International Journal of Mathematics
Vol. 20, No. 10 (2009) 1289–1303

SUBSETS OF THE GENERAL SIERPINSKI CARPET WITH MIXED GROUP FREQUENCIES

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Received 4 October 2007

We consider a class of subsets of the general Sierpinski carpet which is characterized by insisting that the allowed digits in the expansion occur with prescribed mixing group frequencies, determine their Hausdorff dimensions and give the necessary and sufficient conditions for their corresponding Hausdorff measures to be positive and finite.

Keywords: Hausdorff dimension; Sierpinski carpets; mixing group frequencies; Hausdorff measure.

Mathematics Subject Classification 2000: 28A80, 28A78

1. Introduction

Let T be the expanding endomorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the matrix $\mathrm{diag}(n, m)$, where $2 \le m < n$ are integers. The simplest invariant sets for T have the form

$$K(T,D) = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k : d_k \in D \text{ for all } k \ge 1 \right\},$$

where $D \subseteq I \times J$ is a set of digits with $I = \{0, 1, \ldots, n-1\}$ and $J = \{0, 1, \ldots, m-1\}$. Alternatively, define a map $K_T : D^{\mathbb{N}} \longrightarrow \mathbb{T}^2$ by

$$K_T((d_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \begin{pmatrix} n^{-k} & 0 \\ 0 & m^{-k} \end{pmatrix} d_k.$$

Then $K(T, D) = K_T(D^{\mathbb{N}})$ which, as the simplest self-affine set, was named as the general Sierpinski carpet by McMullen in [8]. Hence, each point of $K_T(D^{\mathbb{N}})$ can be encoded by elements of $D^{\mathbb{N}}$ via the map K_T. We call $(d_k)_{k=1}^{\infty} \in D^{\mathbb{N}}$ a location code of $x \in K_T(D^{\mathbb{N}})$ if $K_T((d_k)_{k=1}^{\infty}) = x$. Note that some points in $K_T(D^{\mathbb{N}})$ may have multiple codes. However, one can show that $\sup_{x \in K_T(D^{\mathbb{N}})} \# K_T^{-1}(x) < \infty$. The set $K(T, D)$ was first studied by McMullen [8] and Bedford [2], independently, to determine its Hausdorff and box-counting dimensions. From then on, some further problems related to $K(T, D)$ are proposed and considered by lots of authors. Peres [11, 12] studied its packing and Hausdorff measures. Kenyon and Peres [5, 6] extended the results of McMullen [8] and Bedford [2] to the compact subsets of the 2-torus corresponding to shifts of finite type or sofic shifts and to the Sierpinski sponges. The singular spectrum was studied by King [7] for the general Sierpinski carpet, and later by Olsen [10] for the Sierpinski sponges. Some other results related to the general Sierpinski carpet can be found, e.g. in [13, 14, 15]. As we know, many interesting results have been established for the study of certain subsets of self-similar sets, e.g. in the so-called multifractal analysis. Some detailed description on this topic and recent developments are included in [1]. Unfortunately, lesser analogous results are discovered for the general self-affine sets.

Let $\mathbf{p} = (p_d)_{d\in D}$ be a probability vector on D, i.e. $0 < p_d < 1$ and $\sum_{d\in D} p_d = 1$. For any $x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}$, any positive integer k, and any $d \in D$ let

$$N_k(x, d) = |\{j : 1 \le j \le k \text{ and } x_j = d\}|, \tag{1.1}$$

where and throughout this paper we use $|A|$ to denote the cardinality of a finite set A. Let

$$L(D, \mathbf{p}) = \left\{ x \in D^{\mathbb{N}} : \lim_{k\to\infty} \frac{N_k(x, d)}{k} = p_d \text{ for all } d \in D \right\}. \tag{1.2}$$

Thus $L(D, \mathbf{p})$ consists of those elements of $D^{\mathbb{N}}$ for which each $d \in D$ occurs with the prescribed frequency p_d. The resulting subsets $K_T(L(D, \mathbf{p}))$ of $K_T(D^{\mathbb{N}})$ were studied by Nielsen [9] (recall that the multifractal decompositions of dynamically-defined fractals are, more or less, achieved in this way).

