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Multiscale self-affine Sierpinski carpets

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Abstract
The well-known self-affine Sierpinski carpets, first studied by McMullen and Bedford independently, are constructed geometrically by repeating a single action according to a given pattern. In this paper, we extend them by randomly choosing a pattern from a set of patterns with different scales in each step of their construction process. The Hausdorff and box dimensions of the resulting limit sets are determined explicitly and the sufficient conditions for the corresponding Hausdorff measures to be positive finite are also obtained.

Mathematics Subject Classification: 28A80, 28A78

1. Introduction

Let $2 \leqslant m \leqslant n$ and let $I = \{0, 1, \ldots, n-1\}$ and $J = \{0, 1, \ldots, m-1\}$. For $d \in I \times J$ let

$$f_d(x) = \text{diag}\,(n^{-1}, m^{-1})(x + d), \qquad x \in \mathbb{R}^2.$$

It is well known that for each nonempty subset $D \subseteq I \times J$ the family $\{f_d : d \in D\}$ of contractive functions determines a unique nonempty compact set K_D, called a self-affine set if $m < n$, such that

$$K_D = \bigcup_{d \in D} f_d(K_D). \tag{1}$$

The self-affine set K_D ($2 \leqslant m < n$) were first studied by McMullen [16] and Bedford [2] independently (there K_D was called the general Sierpinski carpet since K_D is the famous Sierpinski carpet when $n = m$). In the past two decades, some further problems related to the general Sierpinski carpet K_D and its various variations have been proposed and considered by

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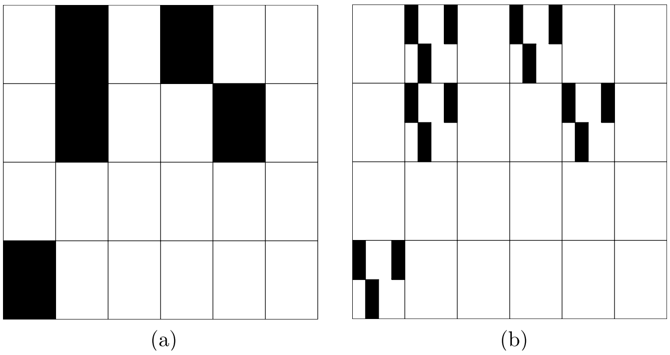

Figure 1. (*a*) The chosen rectangles in the first step according to $D = \{(0,0),(1,2),(1,3),(3,3),(4,2)\}$ with $n = 6$ and $m = 4$. (*b*) The chosen rectangles after the second step where $\Gamma_{i_1} = \{(0,0),(1,2),(1,3),(3,3),(4,2)\}$ with $n_{i_1} = 6$ and $m_{i_1} = 4$ and $\Gamma_{i_2} = \{(0,1),(1,0),(3,1)\}$ with $n_{i_2} = 4$ and $m_{i_2} = 2$.

many authors. For instance, readers can refer to Peres [19, 20], Kenyon and Peres [13, 14], King [15], Nielsen [17], Olsen [18], Barański [1], Gatzouras and Lalley [8], Gui and Li [10], Falconer [7], He *et al* [12], etc, just to list a few.

The general Sierpinski carpet K_D can be constructed geometrically as follows. We divide the unit square $[0,1]^2$ into $n \times m$ congruent (closed) rectangles by drawing n vertical strips of equal width and m horizontal strips of equal height. Then we choose those rectangles according to the pattern D. For each chosen rectangle, we execute the same action as above, i.e. divide each chosen rectangle into $n \times m$ congruent ones, choose the rectangles according to D. Then the nonempty compact K_D is just the limit set by repeating the above procedure. Figure 1(*a*) shows a pattern D and the chosen rectangles at the first step.

In this paper, we extend the above defined self-affine Sierpinski carpets K_D by randomly choosing a pattern from a set of patterns with different scales in each step of their construction process. Fix a $k \in \mathbb{N}$ and take a set $\{(n_i, m_i) : 1 \leqslant i \leqslant k\}$ of pairs of positive integers with $m_i, n_i \geqslant 2$, $1 \leqslant i \leqslant k$. Let

$$\Gamma_i \subseteq \{0, 1, \ldots, n_i - 1\} \times \{0, 1, \ldots, m_i - 1\} \triangleq I_i \times J_i \text{ for } i = 1, 2, \ldots, k.$$

We first take a Γ_{i_1} from $\{\Gamma_1, \ldots, \Gamma_k\}$ randomly and divide the unit square $[0,1]^2$ into $n_{i_1} \times m_{i_1}$ congruent rectangles by drawing n_{i_1} vertical strips of equal width and m_{i_1} horizontal strips of equal height. Then we choose those rectangles according to the pattern Γ_{i_1}. Taking a Γ_{i_2} from $\{\Gamma_1, \ldots, \Gamma_k\}$ randomly, we divide each chosen rectangle into $n_{i_2} \times m_{i_2}$ congruent ones and choose those rectangles according to the pattern Γ_{i_2}. Then we repeat the above procedure by dividing each rectangle chosen by the last step and keeping those rectangles according to the randomly selected pattern from $\{\Gamma_1, \ldots, \Gamma_k\}$. A nonempty compact set is then obtained which we call a *multiscale self-affine Sierpinski carpet*. It can be viewed as a randomized version of the self-affine Sierpinski carpet with respect to $\{\Gamma_1, \ldots, \Gamma_k\}$. Figure 1(*b*) shows the patterns $\Gamma_{i_1}, \Gamma_{i_2}$ and the chosen rectangles after the second steps.

For each $1 \leqslant i \leqslant k$ let

$$f_{i,d}(x) = \operatorname{diag}\,(n_i^{-1}, m_i^{-1})(x + d), \qquad d \in I_i \times J_i \text{ and } x \in \mathbb{R}^2.$$

For an $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$ let $\Pi : \prod_{i=1}^{\infty} \Gamma_{\omega(i)} \longrightarrow [0,1]^2$ be defined by

$$\Pi(\boldsymbol{d}) = \sum_{i=1}^{\infty} \operatorname{diag}\left(\prod_{k=1}^{i} n_{\omega(k)}^{-1}, \prod_{k=1}^{i} m_{\omega(k)}^{-1}\right) d_i \qquad \text{for } \boldsymbol{d} = (d_i)_{i \geqslant 1} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}.$$

So each multiscale self-affine Sierpinski carpet described above can be represented as $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ with a sequence $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1,\ldots,\Gamma_k\}^{\mathbb{N}}$. In particular, when all $\omega(i)$ are identical, $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ reduces to the self-affine Sierpinski carpet defined as in (1).

By $|A|$ we denote the cardinality of a finite set A. Let proj_y (proj_x) denote the projection of $\mathbb{R}^2$ onto its second (first) coordinate. Put

$$h_{i,b} = |\Gamma_i \cap (I_i \times \{b\})| \qquad \text{for } 1 \leqslant i \leqslant k \quad \text{and} \quad b \in \mathrm{proj}_y\Gamma_i. \tag{2}$$

A set Γ_i is said to have *uniformly horizontal fibres* if $h_{i,b}$ = constant for all $b \in \mathrm{proj}_y\Gamma_i$. Two sets Γ_i and Γ_j are said to have *uniformly horizontal fibres of same type* if $|\mathrm{proj}_y\Gamma_i| = |\mathrm{proj}_y\Gamma_j|$ and $h_{i,b} = h_{j,b'}$ for all $b \in \mathrm{proj}_y\Gamma_i$, $b' \in \mathrm{proj}_y\Gamma_j$. For an $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1,\ldots,\Gamma_k\}^{\mathbb{N}}$ and $\ell \in \mathbb{N}$ let

$$N_\ell(\Gamma_j,\omega) = N_\ell(\Gamma_j) := |\{1 \leqslant i \leqslant \ell : \Gamma_{\omega(i)} = \Gamma_j\}|, \qquad 1 \leqslant j \leqslant k.$$

In this paper we fix a probability vector $(p_i)_{i=1}^k$ (i.e. $p_i \in [0,1]$ and $\sum_{i=1}^k p_i = 1$) and assume

$$\zeta := \frac{\sum_{i=1}^k p_i \log m_i}{\sum_{i=1}^k p_i \log n_i} \leqslant 1, \tag{3}$$

where and throughout this paper log denotes the natural logarithm. The requirement $\zeta \leqslant 1$ is not crucial. In fact, one can rewrite all of the results below for the case $\zeta > 1$ by exchanging the roles of x and y axes, e.g. projecting $\mathbb{R}^2$ onto its first coordinate (for the quantity $h_{i,\mathrm{proj}_x d}$), considering vertical fibres, etc.

For the Hausdorff, packing, and box dimensions of $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ we obtain the following results.

Theorem 1.1. *Let ζ be defined as in (3). Let $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1,\ldots,\Gamma_k\}^{\mathbb{N}}$. If*

$$\lim_{\ell\to\infty}\frac{N_\ell(\Gamma_j)}{\ell} = p_j, \qquad 1 \leqslant j \leqslant k, \tag{4}$$

then

$$\dim_H \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \frac{\sum_{i=1}^k p_i \log \sum_{d\in\Gamma_i} h_{i,\,\mathrm{proj}_y d}^{\zeta-1}}{\sum_{i=1}^k p_i \log m_i}$$

and

$$\dim_B \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \dim_P \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \frac{\sum_{i=1}^k p_i(\zeta \log|\Gamma_i| + (1-\zeta)\log|\mathrm{proj}_y\Gamma_i|)}{\sum_{i=1}^k p_i \log m_i}.$$

From above it follows that $\dim_H \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}) = \dim_B \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ if $\zeta = 1$. When $\zeta \neq 1$ we have the following corollary.

