\leq \left| \int_{0}^{-\log\alpha} g(z)dz \right| + \left| \int_{v}^{v+\log\alpha} g(z)dz \right| < +\infty, \end{aligned}$$

since g is bounded on $[0, +\infty)$. Therefore,

$$\limsup_{z\to+\infty} (g(z) - g(z + \log\alpha)) \geq 0.$$

By letting $z = \log t$, this gives

$$\limsup_{t\to+\infty} \left(\frac{T(t)}{t} - \frac{T(\alpha t)}{\alpha t} \right) \geq 0.$$

Note that

$$\begin{aligned} \frac{T(t)}{t} - \frac{T(\alpha t)}{\alpha t} =& \left(\frac{T(t) - T([t])}{t} - \frac{T(\alpha t) - T([\alpha t])}{\alpha t} \right) + \frac{T([t])}{[t]} \left(\frac{[t]}{t} - 1 \right) \\ &+ \left(\frac{S_{[\alpha[t]]}(x)}{\alpha[t]} - \frac{S_{[\alpha t]}(x)}{\alpha t} \right) + \left(\frac{S_{[t]}(x)}{[t]} - \frac{S_{[\alpha[t]]}(x)}{\alpha[t]} \right), \end{aligned} \tag{13}$$

where, as before, $[t]$ with $t \in \mathbb{R}$ denotes the greatest integer function. However, the first three terms on the right side of (13) tend to zero as $t \to +\infty$ by the facts that both functions

$|T(t) - T([t])|$ and $g(z)$ are bounded, and $g(z)$ is uniformly continuous. Hence (11) holds. Therefore, for every $x = (x_j)_{j=1}^{\infty} \in \Omega(d_s, d_t, \beta)$ we have

$$\limsup_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \geq -\gamma,$$

which leads to

$$\limsup_{k\to\infty}(k\delta + \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x))) = \limsup_{k\to\infty} k\left(\delta + \frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x))\right) = +\infty,$$

for any $\delta > \gamma$. Now Lemma 2.1 2) and Corollary 2.3 imply that $\dim_H K_T(\Omega(d_s, d_t, \beta)) \leq \gamma$. □

Proof of Theorem 1.3 As shown in Theorem 1.1, we have

$$\dim_H K_T(\Omega(d_s, d_t, \beta)) = \dim_H K_T(\Xi_{\mathbf{p}^*}) = f(\mathbf{p}^*).$$

Note that the probability vector $\mathbf{p}^* = (p_d^*)_{d\in D}$ is not uniformly distributed on D if $\beta \neq 1$. So both (I) and (III) can be deduced directly from [R3] since $K_T(\Omega(d_s, d_t, \beta)) \supset K_T(\Xi_{\mathbf{p}^*})$.

To prove (II), we first claim that $\mathbf{p}^* = (p_d^*)_{d\in D}$ is uniformly distributed on D, i.e., $\mathbf{p}^* = \left(\frac{1}{\#D}, \frac{1}{\#D}, \dots, \frac{1}{\#D}\right)$, when $\beta = 1$ and D has uniform horizontal fibres. This is done by simply checking that the probability vector $\left(\frac{1}{\#D}, \frac{1}{\#D}, \dots, \frac{1}{\#D}\right) \in \Sigma$ satisfies (9). At this moment, we have

$$\gamma = f(\mathbf{p}^*) = (1-\alpha)\log_m \#B + \alpha \log_m \#D.$$

Therefore,

$$\begin{aligned} k\gamma + \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) &= k((1-\alpha)\log_m \#B + \alpha\log_m \#D) \\ &\quad + [\alpha k]\log_m \frac{1}{\#D} + (k - [\alpha k])\log_m \frac{1}{\#B} \\ &= (\alpha k - [\alpha k])\log_m \frac{\#D}{\#B} \end{aligned}$$

for all $x \in \Omega(d_s, d_t, \beta)$ and all $k \in \mathbb{N}$. Then (II) is justified by Lemma 2.1 3) and Corollary 2.3. □

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