Let σ denote the projection of $\mathbb{R}^2$ onto its second coordinate. Let

$$\alpha = \log_n m, \quad \theta = \frac{\alpha - 1}{\alpha}, \quad B = \sigma(D) \quad \text{and} \quad n_b = |D \cap (I \times \{b\})| \quad \text{for } b \in B.$$

For a given probability vector $\mathbf{p} = (p_d)_{d\in D}$, it induces a probability vector on B,

$$\mathbf{q} = (q_b)_{b\in B}, \quad \text{where } q_b = \sum_{d\in D\cap(I\times\{b\})} p_d.$$

The vector $\mathbf{p}$ is said to be *uniformly distributed* on D if $p_d = |D|^{-1}$ for all $d \in D$ and D is said to have *uniform horizontal fibers* if $n_b = n_{b'}$ for all $b, b' \in B$. Nielson [9] obtained the following results:

(R1) $\dim_H K_T(L(D, \mathbf{p})) = \dim_P K_T(L(D, \mathbf{p})) = -\alpha \sum_{d\in D} p_d \log_m p_d - (1 - \alpha) \sum_{b\in B} q_b \log_m q_b$.

(R2) $\dim_B K_T(L(D, \mathbf{p})) = \dim_B K_T(D^{\mathbb{N}}) = \log_m(|B|^{1-\alpha}|D|^{\alpha})$.

(R3) Let γ denote the common value of $\dim_H K_T(L(D, p))$ and $\dim_P K_T(L(D, \mathbf{p}))$,

  (a) If $\mathbf{p}$ is uniformly distributed on D and D has uniform horizontal fibers then

$$0 < \mathcal{H}^{\gamma}(K_T(L(D, \mathbf{p})) \le \mathcal{P}^{\gamma}(K_T(L(D, \mathbf{p})) < \infty;$$

(b) If $\mathbf{p}$ is not uniformly distributed on D or D does not have any uniform horizontal fiber then

$$\mathcal{H}^{\gamma}(K_T(L(D,\mathbf{p}))) = \mathcal{P}^{\gamma}(K_T(L(D,\mathbf{p}))) = \infty.$$

Although, as a special class of self-affine sets, the general Sierpinski carpet $K_T(D^{\mathbb{N}})$ is generated in a simple way and so has a simple algebraic representation. It essentially differs from self-similar sets, such that unless D has uniform horizontal fibers, $\dim_H K_T(D^{\mathbb{N}}) < \dim_P K_T(D^{\mathbb{N}})$ and $\dim_P K_T(L(D,\mathbf{p})) < \dim_P K_T(D^{\mathbb{N}})$ for each $\mathbf{p} = (p_d)_{d\in D}$.

In the present paper, we like to investigate another class of subsets of the general Sierpinski carpet, which extends Nielson's results, in some sense, into a more complicated case. Now let $\Gamma_1 \subset D, \Gamma_2 \subset D$ such that $\Gamma_1 \bigcap \Gamma_2 \neq \emptyset$ and $\Gamma_1 \bigcup \Gamma_2$ is a proper subset of D. Let $0 < c_j < 1, j = 1,2$. Similar to (1.1) and (1.2), a subset of $D^{\mathbb{N}}$ now can be defined as follows. For any $x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}$, any positive integer k, and any $d \in D$, $\mathbf{c} = (c_1, c_2)$ let

$$L(D,\Gamma_1,\Gamma_2,\mathbf{c}) = \left\{ x \in D^{\mathbb{N}} : \lim_{k\to\infty} \frac{N_k(x,\Gamma_j)}{k} = c_j \text{ for } j = 1,2, \right\}, \tag{1.3}$$

where

$$N_k(x,\Gamma_j) = |\{i : 1 \le i \le k \text{ and } x_i \in \Gamma_j\}|. \tag{1.4}$$