Corollary 1.2. *Let $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1,\ldots,\Gamma_k\}^{\mathbb{N}}$ and (4) hold. Let $0 < \zeta < 1$. Then $\dim_H \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}) = \dim_B \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ if and only if each Γ_i, $1 \leqslant i \leqslant k$ with $p_i \neq 0$ has uniformly horizontal fibres.*

Proof. Note that $x \to x^\zeta$ is a strictly concave function because of $0 < \zeta < 1$. Thus, for each $1 \leqslant i \leqslant k$ with $p_i \neq 0$

$$|\mathrm{proj}_y\Gamma_i|^{1-\zeta}|\Gamma_i|^\zeta = |\mathrm{proj}_y\Gamma_i|\left(\sum_{b\in\mathrm{proj}_y\Gamma_i}\frac{h_{i,b}}{|\mathrm{proj}_y\Gamma_i|}\right)^\zeta \geqslant |\mathrm{proj}_y\Gamma_i|\sum_{b\in\mathrm{proj}_y\Gamma_i}\frac{h_{i,b}^\zeta}{|\mathrm{proj}_y\Gamma_i|}$$
$$= \sum_{b\in\mathrm{proj}_y\Gamma_i} h_{i,b}^\zeta = \sum_{d\in\Gamma_i} h_{i,\,\mathrm{proj}_y d}^{\zeta-1},$$

where the equality holds if and only if all $h_{i,b}$, $b \in \mathrm{proj}_y\Gamma_i$ are identical. □

When all $\omega(i)$ are identical, say all $\Gamma_{\omega(i)} = D$, this gives the Hausdorff, packing, and box dimensions of the self-affine Sierpinski carpet K_D defined by (1)(cf [2, 16, 19])

$$\dim_H K_D = \log_m \sum_{b\in \mathrm{proj}_y D} h_b^{\log_n m} = \log_m \sum_{d\in D} h_{\mathrm{proj}_y d}^{\log_n m-1}$$

and

$$\dim_B K_D = \dim_P K_D = \log_m(|\mathrm{proj}_y D|^{1-\log_n m}||D|^{\log_n m}).$$

Thus, the following corollary is direct ([11, theorem 1.1 and proposition 3.1]) and so the results in this paper greatly extend those in [11] where $(n_i, m_i) = (n, m)$ for all $1 \leqslant i \leqslant k$.

Corollary 1.3. *Let* $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$. *If* (4) *holds and* $(n_i, m_i) = (n, m)$ *(in this case* $\zeta = \log_n m \leqslant 1$*) for all* $1 \leqslant i \leqslant k$ *with* $p_i \neq 0$*, then*

$$\dim_H \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \sum_{i=1}^{k} p_i \log_m \sum_{d\in\Gamma_i} h_{i,\mathrm{proj}_y d}^{\log_n m-1} = \sum_{i=1}^{k} p_i \dim_H K_{\Gamma_i}$$

and

$$\dim_B \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \dim_P \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \sum_{i=1}^{k} \log_m(|\mathrm{proj}_y \Gamma_i|^{1-\log_n m}||\Gamma_i|^{\log_n m})$$

$$= \sum_{i=1}^{k} p_i \dim_B K_{\Gamma_i} = \sum_{i=1}^{k} p_i \dim_P K_{\Gamma_i}.$$

Corollary 1.3 shows that when all patterns Γ_i with $p_i \neq 0$ are of same scale, i.e. all (n_i, m_i) with $p_i \neq 0$ are identical, the Hausdorff (box, packing) dimension of $\prod_{i=1}^{\infty} \Gamma_{\omega(i)}$ is just a weighted average of the Hausdorff (box, packing) dimensions of K_{Γ_i} according to the weights $(p_i)_{i=1}^{k}$ of the frequencies of Γ_is occurring in $\prod_{i=1}^{\infty} \Gamma_{\omega(i)}$. However, it is not true for the general multiscale case shown in theorem 1.1. This is a bit surprising and quite different from the self-similar case.

In 1994, Gatzouras and Lalley [9] studied the randomization of the general Sierpinski carpet by means of branching processes. They gave exact expressions for the Hausdorff and box dimensions of the random general Sierpinski carpet. The multiscale self-affine Sierpinski carpets discussed in this paper can be also considered as a random version of the general Sierpinski carpets if in each step one independently takes the pattern Γ_i $(1 \leqslant i \leqslant k)$ with probability p_i. More precisely, this can be described as follows.

We endow $\{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$ with a probability measure $\mathbb{P}$. For the probability vector $(p_i)_{i=1}^{k}$ (cf (3)), let

$$\mathbb{P} := \prod_{\mathbb{N}} \left(\sum_{i=1}^{k} p_i \delta_i\right), \tag{5}$$

where δ_i denotes the Dirac measure concentrated at Γ_i. From the ergodic theorem (or the law of large number) it follows that (4) holds for $\mathbb{P}$-a.e $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$. Thus theorem 1.1 can be formulated in a random way.

Corollary 1.4. *Let* ζ *be defined as in* (3)*. Then for* $\mathbb{P}$*-a.e* $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$

$$\dim_H \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \frac{\sum_{i=1}^{k} p_i \log \sum_{d\in\Gamma_i} h_{i,\ \mathrm{proj}_y d}^{\zeta-1}}{\sum_{i=1}^{k} p_i \log m_i}$$

and

$$\dim_B \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \dim_P \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \frac{\sum_{i=1}^{k} p_i(\zeta\log|\Gamma_i| + (1-\zeta)\log|\mathrm{proj}_y\Gamma_i|)}{\sum_{i=1}^{k} p_i\log m_i}.$$

As to the Hausdorff measure of $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ in its dimension we have the following theorem.

Theorem 1.5. *Let* $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$ *and* (4) *hold. Let* $\gamma = \dim_H \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$.

(I) *Suppose* $\zeta < 1$ *(see* (3) *for its definition). If* $0 < \mathcal{H}^{\gamma}(\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})) < \infty$*, then each* Γ_i *with* $p_i \neq 0$ *has uniformly horizontal fibres;*

(II) *If all patterns* Γ_i *with* $p_i \neq 0$ *have uniformly horizontal fibres of same type and* $(n_i, m_i) = (n, m)$ *and the patterns* Γ_i *with* $p_i = 0$ *only occur finitely many times in* $(\Gamma_{\omega(i)})_{i=1}^{\infty}$*, then* $0 < \mathcal{H}^{\gamma}(\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})) < \infty$.

The above (II) is almost theorem 4.1 (II) in [11] where all Γ_i are required to have uniformly horizontal fibres of same type.

By theorem 1.5 and corollary 1.2 we have that in the case of the general Sierpinski carpet (i.e. the set K_D defined as in (1) with $2 \leqslant m < n$) $\dim_H K_D = \dim_B K_D$ if and only if K_D has positive Hausdorff measure in its dimension.

The rest of this paper is organized as follows. Section 2 is mainly devoted to establishing a reformation of Rogers–Taylor density theorem. The proofs of theorems 1.1 and 1.5 are arranged in section 3. Some examples are given to show that conditions in (I) and (II) of theorem 1.5 are not necessary.

2. Preliminaries

Fix an $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$. For $\ell \in \mathbb{N}$ let

$$\ell^* = \max\left\{v \in \mathbb{N} : \prod_{i=1}^{\ell} m_{\omega(i)} \geqslant \prod_{i=1}^{v} n_{\omega(i)}\right\} \tag{6}$$

with the convention $\max\emptyset = 0$. Clearly, ℓ^* is uniquely determined by ℓ and $\ell^* \in \mathbb{N}$ if ℓ is big enough. In the following we always assume $\ell^* \in \mathbb{N}$. Thus

$$1 \leqslant \frac{\prod_{i=1}^{\ell^*} n_{\omega(i)}^{-1}}{\prod_{i=1}^{\ell} m_{\omega(i)}^{-1}} < \max_{1\leqslant i\leqslant k} n_i. \tag{7}$$

For $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty}\Gamma_{\omega(i)}$ let

$$\begin{aligned} Q_\ell(x) = \Bigg\{\Pi(z) : z = (z_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty}(I_{\omega(i)} \times J_{\omega(i)}),\quad z_i = x_i \text{ for } 1 \leqslant i \leqslant \ell^*;\\ \mathrm{proj}_y z_i = \mathrm{proj}_y x_i \text{ for } \ell^*+1 \leqslant i \leqslant \ell\Bigg\},\qquad \text{if } \ell^* \leqslant \ell \end{aligned} \tag{8}$$

or

$$\begin{aligned} Q_\ell(x) = \Bigg\{\Pi(z) : z = (z_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty}(I_{\omega(i)} \times J_{\omega(i)}),\quad z_i = x_i \text{ for } 1 \leqslant i \leqslant \ell;\\ \mathrm{proj}_x z_i = \mathrm{proj}_x x_i \text{ for } \ell+1 \leqslant i \leqslant \ell^*\Bigg\},\qquad \text{if } \ell^* > \ell. \end{aligned} \tag{9}$$

The rectangle $Q_\ell(x)$ is an approximate square in $[0,1]^2$ since whose sides have length $\prod_{i=1}^{\ell^*} n_{\omega(i)}^{-1}$, $\prod_{i=1}^{\ell} m_{\omega(i)}^{-1}$ comparable by (7). Thus, to evaluate the Hausdorff dimension of $\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})$ one can restrict attention to covers by such approximate squares. We will use the following lemma which is just a reformation of Rogers–Taylor density theorem [21].

Lemma 2.1. *Let $\omega = (\Gamma_{\omega(i)})_{i=1}^\infty \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$. Suppose that μ is a finite Borel measure on $[0,1]^2$ such that $\mu(\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})) > 0$. Let δ be a positive number. For each point $x = (x_i)_{i=1}^\infty \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$, put*

$$A(x) = \limsup_{\ell\to\infty} \left(\delta \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \mu(Q_\ell(x)) \right).$$

(1) *If $A(x) = -\infty$ for all $x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$, then $\mathcal{H}^\delta(\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})) = +\infty$;*
(2) *If $A(x) = +\infty$ for all $x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$, then $\mathcal{H}^\delta(\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})) = 0$;*
(3) *If there are real numbers a and b such that $a \leqslant A(x) \leqslant b$ for all $x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$, then $0 < \mathcal{H}^\delta(\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})) < +\infty$.*

Proof. For each $\ell \in \mathbb{N}$ put

$$\mathcal{Q}_\ell = \left\{ Q_j(x) : x \in \prod_{i=1}^\infty \Gamma_{\omega(i)} \text{ and } j \geqslant \ell \right\} \qquad \text{and} \qquad \mathcal{M} = \bigcup_{\ell\geqslant 1} \mathcal{Q}_\ell.$$

Let $\mathcal{I}$ be a finite or countable index set. A covering $\{E_i : i \in \mathcal{I}\}$ of $\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})$ is called an *ℓ-level Q-type covering* if all $E_i \in \mathcal{Q}_\ell$. Then for each E_i in an ℓ-level Q-type covering there exists a unique positive integer, denoted by $\theta(E_i)(\geqslant\ell)$, and $x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$ (not unique) such that $E_i = Q_{\theta(E_i)}(x)$. By (7) the diameter of E_i is comparable to $\prod_{j=1}^{\theta(E_i)} m_{\omega(j)}^{-1}$, more exactly

$$\sqrt{2} \leqslant \frac{\operatorname{diam} E_i}{\prod_{j=1}^{\theta(E_i)} m_{\omega(j)}^{-1}} = \frac{\operatorname{diam} Q_{\theta(E_i)}(x)}{\prod_{j=1}^{\theta(E_i)} m_{\omega(j)}^{-1}} \leqslant \sqrt{1 + (\max_{1\leqslant i\leqslant k} n_i)^2}.$$

Now for $\delta > 0$ let

$$\mathcal{H}_{\mathcal{M}}^\delta \left(\Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) \right) = \lim_{\ell\to\infty} \inf_{\{E_i : i\in\mathcal{I}\}} \sum_{i\in\mathcal{I}} \left(\prod_{j=1}^{\theta(E_i)} m_{\omega(j)}^{-1} \right)^\delta,$$

where the infimum is taken over all ℓ-level Q-type coverings of $\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})$. It is easy to check that there exist positive numbers c_1 and c_2 (they depend on δ) such that

$$c_1 \mathcal{H}_{\mathcal{M}}^\delta \left(\Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) \right) \leqslant \mathcal{H}^\delta \left(\Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) \right) \leqslant c_2 \mathcal{H}_{\mathcal{M}}^\delta \left(\Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) \right).$$

This implies that

$$\begin{aligned} \dim_H \Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) &= \sup \left\{ \delta > 0 : \mathcal{H}_{\mathcal{M}}^\delta \left(\Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) \right) = +\infty \right\} \\ &= \inf \left\{ \delta > 0 : \mathcal{H}_{\mathcal{M}}^\delta \left(\Pi \left(\prod_{i=1}^\infty \Gamma_{\omega(i)} \right) \right) = 0 \right\}. \end{aligned}$$

For each $x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$ and $\delta > 0$ put

$$C(x) = \liminf_{\ell\to\infty} \frac{\left(\prod_{i=1}^{\ell} m_{\omega(i)}^{-1} \right)^\delta}{\mu(Q_\ell(x))},$$

where we adopt the convention $c/0 = +\infty$ for positive real number c.

The following relations between $A(x)$ and $C(x)$ are clear.

(I) $A(x) = -\infty$ if and only if $C(x) = +\infty$;
(II) $A(x) = +\infty$ if and only if $C(x) = 0$;
(III) $a < A(x) < b < +\infty$ if and only if $\mathrm{e}^{-b} \leqslant C(x) \leqslant \mathrm{e}^{-a}$.

Therefore, to finish the proof it suffices to show that there exist positive constants c_3, c_4 such that

$$c_3\mu\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right)\right)\inf_{x\in\prod_{i=1}^{\infty}\Gamma_{\omega(i)}} C(x) \leqslant \mathcal{H}^{\delta}_{\mathcal{M}}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right)\right)$$
$$\leqslant c_4\mu([0,1]^2)\sup_{x\in\prod_{i=1}^{\infty}\Gamma_{\omega(i)}} C(x).$$

We first show the right inequality. Without loss of generality, we assume $K \triangleq \sup_{x\in\prod_{i=1}^{\infty}\Gamma_{\omega(i)}} C(x) < \infty$. Then for all $x \in \prod_{i=1}^{\infty}\Gamma_{\omega(i)}$

$$\liminf_{\ell\to\infty}\frac{\left(\prod_{i=1}^{\ell} m_{\omega(i)}^{-1}\right)^{\delta}}{\mu(Q_\ell(x))} \leqslant K \qquad \text{or} \qquad \limsup_{\ell\to\infty}\frac{\mu(Q_\ell(x))}{\left(\prod_{i=1}^{\ell} m_{\omega(i)}^{-1}\right)^{\delta}} \geqslant K^{-1}.$$

This implies the right inequality (cf [21, lemma 2] and [6, proposition 2.2(b)]). The left inequality can be shown by the same argument. □

The Borel measures on $[0,1]^2$ to which the above lemma will be applied are constructed as follows. Let $\boldsymbol{p}_i = (p_{i,d})_{d\in\Gamma_i}$ be a probability vector on Γ_i, $1 \leqslant i \leqslant k$, i.e. $\sum_{d\in\Gamma_i} p_{i,d} = 1$ with each $p_{i,d} \in (0,1)$. This, for each $1 \leqslant i \leqslant k$, induces a probability vector $(q_{i,b})_{b\in\mathrm{proj}_y\Gamma_i}$ on $\mathrm{proj}_y\Gamma_i$ and a probability vector $(\bar{q}_{i,b})_{b\in\mathrm{proj}_x\Gamma_i}$ on $\mathrm{proj}_x\Gamma_i$ by letting

$$q_{i,b} = \sum_{d\in\Gamma_i,\ \mathrm{proj}_y d=b} p_{i,d} \qquad \text{and} \qquad \bar{q}_{i,b} = \sum_{d\in\Gamma_i,\ \mathrm{proj}_x d=b} p_{i,d}. \tag{10}$$

Denote $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_k)$. Then $\boldsymbol{P}$ determines a unique infinite product Borel probability measure, denoted by $\mu_{\boldsymbol{P}}$, on $\prod_{i=1}^{\infty}\Gamma_{\omega(i)}$: For any finite sequence $(x_1, x_2, \ldots, x_\ell) \in \prod_{i=1}^{\ell}\Gamma_{\omega(i)}$ let

$$\mu_{\boldsymbol{P}}([x_1, x_2, \ldots, x_\ell]) = \prod_{i=1}^{\ell} p_{\omega(i),x_i}, \tag{11}$$

where $[x_1, x_2, \ldots, x_\ell] \triangleq \{y = (y_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty}\Gamma_{\omega(i)} : y_i = x_i \text{ for } 1 \leqslant i \leqslant \ell\}$ is a cylinder set of $\prod_{i=1}^{\infty}\Gamma_{\omega(i)}$ with base $(x_1, x_2, \ldots, x_\ell)$. Let $\tilde{\mu}_{\boldsymbol{P}}$ be the Borel probability measure on $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ which is the image measure of $\mu_{\boldsymbol{P}}$ under Π, i.e.

$$\tilde{\mu}_{\boldsymbol{P}}(A) = \mu_{\boldsymbol{P}}(\Pi^{-1}A) \qquad \text{for Borel set } A \subseteq \mathbb{R}^2. \tag{12}$$

From the definition of the approximate square $Q_\ell(x)$ it follows that for any $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty}\Gamma_{\omega(i)}$

$$\tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = \begin{cases} \displaystyle\prod_{i=1}^{\ell^*} p_{\omega(i),\,x_i}\cdot\prod_{i=\ell^*+1}^{\ell} q_{\omega(i),\,\mathrm{proj}_y x_i} & \text{if } \ell^* \leqslant \ell \\ \displaystyle\prod_{i=1}^{\ell} p_{\omega(i),\,x_i}\cdot\prod_{i=\ell+1}^{\ell^*} \bar{q}_{\omega(i),\,\mathrm{proj}_x x_i} & \text{if } \ell^* > \ell, \end{cases} \tag{13}$$

where ℓ^*, $q_{\omega(i),\,\mathrm{proj}_y x_i}$ and $\bar{q}_{\omega(i),\,\mathrm{proj}_x x_i}$ are defined as in (6) and (10), respectively.