Thus $L(D,\Gamma_1,\Gamma_2,\mathbf{c})$ consists of those elements of $D^{\mathbb{N}}$ for which the digits in each Γ_j occur with the prescribed mixing *group* frequency c_j. It is clear that $K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c}))$ is T-invariant, dense in $D^{\mathbb{N}}$ but not compact . Let

$$\Xi = \left\{ \mathbf{p} = (p_d)_{d\in D} : p_d \in (0,1),\ \sum_{d\in D} p_d = 1 \text{ and } \sum_{d\in\Gamma_j} p_d = c_j,\ j = 1,2 \right\}. \tag{1.5}$$

Obviously, $\Xi \neq \emptyset$. Let

$$\Delta_3 = \Gamma_1 \cap \Gamma_2; \quad \Delta_1 = \Gamma_1 \backslash \Delta_3; \quad \Delta_2 = \Gamma_2 \backslash \Delta_3 \quad \text{and} \quad \Delta_4 = D \backslash \Gamma_1 \cup \Gamma_2.$$

Then $\{\Delta_i\}_{i=1}^4$ is a partition of D and $\Delta_1 \cup \Delta_3 = \Gamma_1$, $\Delta_2 \cup \Delta_3 = \Gamma_2$. Clearly, we have

$$L(D,\Gamma_1,\Gamma_2,\mathbf{c}) \supseteq \bigcup_{\mathbf{p}\in\Xi} L(D,\mathbf{p}) \tag{1.6}$$

with proper inclusion and uncountable union on the right side. From (1.6) and (R2) it follows that

$$\dim_B K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c})) = \log_m(|B|^{1-\alpha}|D|^{\alpha}).$$

For each $\mathbf{p} \in \Xi$, let

$$\begin{aligned} f(\mathbf{p}) &= -\alpha \sum_{d \in D} p_d \log_m p_d - (1-\alpha) \sum_{b \in B} q_b \log_m q_b \\ &= -\alpha \sum_{d \in D} p_d \log_m p_d - (1-\alpha) \sum_{d \in D} p_d \log_m q_{\sigma(d)}. \end{aligned} \tag{1.7}$$

One can continuously extend $f(\mathbf{p})$ to $\mathrm{cl}(\Xi)$ by taking $0 \log 0 = 0$. Then $f(\mathbf{p})$ can obtain its maximum $f_{\max}$ on $\mathrm{cl}(\Xi)$. In fact, as shown in Proposition 2.1 $f_{\max}$ is uniquely attained at some $\mathbf{p}^* = (p_d^*)_{d \in D} \in \Xi$ which is determined by (2.1) in a more explicit way. Therefore, $\dim_H K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c})) \geq f_{\max} = f(\mathbf{p}^*)$ by (1.6) and (R1). However, our main result shows that the opposite inequality also holds. Therefore,

$$\dim_H K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c})) = \max_{\mathbf{p} \in \Xi} \dim_H K_T(L(D, \mathbf{p})), \tag{1.8}$$

i.e. the Hausdorff dimension of $K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c}))$ is carried by a single subset for which all the digit frequencies are known. In this paper, we obtain the following results.

Theorem 1.1. *Let* $\theta = \frac{\alpha - 1}{\alpha}$. *Let* $\mathbf{p}^* = (p_d^*)_{d \in D} \in \Xi$ *be the unique point such that* $f(\mathbf{p}^*) = f_{\max}$ *and* $(q_b^*)_{b \in B}$ *be the probability vector induced by* $\mathbf{p}^*$. *Then*