Let us recall the definition of the Hausdorff dimension of a measure μ. It is defined as the infimum of Hausdorff dimensions of sets of full μ-measure. The following lemma is a version of the well-known Billingsley lemma [3] for which the ball is replaced by the approximate square.

Lemma 2.2. *Let $\tilde{\mu}_{\boldsymbol{P}}$ be the probability Borel measure on $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ defined by (12). If*

$$\liminf_{\ell\to\infty}\frac{\log\tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x))}{\sum_{i=1}^{\ell}\log m_{\omega(i)}^{-1}}=\beta$$

for $\mu_{\boldsymbol{P}}$-almost every $x\in\prod_{i=1}^{\infty}\Gamma_{\omega(i)}$, then $\dim_H\tilde{\mu}_{\boldsymbol{P}}=\beta$.

We now point out that when an $\omega=(\Gamma_{\omega(i)})_{i=1}^{\infty}\in\{\Gamma_1,\ldots,\Gamma_k\}^{\mathbb{N}}$ satisfies (4)

$$\lim_{\ell\to\infty}\frac{\ell^*}{\ell}=\lim_{\ell\to\infty}\frac{\ell^{-1}\left(\sum_{i=1}^{\ell^*}\log n_{\omega(i)}-\sum_{i=1}^{\ell}\log m_{\omega(i)}\right)+\ell^{-1}\sum_{i=1}^{\ell}\log m_{\omega(i)}}{\ell^{*-1}\sum_{i=1}^{\ell^*}\log n_{\omega(i)}}$$

$$=\frac{\sum_{i=1}^{k}p_i\log m_i}{\sum_{i=1}^{k}p_i\log n_i}=\zeta\leqslant 1 \tag{14}$$

by (3) and (7). For this case we have $\ell^*<\ell$ if $\zeta<1$ and ℓ big enough, and so $Q_\ell(x)$ takes the form (8). While $Q_\ell(x)$ may take the form (8) or the form (9) when $\zeta=1$.

By means of lemma 2.2 the Hausdorff dimension of $\tilde{\mu}_{\boldsymbol{P}}$ defined by (12) can be obtained explicitly.

Proposition 2.3. *Let $\omega=(\Gamma_{\omega(i)})_{i=1}^{\infty}\in\{\Gamma_1,\ldots,\Gamma_k\}^{\mathbb{N}}$ satisfy (4). Let $\tilde{\mu}_{\boldsymbol{P}}$ be the probability Borel measure on $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ defined by (12). Then*

$$\dim_H\tilde{\mu}_{\boldsymbol{P}}=\frac{\sum_{i=1}^{k}p_i\left(-\zeta\sum_{d\in\Gamma_i}p_{i,d}\log p_{i,d}-(1-\zeta)\sum_{d\in\Gamma_i}p_{i,d}\log q_{i,\,\mathrm{proj}_y d}\right)}{\sum_{i=1}^{k}p_i\log m_i},$$

where $\zeta=\frac{\sum_{i=1}^{k}p_i\log m_i}{\sum_{i=1}^{k}p_i\log n_i}$.

Proof. For $j\in\mathbb{N}$ let X_j be the random variable on $(\prod_{i=1}^{\infty}\Gamma_{\omega(i)},\mathcal{B},\mu_{\boldsymbol{P}})$ such that for $x=(x_i)_{i=1}^{\infty}\in\prod_{i=1}^{\infty}\Gamma_{\omega(i)}$

$$X_j(x)=\log p_{\omega(j),x_j}-\sum_{d\in\Gamma_{\omega(j)}}p_{\omega(j),d}\log p_{\omega(j),d}.$$

Then, $\{X_j\}_{j=1}^{\infty}$ is a sequence of independent random variables with

$$\mathcal{E}(X_j)=0\qquad\text{and}\qquad\sum_{j=1}^{\infty}\frac{\mathcal{E}(X_j^2)}{j^2}<\infty,$$

where $\mathcal{E}(X_j)$ denotes the expectation of the random variable X_j. From the strong law of large numbers (cf [4, theorem 1 in section 5.2]) it follows that

$$\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{j=1}^{\ell}X_j(x)=0\qquad\text{for }\mu_{\boldsymbol{P}}\text{-a.e }x=(x_i)_{i=1}^{\infty}\in\prod_{i=1}^{\infty}\Gamma_{\omega(i)}.$$

Similarly, by letting

$$Y_j(x)=\log q_{\omega(j),\,\mathrm{proj}_y x_j}-\sum_{d\in\Gamma_{\omega(j)}}p_{\omega(j),\,d}\log q_{\omega(j),\,\mathrm{proj}_y d}$$

and

$$\overline{Y}_j(x) = \log q_{\omega(j),\ \mathrm{proj}_x x_j} - \sum_{d\in\Gamma_{\omega(j)}} p_{\omega(j),\ d} \log q_{\omega(j),\ \mathrm{proj}_x d}$$

we have

$$\lim_{\ell\to\infty} \frac{1}{\ell}\sum_{j=1}^{\ell} Y_j(x) = 0 \qquad \text{and}$$

$$\lim_{\ell\to\infty} \frac{1}{\ell}\sum_{j=1}^{\ell} \overline{Y}_j(x) = 0 \ \text{ for } \mu_{\boldsymbol{P}}\text{-a.e } x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}.$$

By (13) we have (if $\ell^* \leqslant \ell$)

$$\log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = \sum_{i=1}^{\ell^*} X_i(x) + \sum_{i=\ell^*+1}^{\ell} Y_i(x) + \sum_{i=1}^{\ell^*}\sum_{d\in\Gamma_{\omega(i)}} p_{\omega(i),d}\log p_{\omega(i),d} + \sum_{i=\ell^*+1}^{\ell}\sum_{d\in\Gamma_{\omega(i)}} p_{\omega(i),d}\log q_{\omega(i),\mathrm{proj}_y d}$$

or (if $\ell^* > \ell$)

$$\log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = \sum_{i=1}^{\ell} X_i(x) + \sum_{i=\ell+1}^{\ell^*} \overline{Y}_i(x) + \sum_{i=1}^{\ell}\sum_{d\in\Gamma_{\omega(i)}} p_{\omega(i),d}\log p_{\omega(i),d} + \sum_{i=\ell+1}^{\ell^*}\sum_{d\in\Gamma_{\omega(i)}} p_{\omega(i),d}\log q_{\omega(i),\ \mathrm{proj}_x d}.$$

Therefore, we have that for $\mu_{\boldsymbol{P}}$-almost every $x \in \prod_{i=1}^{\infty}\Gamma_{\omega(i)}$ (recall that $\zeta < 1$ implies $\ell^* < \ell$ for ℓ big enough)

$$\lim_{\ell\to\infty}\frac{\log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x))}{-\ell} = \sum_{i=1}^{k} p_i\left(-\zeta\sum_{d\in\Gamma_i} p_{i,d}\log p_{i,d} - (1-\zeta)\sum_{d\in\Gamma_i} p_{i,d}\log q_{i,\ \mathrm{proj}_y d}\right)$$

by (14). This gives

$$\lim_{\ell\to\infty}\frac{\log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x))}{\sum_{i=1}^{\ell}\log m_{\omega(i)}^{-1}} = \frac{\sum_{i=1}^{k} p_i\left(-\zeta\sum_{d\in\Gamma_i} p_{i,d}\log p_{i,d} - (1-\zeta)\sum_{d\in\Gamma_i} p_{i,d}\log q_{i,\ \mathrm{proj}_y d}\right)}{\sum_{i=1}^{k} p_i\log m_i},$$

leading to the desired result by lemma 2.2. □

The following proposition shows that $\dim_H \tilde{\mu}_{\boldsymbol{P}}$ can attain its maximum at some $\boldsymbol{P}^* = (\boldsymbol{p}_1^*, \boldsymbol{p}_2^*, \ldots, \boldsymbol{p}_k^*)$.

Proposition 2.4. *Let* $\omega = (\Gamma_{\omega(i)})_{i=1}^{\infty} \in \{\Gamma_1, \ldots, \Gamma_k\}^{\mathbb{N}}$ *satisfy* (4). *Let* $\tilde{\mu}_{\boldsymbol{P}}$ *be the probability Borel measure on* $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ *defined by* (12). *Let* $\boldsymbol{P}^* = (\boldsymbol{p}_1^*, \boldsymbol{p}_2^*, \ldots, \boldsymbol{p}_k^*)$ *where* $\boldsymbol{p}_i^* = (p_{i,d}^*)_{d\in\Gamma_i} = (h_{i,\ \mathrm{proj}_y d}^{\zeta-1}/\sum_{e\in\Gamma_i} h_{i,\ \mathrm{proj}_y e}^{\zeta-1})_{d\in\Gamma_i}$, $1 \leqslant i \leqslant k$ *with* ζ *defined as in* (3). *Then* $\boldsymbol{P}^*$ *is the unique point such that*

$$\dim_H \tilde{\mu}_{\boldsymbol{P}^*} = \max_{\boldsymbol{P}} \dim_H \tilde{\mu}_{\boldsymbol{P}} = \frac{\sum_{i=1}^{k} p_i\log\sum_{d\in\Gamma_i} h_{i,\ \mathrm{proj}_y d}^{\zeta-1}}{\sum_{i=1}^{k} p_i\log m_i}.$$

Proof. Fix an $i \in \{1, \ldots, k\}$. Consider the function

$$R(\boldsymbol{p}_i) \triangleq -\zeta \sum_{d\in\Gamma_i} p_{i,d} \log p_{i,d} - (1-\zeta) \sum_{d\in\Gamma_i} p_{i,d} \log q_{i,\ \mathrm{proj}_y d}.$$

We adopt the convention that $0 \log 0 = 0$. Thus $R(\boldsymbol{p}_i)$ is a strictly concave function in $p_{i,d}, d \in \Gamma_i$ on the closed convex set $\{\boldsymbol{p}_i = (p_{i,d})_{d\in\Gamma_i} : \sum_{d\in\Gamma_i} p_{i,d} = 1, 0 \leqslant p_{i,d} \leqslant 1\}$. By a well-known property of convex programming, there exists a unique $\boldsymbol{p}_i^* = (p_{i,d}^*)_{d\in\Gamma_i} \in \{\boldsymbol{p}_i = (p_{i,d})_{d\in\Gamma_i} : \sum_{d\in\Gamma_i} p_{i,d} = 1, 0 \leqslant p_{i,d} \leqslant 1\}$ such that $R(\boldsymbol{p}_i)$ attains its maximum at $\boldsymbol{p}_i^*$. However, it is easy to prove that $\boldsymbol{p}_i^* = (p_{i,d}^*)_{d\in\Gamma_i} \in \{\boldsymbol{p}_i = (p_{i,d})_{d\in\Gamma_i} : \sum_{d\in\Gamma_i} p_{i,d} = 1, 0 < p_{i,d} < 1\}$. A direct calculation by Lagrange's method of multipliers gives $\boldsymbol{p}_i^* = (p_{i,d}^*)_{d\in\Gamma_i} = (h_{i,\ \mathrm{proj}_y d}^{\zeta-1} / \sum_{e\in\Gamma_i} h_{i,\ \mathrm{proj}_y e}^{\zeta-1})_{d\in\Gamma_i}$. □

3. Proofs

This section is devoted to the proofs of theorems 1.1 and 1.5.

Proof of theorem 1.1. The lower bound of the Hausdorff dimension of $\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})$ just follows from proposition 2.4, i.e.

$$\dim_H \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) \geqslant \dim_H \tilde{\mu}_{\boldsymbol{P}^*} = \frac{\sum_{i=1}^{k} p_i \log \sum_{d\in\Gamma_i} h_{i,\ \mathrm{proj}_y d}^{\zeta-1}}{\sum_{i=1}^{k} p_i \log m_i}.$$

For $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}$ and $\ell \in \mathbb{N}$ let

$$S_\ell(x) = \sum_{i=1}^{\ell} \log h_{\omega(i),\ \mathrm{proj}_y x_i}. \tag{15}$$

Let $\boldsymbol{P}^* = (\boldsymbol{p}_1^*, \boldsymbol{p}_2^*, \ldots, \boldsymbol{p}_k^*)$ be the unique maximum point given in proposition 2.4. Recall that each $\boldsymbol{p}_i^* = (p_{i,d}^*)_{d\in\Gamma_i}$ induces a probability vector $(q_{i,\ b}^*)_{b\in \mathrm{proj}_y \Gamma_i}$ on $\mathrm{proj}_y \Gamma_i$ via (10), and each $\ell \in \mathbb{N}$ determines ℓ^* via (6).

Case I: $\zeta < 1$. By (13) we have (for ℓ big enough)

$$\begin{aligned}
\log \tilde{\mu}_{\boldsymbol{P}^*}(Q_\ell(x)) &= \sum_{i=1}^{\ell^*} \log p_{\omega(i),\ x_i}^* + \sum_{i=\ell^*+1}^{\ell} \log q_{\omega(i),\ \mathrm{proj}_y x_i}^* \\
&= \sum_{i=1}^{\ell^*} \log \frac{h_{\omega(i),\ \mathrm{proj}_y x_i}^{\zeta-1}}{\sum_{e\in\Gamma_{\omega(i)}} h_{\omega(i),\ \mathrm{proj}_y e}^{\zeta-1}} + \sum_{i=\ell^*+1}^{\ell} \log \frac{h_{\omega(i),\ \mathrm{proj}_y x_i}^{\zeta}}{\sum_{e\in\Gamma_{\omega(i)}} h_{\omega(i),\ \mathrm{proj}_y e}^{\zeta-1}} \\
&= \zeta \sum_{i=1}^{\ell} \log h_{\omega(i),\ \mathrm{proj}_y x_i} - \sum_{i=1}^{\ell^*} \log h_{\omega(i),\ \mathrm{proj}_y x_i} - \sum_{i=1}^{\ell} \log \sum_{e\in\Gamma_{\omega(i)}} h_{\omega(i),\ \mathrm{proj}_y e}^{\zeta-1}.
\end{aligned}$$

Thus, by (15)

$$\begin{aligned}
\limsup_{\ell\to\infty} \frac{1}{\ell} \log \tilde{\mu}_{\boldsymbol{P}^*}(Q_\ell(x)) &= \limsup_{\ell\to\infty} \frac{1}{\ell} \left(\zeta S_\ell(x) - S_{\ell^*}(x) - \sum_{i=1}^{\ell} \log \sum_{e\in\Gamma_{\omega(i)}} h_{\omega(i),\mathrm{proj}_y e}^{\zeta-1} \right) \\
&= \limsup_{\ell\to\infty} \frac{1}{\ell} (\zeta S_\ell(x) - S_{\ell^*}(x)) - \sum_{i=1}^{k} p_i \log \sum_{e\in\Gamma_i} h_{i,\mathrm{proj}_y e}^{\zeta-1}.
\end{aligned}$$

Note that by (15) we have $\limsup_{\ell\to\infty} S_\ell(x)/\ell$ is finite for each $x = (x_i)_{i=1}^\infty \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$. Thus

$$\limsup_{\ell\to\infty} \frac{1}{\ell}(\zeta S_\ell(x) - S_{\ell^*}(x)) \geqslant \zeta \limsup_{\ell\to\infty} \frac{S_\ell(x)}{\ell} - \limsup_{\ell\to\infty} \frac{S_{\ell^*}(x)}{\ell}$$
$$= \zeta \limsup_{\ell\to\infty} \frac{S_\ell(x)}{\ell} - \limsup_{\ell\to\infty} \left(\frac{S_{\ell^*}(x)}{\ell^*} \cdot \frac{\ell^*}{\ell}\right) = 0,$$

by (14). Therefore

$$\limsup_{\ell\to\infty} \frac{1}{\ell} \log \tilde{\mu}_{\boldsymbol{P}^*}(Q_\ell(x)) \geqslant -\sum_{i=1}^k p_i \log \sum_{d\in\Gamma_i} h_{i,\ \mathrm{proj}_y d}^{\zeta-1}.$$

Fix an arbitrary $\delta > \frac{\sum_{i=1}^k p_i \log \sum_{d\in\Gamma_i} h_{i,\ \mathrm{proj}_y d}^{\zeta-1}}{\sum_{i=1}^k p_i \log m_i}$. Then for each $x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}$ we have

$$A(x) = \limsup_{\ell\to\infty} \left(\delta \sum_{i=1}^\ell \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}^*}(Q_\ell(x))\right)$$
$$= \limsup_{\ell\to\infty} \ell \left(\frac{\delta \sum_{i=1}^\ell \log m_{\omega(i)}}{\ell} + \frac{\log \tilde{\mu}_{\boldsymbol{P}^*}(Q_\ell(x))}{\ell}\right) = +\infty,$$

Now lemma 2.1 (2) implies that

$$\dim_H \Pi\left(\prod_{i=1}^\infty \Gamma_{\omega(i)}\right) \leqslant \frac{\sum_{i=1}^k p_i \log \sum_{d\in\Gamma_i} h_{i,\ \mathrm{proj}_y d}^{\zeta-1}}{\sum_{i=1}^k p_i \log m_i}.$$

Case II: $\zeta = 1$. For this case we have $\boldsymbol{p}_i^* = (p_{i,d}^*)_{d\in\Gamma_i} = (\frac{1}{|\Gamma_i|}, \ldots, \frac{1}{|\Gamma_i|})$, the uniformly distributed probability measure. By (13) we have

$$\log \tilde{\mu}_{\boldsymbol{P}^*}(Q_\ell(x)) = \begin{cases} -\sum_{i=1}^{\ell} \log|\Gamma_{\omega(i)}| + \sum_{i=\ell^*+1}^{\ell} \log h_{\omega(i),\mathrm{proj}_y x_i} & \text{if } \ell^* \leqslant \ell \\ -\sum_{i=1}^{\ell^*} \log|\Gamma_{\omega(i)}| + \sum_{i=\ell+1}^{\ell^*} \log h_{\omega(i),\mathrm{proj}_x x_i} & \text{if } \ell^* > \ell. \end{cases}$$

By the same argument as above we get

$$\dim_H \Pi\left(\prod_{i=1}^\infty \Gamma_{\omega(i)}\right) \leqslant \frac{\sum_{i=1}^k p_i \log|\Gamma_i|}{\sum_{i=1}^k p_i \log m_i}.$$