$$\begin{aligned} &\dim_H K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c})) \\ &\quad = \max_{\mathbf{p} \in \Xi} \dim_H K_T(L(D, \mathbf{p})) = f(\mathbf{p}^*) \\ &\quad = \alpha \sum_{j=1}^{2} c_j \log_m \frac{\sum_{d \in \Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d \in \Delta_j} p_d^*} + \alpha(1 - c_1 - c_2) \log_m \frac{\sum_{d \in \Delta_4} q_{\sigma(d)}^{*\theta}}{\sum_{d \in \Delta_4} p_d^*} \\ &\quad = \sum_{j=1}^{2} c_j \log_n \frac{\sum_{d \in \Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d \in \Delta_j} p_d^*} + (1 - c_1 - c_2) \log_n \frac{\sum_{d \in \Delta_4} q_{\sigma(d)}^{*\theta}}{\sum_{d \in \Delta_4} p_d^*}, \end{aligned}$$

where $L(D, \Gamma_1, \Gamma_2, \mathbf{c})$ *is defined as in* (1.3).

As done in [9], the proof of our Theorem 1.1 is also based on Lemma 2.1 (cf. [9, Lemma 4; 12, Sec. 2]). Hence, we need to estimate $\limsup_{k \to \infty} k^{-1} \log_m \widetilde{\mu}_{\mathbf{p}}(Q_k(x))$ for each $x \in K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c}))$. When $x \in K_T(L(D, \mathbf{p}))$ (the case considered in [9]), this supremum limit (in fact, the limit) equals to a constant independent of x. Unfortunately, it is not the case if $x \in K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c}))$ (the case considered in the present paper), and furthermore it is hard to investigate the infimum limit. This, as well as the overlap of $\Gamma_i, i = 1, 2$, makes the present problem more complicated, and so some delicate techniques are needed. This is why we only consider the relatively simple case related to only two sets Γ_1 and Γ_2 and lack obtaining its packing dimension. However, we believe that the property shown in (1.8) still holds for multiple Γ_i's (this is true, given by the authors in [3], when all Γ_i's are pairwise disjoint). To prove it, some new ideas and techniques may be needed since the method used in the present paper depends on the relatively more explicit expression, shown in (2.1), for $\mathbf{p}^* = (p_d^*)_{d\in D}$. For the general case, such expressions are hard to find. As to the Hausdorff measure of $K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c}))$ in its dimension, we have the following theorem. The first part can be deduced similarly as [9], while the second part follows directly from (b) in (R3).

Theorem 1.2. *Let* $\gamma = \dim_H K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c}))$.

(I) *If* $c_j = \frac{|\Gamma_j|}{|D|}$ *for all* $1 \le j \le 2$ *and* D *has uniform horizontal fibers, then*

$$0 < \mathcal{H}^\gamma(K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c}))) < \infty;$$

(II) *If there exists some* $1 \le j \le 2$ *such that* $c_j \ne \frac{|\Gamma_j|}{|D|}$ *or* D *does not have uniform horizontal fibers, then* $\mathcal{H}^\gamma(K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c}))) = \infty$.

The rest of this paper is organized as follows. In Sec. 2, some basic facts and known results needed in the proof of our theorems are described. Proofs of Theorems 1.1 and 1.2 are arranged in Sec. 3.

2. Preliminaries

Following [9], one can use approximate squares as covering sets to calculate dimension. For each point $x = (x_n)_{n=1}^\infty \in (I \times J)^{\mathbb{N}}$ and each positive integer k, we denote by $Q_k(x)$, the set consisting of all points $K_T(y)$, with $y \in (I\times J)^{\mathbb{N}}$ satisfying $y_j = x_j$ for $1 \le j \le [\alpha k]$, and $\sigma(y_j) = \sigma(x_j)$ for $[\alpha k]+1 \le j \le k$. Here $[t]$, as usual, denotes the greatest integer not more than t ($\in \mathbb{R}$). The sets $Q_k(x)$ are approximate squares in $[0,1]^2$, whose sides have length $n^{-[\alpha k]}$ and m^{-k}. Note that the radio of the sizes of $Q_k(x)$ is at most n, and their diameters $\mathrm{diam} Q_k(x)$ satisfy

$$\sqrt{2}m^{-k} \le \mathrm{diam} Q_k(x) \le \sqrt{2}nm^{-k},$$

and so in the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares $Q_k(x)$.