As to the box dimension of $\Pi(\prod_{i=1}^\infty \Gamma_{\omega(i)})$ it is easy to see

$$\dim_B \Pi\left(\prod_{i=1}^\infty \Gamma_{\omega(i)}\right) = \lim_{\ell\to\infty} \frac{\log|\{Q_\ell(x) : x \in \prod_{i=1}^\infty \Gamma_{\omega(i)}\}|}{\sum_{i=1}^\ell \log m_{\omega(i)}}$$
$$= \begin{cases} \lim_{\ell\to\infty} \frac{\log\left(\prod_{i=1}^{\ell^*} |\Gamma_{\omega(i)}| \prod_{i=\ell^*+1}^{\ell} |\mathrm{proj}_y \Gamma_{\omega(i)}|\right)}{\sum_{i=1}^\ell \log m_{\omega(i)}} & \text{if } \ell^* \leqslant \ell \\ \lim_{\ell\to\infty} \frac{\log\left(\prod_{i=1}^{\ell} |\Gamma_{\omega(i)}| \prod_{i=\ell+1}^{\ell^*} |\mathrm{proj}_x \Gamma_{\omega(i)}|\right)}{\sum_{i=1}^\ell \log m_{\omega(i)}} & \text{if } \ell^* > \ell \end{cases}$$
$$= \frac{\sum_{i=1}^k p_i(\zeta \log|\Gamma_i| + (1-\zeta)\log|\mathrm{proj}_y \Gamma_i|)}{\sum_{i=1}^k p_i \log m_i}.$$

Finally, for any open set V with $V \cap \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}) \neq \emptyset$ we have $\dim_B(V \cap \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})) = \dim_B \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$, leading to $\dim_P \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}) = \dim_B \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ by [5, corollary 3.9] □

Proof of theorem 1.5. Without loss of generality, we assume that $p_i > 0$ for $1 \leqslant i \leqslant t$ $(t \leqslant k)$ and $p_i = 0$ for $t < i \leqslant k$.

(I) By the structure of $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ we have

$$\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \bigcup_{d\in\Gamma_{\omega(1)}} \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) \cap f_{\omega(1),d}([0,1]^2),$$

where the sets on the right side are translations of each other. Thus,

$$\mathcal{H}^{\gamma}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) \cap f_{\omega(1),\,d}([0,1]^2)\right) = \frac{1}{|\Gamma_{\omega(1)}|}\mathcal{H}^{\gamma}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right)\right),$$

for all $d \in \Gamma_{\omega(1)}$. For any $\boldsymbol{d} = (d_i)_{i=1}^{\ell} \in \prod_{i=1}^{\ell}\Gamma_{\omega(i)}$ the same argument yields that

$$\mathcal{H}^{\gamma}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) \cap f_{\boldsymbol{d}}([0,1]^2)\right) = \frac{1}{\prod_{i=1}^{\ell}|\Gamma_{\omega(i)}|}\mathcal{H}^{\gamma}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right)\right),$$

where $f_{\boldsymbol{d}} \triangleq f_{\omega(1),d_1} \circ f_{\omega(2),d_2} \circ \cdots \circ f_{\omega(\ell),d_\ell}$. Now we take $\overline{\boldsymbol{P}} = (\overline{\boldsymbol{p}}_1, \overline{\boldsymbol{p}}_2, \ldots, \overline{\boldsymbol{p}}_k)$ where $\overline{\boldsymbol{p}}_i = (|\Gamma_i|^{-1}, \ldots, |\Gamma_i|^{-1})$ is a probability vector on Γ_i, $1 \leqslant i \leqslant k$. Let $\tilde{\mu}_{\overline{\boldsymbol{P}}}$ be the probability measure on $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ defined as in (12). Then

$$\tilde{\mu}_{\overline{\boldsymbol{P}}}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) \cap f_{\boldsymbol{d}}([0,1]\times[0,1])\right) = \frac{1}{\prod_{i=1}^{\ell}|\Gamma_{\omega(i)}|},$$

for any $\boldsymbol{d} = (d_i)_{i=1}^{\ell} \in \prod_{i=1}^{\ell}\Gamma_{\omega(i)}$. Therefore, for any Borel set $A \subset [0,1]^2$

$$\tilde{\mu}_{\overline{\boldsymbol{P}}}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) \cap A\right) = \frac{1}{\mathcal{H}^{\gamma}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right)\right)}\mathcal{H}^{\gamma}\left(\Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) \cap A\right).$$

Note that

$$\dim_H \tilde{\mu}_{\overline{\boldsymbol{P}}} = \inf_{\substack{A \subset \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}) \\ \tilde{\mu}_{\overline{\boldsymbol{P}}}(A) = 1}} \dim_H A = \inf_{\substack{A \subset \Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}) \\ \mathcal{H}^{\gamma}(A) = \mathcal{H}^{\gamma}(\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}))}} \dim_H A = \gamma.$$

On the other hand, from proposition 2.3 it follows

$$\dim_H \tilde{\mu}_{\overline{\boldsymbol{P}}} = \frac{\sum_{i=1}^{t} p_i \sum_{d\in\Gamma_i}|\Gamma_i|^{-1}\log(|\Gamma_i|h_{i,\,\mathrm{proj}_y d}^{\zeta-1})}{\sum_{i=1}^{t} p_i \log m_i}.$$

Recall that $\gamma = (\sum_{i=1}^{t} p_i \log \sum_{d\in\Gamma_i} h_{i,\mathrm{proj}_y d}^{\zeta-1})/(\sum_{i=1}^{t} p_i \log m_i)$ by theorem 1.1. Since $\log x$ is a strictly concave function in x we obtain that for each $1 \leqslant i \leqslant t$ all $h_{i,\mathrm{proj}_y d}$, $d \in \Gamma_i$ are identical, i.e. each Γ_i, $i = 1, 2, \ldots, t$ has uniformly horizontal fibres.

(II) This essentially is just theorem 4.1 (II) in [11] with a bit difference. For reader's convenience, we still give its proof. Under the conditions, we have

$$\zeta = \log_n m \leqslant 1,\ c_1 + \ell\log_n m \leqslant \ell^* \leqslant c_2 + \ell\log_n m \qquad \text{for some constants } c_1, c_2 \in \mathbb{R}$$

and

$$\gamma = \dim_H \Pi\left(\prod_{i=1}^{\infty}\Gamma_{\omega(i)}\right) = \frac{\zeta\log h + \log\eta}{\log m},$$

where $h = h_{i,b}$, $\eta = \mathrm{proj}_y\Gamma_i$ for $p_i \neq 0$.

We take $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_k)$ where $\boldsymbol{p}_i = (p_{d,i})_{d\in\Gamma_i} = (|\Gamma_i|^{-1}, |\Gamma_i|^{-1}, \ldots, |\Gamma_i|^{-1})$ is a probability vector on Γ_i, $1 \leqslant i \leqslant k$. The probability measure $\tilde{\mu}_{\boldsymbol{P}}$ on $\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})$ is constructed in the way shown by (11) and (12).

Case I: $\ell^* \leqslant \ell$ for ℓ big enough. This happens when $\zeta = \log_n m < 1$ (and may happen when $\zeta = \log_n m = 1$).

Then there exists some constant c such that for any $x = (x_i)_{i\geqslant 1} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}$ and any $\ell \in \mathbb{N}$ (ℓ big enough)

$$\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = \ell(\zeta \log h + \log \eta) - \ell \log \eta - \ell^* \log h + c$$
$$= (\ell \log_n m - \ell^*) \log h + c,$$

which gives $0 < \mathcal{H}^\gamma(\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})) < \infty$ by lemma 2.1 (3).

Case II: $\ell^* > \ell$ for ℓ big enough. For this case it has to be $\zeta = \log_n m = 1$, i.e. $n = m$.

In this case there exists a $q \in \mathbb{N}$ such that $\ell^* = \ell + q$ when ℓ big enough. Thus the desired result is obtained since for ℓ big enough

$$\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = c,$$

with $c \in \mathbb{R}$ a constant. □

Let $0 < \zeta < 1$. Combining theorem 1.5 (I) with corollary 1.2 we have that $\dim_H \Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}) = \dim_B \Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})$ if $\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})$ has positive Hausdorff measure in its dimension. Let us recall that in the case of general Sierpinski carpet (i.e. the set K_D defined as in (1) with $2 \leqslant m < n$) we have $\dim_H K_D = \dim_B K_D$ if and only if K_D has positive Hausdorff measure in its dimension. However, this is not true for the case discussed in this paper. In other words, the condition shown in theorem 1.5 (I) is not necessary. This is shown by the following examples 1 and 2. Moreover, the following example 3 shows that the condition shown in theorem 1.5 (II) is also not necessary. For simplicity, we only consider the case that $k = 2$ although it becomes more complicated for $k > 2$.

Example 1. Let $\omega = (\Gamma_{\omega(j)})_{j\geqslant 1} \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}}$ satisfy (4) with $p_1 = 1$ and $p_2 = 0$. Suppose that

(H1) $2 \leqslant m_1 \leqslant n_1$ and $2 \leqslant m_2 \leqslant n_2$ (so $\ell^* \leqslant \ell$);
(H2) $\lim_{\ell\to\infty} N_\ell(\Gamma_2) = \infty$;
(H3) Γ_1 have uniformly horizontal fibres.

Then $\zeta = \log_{n_1} m_1$ and

$$\gamma := \dim_H \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \dim_B \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right)$$
$$= \log_{m_1} |\mathrm{proj}_y \Gamma_1| + \log_{m_1} \left(\frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|}\right)^{\log_{n_1} m_1}.$$

We claim that $\mathcal{H}^\gamma(\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}))$ may take ∞ or 0 if Γ_1 and Γ_2 are properly selected.

Proof. Take the probability vectors $\boldsymbol{p}_i$ on Γ_i as $\boldsymbol{p}_i = (p_{i,d})_{d\in\Gamma_i} = (|\Gamma_i|^{-1}, \ldots, |\Gamma_i|^{-1})$, $i = 1, 2$. The probability measure $\tilde{\mu}_{\boldsymbol{P}}$ on $\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})$ is constructed in the way shown by (11) and (12). Then for any $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}$ and any $\ell \in \mathbb{N}$

$$\begin{aligned}
\log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) &= N_{\ell^*}(\Gamma_1) \log \frac{1}{|\Gamma_1|} + N_{\ell^*}(\Gamma_2) \log \frac{1}{|\Gamma_2|} + (N_\ell(\Gamma_1) - N_{\ell^*}(\Gamma_1)) \log \frac{1}{|\mathrm{proj}_y \Gamma_1|} \\
&\quad + \sum_{\ell^*+1 \leqslant i \leqslant \ell,\ \omega(i)=2} \log q_{2,\mathrm{proj}_y x_i} \\
&= -\log \left(\frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} \right)^{N_{\ell^*}(\Gamma_1)} - \log |\mathrm{proj}_y \Gamma_1|^{N_\ell(\Gamma_1)} - N_{\ell^*}(\Gamma_2) \log |\Gamma_2| \\
&\quad + \sum_{\ell^*+1 \leqslant i \leqslant \ell, \omega(i)=2} \log q_{2,\mathrm{proj}_y x_i} \qquad (16)
\end{aligned}$$

and so

$$\begin{aligned}
&\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) \\
&\quad = (N_\ell(\Gamma_1) \log_{n_1} m_1 - N_{\ell^*}(\Gamma_1)) \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} + \gamma N_\ell(\Gamma_2) \log m_2 \\
&\quad - N_{\ell^*}(\Gamma_2) \log |\Gamma_2| + \sum_{\ell^*+1 \leqslant i \leqslant \ell, \omega(i)=2} \log q_{2,\mathrm{proj}_y x_i},
\end{aligned}$$

where ℓ^* is determined by (6) and $\lim_{\ell\to\infty} \ell^*/\ell = \log_{n_1} m_1$ by (14). From (7) it follows that

$$1 \leqslant \frac{m_1^{N_\ell(\Gamma_1)} m_2^{N_\ell(\Gamma_2)}}{n_1^{N_{\ell^*}(\Gamma_1)} n_2^{N_{\ell^*}(\Gamma_2)}} < \max\{n_1, n_2\} := n_{\max},$$

i.e.,

$$\begin{aligned}
N_{\ell^*}(\Gamma_2) \log n_2 - N_\ell(\Gamma_2) \log m_2 &\leqslant N_\ell(\Gamma_1) \log m_1 - N_{\ell^*}(\Gamma_1) \log n_1 \\
&< N_{\ell^*}(\Gamma_2) \log n_2 - N_\ell(\Gamma_2) \log m_2 + \log n_{\max}.
\end{aligned}$$

Let $h_{\max} = \max\{h_{2,b} : b \in \mathrm{proj}_y \Gamma_2\}$ and $h_{\min} = \min\{h_{2,b} : b \in \mathrm{proj}_y \Gamma_2\}$ where $h_{2,b}$ is defined as in (2). Then $q^* := \max_{b \in \mathrm{proj}_y \Gamma_2} q_{2,b} = \frac{h_{\max}}{|\Gamma_2|}$ and $q_* := \min_{b \in \mathrm{proj}_y \Gamma_2} q_{2,b} = \frac{h_{\min}}{|\Gamma_2|}$. Then

$$\begin{aligned}
&\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) \\
&\quad \leqslant \frac{N_{\ell^*}(\Gamma_2) \log n_2 - N_\ell(\Gamma_2) \log m_2 + \log n_{\max}}{\log n_1} \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} \\
&\quad + \gamma N_\ell(\Gamma_2) \log m_2 - N_{\ell^*}(\Gamma_2) \log |\Gamma_2| + (N_\ell(\Gamma_2) - N_{\ell^*}(\Gamma_2)) \log q^* \\
&\quad = \left(\log_{n_1} n_2 \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} - \log |\Gamma_2| - \log q^* \right) N_{\ell^*}(\Gamma_2) \\
&\quad + \log_{n_1} n_{\max} \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} \\
&\quad + \left(\gamma \log m_2 + \log q^* - \log_{n_1} m_2 \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} \right) N_\ell(\Gamma_2) \qquad (17)
\end{aligned}$$

and

$$\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) \geqslant \frac{N_{\ell^*}(\Gamma_2)\log n_2 - N_\ell(\Gamma_2)\log m_2}{\log n_1} \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|}$$

$$+ \gamma N_\ell(\Gamma_2)\log m_2 - N_{\ell^*}(\Gamma_2)\log|\Gamma_2| + (N_\ell(\Gamma_2) - N_{\ell^*}(\Gamma_2))\log q_*$$

$$= \left(\log_{n_1} n_2 \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} - \log|\Gamma_2| - \log q_* \right) N_{\ell^*}(\Gamma_2)$$

$$+ \left(\gamma \log m_2 + \log q_* - \log_{n_1} m_2 \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} \right) N_\ell(\Gamma_2) \tag{18}$$

Note that

$$\gamma \log m_2 = \log_{m_1} m_2 \cdot \log|\mathrm{proj}_y \Gamma_1| + \log_{n_1} m_2 \cdot \log \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|}. \tag{19}$$

Furthermore we require that

$$n_1 = n_2, \ \frac{|\Gamma_1|}{|\mathrm{proj}_y \Gamma_1|} = 2, \quad |\Gamma_2| q^* = h_{\max} = 3 \qquad \text{and}$$

$$\log_{m_1} m_2 \cdot \log|\mathrm{proj}_y \Gamma_1| + \log \frac{h_{\max}}{|\Gamma_2|} < 0.$$

Then for any $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}$

$$A(x) = \limsup_{\ell \to \infty} \left(\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) \right) = -\infty$$

by (17), (19) and (H2). Thus $\mathcal{H}^\gamma(\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})) = +\infty$ by lemma 2.1 (1). Similarly one can put additional conditions on Γ_1, Γ_2 such that $A(x) = +\infty$ for all $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \Gamma_{\omega(i)}$, leading $\mathcal{H}^\gamma(\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)})) = 0$. □

Let (p_1, p_2) be a probability vector with $0 < p_1 < 1$. Let $\mathbb{P}$ be the probability measure defined as in (5) on $\{\Gamma_1, \Gamma_2\}^{\mathbb{N}}$. Define random variables X_j on $\{\Gamma_1, \Gamma_2\}^{\mathbb{N}}$ by letting

$$X_j(\omega) = \begin{cases} 1 - p_1 & \text{if } \omega(j) = 1 \\ -p_1 & \text{if } \omega(j) = 2 \end{cases} \qquad \text{for } \omega = (\Gamma_{\omega(j)})_{j \geqslant 1} \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}}.$$

Then $\{X_j\}_{j \geqslant 1}$ are independent (identical distributed) r.v.s with $EX_j = 0$. From law of the iterated logarithm (cf [4, theorem 1 in section 10.2]) it follows that

$$1 = \mathbb{P}\left\{ \omega \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}} : \overline{\lim}_{\ell \to \infty} \frac{\sum_{j=1}^{\ell} X_j(\omega)}{\sqrt{\ell p_1 p_2}\sqrt{\log_2 \sqrt{\ell p_1 p_2}}} = \sqrt{2} \right\}$$

$$= \mathbb{P}\left\{ \omega \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}} : \underline{\lim}_{\ell \to \infty} \frac{\sum_{j=1}^{\ell} X_j(\omega)}{\sqrt{\ell p_1 p_2}\sqrt{\log_2 \sqrt{\ell p_1 p_2}}} = -\sqrt{2} \right\}.$$

Note that $\sum_{j=1}^{\ell} X_j(\omega) = N_\ell(\Gamma_1) - p_1 \ell$. Thus

$$1 = \mathbb{P}\{\omega \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}} : \overline{\lim}_{\ell \to \infty} (N_\ell(\Gamma_1) - p_1 \ell) = +\infty\}$$

$$= \mathbb{P}\{\omega \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}} : \underline{\lim}_{\ell \to \infty} (N_\ell(\Gamma_1) - p_1 \ell) = -\infty\}. \tag{20}$$

Example 2. Let $\omega = (\Gamma_{\omega(j)})_{j\geqslant 1} \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}}$ satisfy (4) and (20) with $0 < p_1 < 1$ (so $p_2 = 1 - p_1 > 0$). Let both Γ_1 and Γ_2 have uniformly horizontal fibres. Denote

$$t_i := \frac{|\Gamma_i|}{|\mathrm{proj}_y \Gamma_i|} \qquad \text{and} \qquad s_i := |\mathrm{proj}_y \Gamma_i| \quad \text{for } i = 1, 2.$$

Then by theorem 1.1 and corollary 1.2

$$\gamma := \dim_H \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \dim_B \Pi\left(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}\right) = \frac{p_1 \log(s_1 t_1^{\zeta}) + (1-p_1)\log(s_2 t_2^{\zeta})}{p_1 \log m_1 + (1-p_1)\log m_2}.$$

We claim that $\mathcal{H}^{\gamma}(\Pi(\prod_{i=1}^{\infty} \Gamma_{\omega(i)}))$ may take 0 for some properly selected Γ_1, Γ_2.

Proof. Let $2 \leqslant m_i \leqslant n_i, i = 1, 2$. We first assume that $m_1 = m_2 := m$. With the same probability measure $\tilde{\mu}_{\boldsymbol{P}}$ as that in example 1, it follows from (16) that for any $x = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} \Gamma_{\omega(j)}$ and any $\ell \in \mathbb{N}$

$$\begin{aligned}
&\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) \\
&\qquad = \frac{p_1 \log(s_1 t_1^{\zeta}) + (1-p_1)\log(s_2 t_2^{\zeta})}{p_1 \log m_1 + (1-p_1)\log m_2} (N_\ell(\Gamma_1)\log m_1 + N_\ell(\Gamma_2)\log m_2) \\
&\qquad - N_{\ell^*}(\Gamma_1)\log t_1 - N_\ell(\Gamma_1)\log s_1 - N_{\ell^*}(\Gamma_2)\log t_2 - N_\ell(\Gamma_2)\log s_2 \\
&\qquad = (p_1\ell - N_\ell(\Gamma_1))\log\frac{s_1}{s_2} + (p_1\ell^* - N_{\ell^*}(\Gamma_1))\log\frac{t_1}{t_2} \\
&\qquad + (\zeta\ell - \ell^*)(p_1\log t_1 + p_2 \log t_2).
\end{aligned} \tag{21}$$

We further assume $t_1 = t_2 = t$. So for any $x = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty} \Gamma_{\omega(j)}$ and any $\ell \in \mathbb{N}$

$$\gamma \sum_{i=1}^{\ell} \log m_{\omega(i)} + \log \tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = (p_1\ell - N_\ell(\Gamma_1))\log\frac{s_1}{s_2} + (\zeta\ell - \ell^*)\log t. \tag{22}$$

By (7) we have

$$0 \leqslant \ell\log m - N_{\ell^*}(\Gamma_1)\log n_1 - N_{\ell^*}(\Gamma_2)\log n_2 < \log\max\{n_1, n_2\} := \log n_{\max}$$

and so

$$0 \leqslant \zeta\ell - \frac{N_{\ell^*}(\Gamma_1)\log n_1 + N_{\ell^*}(\Gamma_2)\log n_2}{p_1\log n_1 + p_2 \log n_2} < \frac{\log n_{\max}}{p_1 \log n_1 + p_2 \log n_2}$$

since $\zeta = \frac{\log m}{p_1 \log n_1 + p_2 \log n_2}$. Therefore,

$$\begin{aligned}
\frac{\log n_1 - \log n_2}{p_1\log n_1 + p_2\log n_2}(N_{\ell^*}(\Gamma_1) - p_1\ell^*) &= \frac{N_{\ell^*}(\Gamma_1)\log n_1 + N_{\ell^*}(\Gamma_2)\log n_2}{p_1\log n_1 + p_2\log n_2} - \ell^* \\
&\leqslant \zeta\ell - \ell^* < \frac{N_{\ell^*}(\Gamma_1)\log n_1 + N_{\ell^*}(\Gamma_2)\log n_2}{p_1\log n_1 + p_2\log n_2} - \ell^* + \frac{\log n_{\max}}{p_1\log n_1 + p_2\log n_2} \\
&= \frac{\log n_1 - \log n_2}{p_1\log n_1 + p_2\log n_2}(N_{\ell^*}(\Gamma_1) - p_1\ell^*) + \frac{\log n_{\max}}{p_1\log n_1 + p_2\log n_2}.
\end{aligned} \tag{23}$$

By (20) one can take $\ell_j \uparrow \infty$ such that $\lim_{j\to\infty}(p_1\ell_j - N_{\ell_j}(\Gamma_1)) = +\infty$. With ℓ_j^* being determined by (6), let

$$\eta = \limsup_{j\to\infty}(N_{\ell_j^*}(\Gamma_1) - p_1\ell_j^*).$$

Thus, by (22) and (23) we have for any $x = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty}\Gamma_{\omega(j)}$

$$A(x) = \limsup_{\ell\to\infty}\left(\gamma\sum_{i=1}^{\ell}\log m_{\omega(i)} + \log\tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x))\right) = +\infty$$

if $s_1 > s_2$ and $(\log n_1 - \log n_2)\eta \geqslant 0$. So $\mathcal{H}^\gamma(\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})) = 0$ by lemma 2.1 (2). □

Example 3. There exists $\omega = (\Gamma_{\omega(j)})_{j\geqslant 1} \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}}$ satisfying (4) with $0 < p_1 < 1$ such that the Hausdorff measure of $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ in its dimension is positive finite but Γ_1 and Γ_2 are not of same type.

Proof. Let $2 \leqslant m_i \leqslant n_i, i = 1, 2$. Take $\omega = (\Gamma_{\omega(j)})_{j\geqslant 1} \in \{\Gamma_1, \Gamma_2\}^{\mathbb{N}}$ such that $-c \leqslant p_1\ell - N_\ell(\Gamma_1) \leqslant c$ for some positive constant c (this kind of ω always exists, e.g. one can take $\omega(2\ell - 1) = 1$ and $\omega(2\ell) = 2$ for $\ell \in \mathbb{N}$ when $p_1 = p_2$). Let both Γ_1 and Γ_2 have uniformly horizontal fibres. Let $s_i, t_i, i = 1, 2$ be defined as in example 2. Let $m_1 = m_2 := m$. Then by (21)

$$\gamma\sum_{i=1}^{\ell}\log m_{\omega(i)} + \log\tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x)) = (p_1\ell - N_\ell(\Gamma_1))\log\frac{s_1}{s_2} + (p_1\ell^* - N_{\ell^*}(\Gamma_1))\log\frac{t_1}{t_2}$$
$$+ (\zeta\ell - \ell^*)(p_1\log t_1 + p_2\log t_2).$$

for any $x = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty}\Gamma_{\omega(j)}$ and any $\ell \in \mathbb{N}$. By (23) we have that for any $x = (x_j)_{j\geqslant 1} \in \prod_{j=1}^{\infty}\Gamma_{\omega(j)}$

$$A(x) = \limsup_{\ell\to\infty}\left(\gamma\sum_{i=1}^{\ell}\log m_{\omega(i)} + \log\tilde{\mu}_{\boldsymbol{P}}(Q_\ell(x))\right) \in [a, b]$$

for some (finite) real numbers a and b. This implies that the Hausdorff measure of $\Pi(\prod_{i=1}^{\infty}\Gamma_{\omega(i)})$ in its dimension is positive finite by lemma 2.1 (3). □

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References

[1] Barański K 2007 Hausdorff dimension of the limit sets of some planar geometric constructions *Adv. Math.* **210** 215–45
[2] Bedford T 1984 Crinkly curves, Markov partitions and box dimension in self-similar sets *PhD Thesis* University of Warwick
[3] Billingsley P 1965 *Ergodic Theory and Information* (New York: Wiley)
[4] Chow Y S and Teicher H 1978 *Probability Theory* (New York: Springer)
[5] Falconer K J 1990 *Fractal Geometry–Mathematical Foundations and Applications* (Chichester: Wiley)
[6] Falconer K J 1997 *Techniques in Fractal Geometry* (New York: Wiley)
[7] Falconer K J 1988 The Hausdorff dimension of self-affine fractals *Math. Proc. Camb. Phil. Soc.* **103** 339–50

[8] Gatzouras D and Lalley S P 1992 Hausdorff and box dimensions of certain self-affine fractals *Indiana University Math. J.* **41** 533–68
[9] Gatzouras D and Lalley S P 1994 Statistically self-affine sets: Hausdorff and box dimensions *J. Theor. Probab.* **7** 437–68
[10] Gui Y X and Li W X 2007 Hausdorff dimension of subsets with proportional fibre frequencies of the general Sierpinski carpet *Nonlinearity* **20** 2353–64
[11] Gui Y X and Li W X 2008 A random version of McMullen–Bedford general Sierpinski carpets and its application *Nonlinearity* **21** 1745–58
[12] He X G, Lau K S and Rao H 2003 Self-affine sets and graph-directed systems *Constr. Approx.* **19** 373–97
[13] Kenyon R and Peres Y 1996 Measures of full dimension on affine-invariant sets *Ergodic Theory Dyn. Syst.* **16** 307–23
[14] Kenyon R and Peres Y 1996 Hausdorff dimensions of sofic affine-invariant sets *Israel J. Math.* **122** 540–74
[15] King J 1995 The singular spectrum for general Sierpinski carpets *Adv. Math.* **116** 1–8
[16] McMullen C 1984 The Hausdorff dimension of general Sierpinski carpets *Nagoya Math. J.* **96** 1–9
[17] Nielsen O A 1999 The hausdorff and packing dimensions of some sets related to sierpinski carpets *Can. J. Math.* **51** 1073–88
[18] Olsen L 1998 Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$ *Pac. J. Math.* **183** 99–143
[19] Peres Y 1994 The packing measure of self-affine carpets *Math. Proc. Camb. Phil. Soc.* **115** 437–50
[20] Peres Y 1994 The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure *Math. Proc. Camb. Phil. Soc.* **116** 513–26
[21] Rogers C and Taylor S J 1961 Function continuous and singular with respect to a Hausdorff measure *Mathematika* **15** 1–31