The following lemma comes from Lemma 4 in [9] which will be used in this paper.

Lemma 2.1. *Suppose that* μ *is a finite Borel measure* $[0,1]^2$, *and that* E *is a subset of* $(I \times J)^{\mathbb{N}}$ *such that* $K_T(E)$ *is a Borel subset of* $[0,1]^2$ *with* $\mu(K_T(E)) > 0$. *Let* δ *be a positive number. For each point* $x \in E$, *put*

$$A(x) = \limsup_{k\to\infty}(k\delta + \log_m \mu(Q_k(x))).$$

(1) *If* $A(x) = -\infty$ *for all* $x \in E$, *then* $\mathcal{H}^\delta(K_T(E)) = +\infty$;
(2) *If* $A(x) = +\infty$ *for all* $x \in E$, *then* $\mathcal{H}^\delta(K_T(E)) = 0$;
(3) *If there are real numbers* a *and* b *such that* $a \le A(x) \le b$ *for all* $x \in E$, *then* $0 < \mathcal{H}^\delta(K_T(E)) < +\infty$.

The Borel measures on $[0,1]^2$, to which the above lemmas will be applied, are constructed as follows. Let $\mathbf{p}=(p_d)_{d\in D}$ be a probability vector on D, i.e. $\sum_{d\in D}p_d=1$ with each $p_d\in(0,1)$. Then $\mathbf{p}$ determines a unique infinite product Borel probability measure, denoted by $\mu_{\mathbf{p}}$, on $D^{\mathbb{N}}$. For any finite sequence $(x_1,x_2,\ldots,x_k)\in D^k$

$$\mu_{\mathbf{p}}(\mathbb{C}(x_1,x_2,\ldots,x_k))=\prod_{j=1}^{k}p_{x_j},$$

where $\mathbb{C}(x_1,x_2,\ldots,x_k):=\{d=(d_j)_{j=1}^{\infty}\in D^{\mathbb{N}}:d_j=x_j \text{ for } 1\le j\le k\}$ is a cylinder set of $D^{\mathbb{N}}$ with base $(x_1,x_2,\ldots,x_k)$. Let $\widetilde{\mu}_{\mathbf{p}}$ be the Borel probability measure on $K_T(D^{\mathbb{N}})$ which is the image measure of $\mu_{\mathbf{p}}$ under K_T, i.e. $\widetilde{\mu}_{\mathbf{p}}(A)=\mu_{\mathbf{p}}(K_T^{-1}A)$ for Borel set $A\subseteq\mathbb{R}^2$. From the definition of $Q_k(x)$, it follows that for any $x\in D^{\mathbb{N}}$ (cf. [9, formula (4)], also [4, formula (4.4)])

$$\widetilde{\mu}_{\mathbf{p}}(Q_k(x))=\prod_{j=1}^{[\alpha k]}p_{x_j}\cdot\prod_{j=[\alpha k]+1}^{k}q_{\sigma(x_j)}.$$

By means of the Law of Large Numbers, we have $\widetilde{\mu}_{\mathbf{p}}(K_T(L(D,\mathbf{p}))=1$ for any $\mathbf{p}\in\Xi$. So

$$\widetilde{\mu}_{\mathbf{p}}(K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c})))=1\quad\text{for any } \mathbf{p}\in\Xi,$$

since $K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c}))\supseteq K_T(L(D,\mathbf{p}))$.

The following proposition shows that the function $f(\mathbf{p})$ attains its maximum $f_{\max}$ uniquely at some point $\mathbf{p}^*=(p_d^*)_{d\in D}\in\Xi$ and characterizes $\mathbf{p}^*$ in a more explicit way.

Proposition 2.1. *Let $f(\mathbf{p})$ be given by* (1.7)*. Then there exists a unique point $\mathbf{p}^*=(p_d^*)_{d\in D}\in\Xi$ (Ξ is defined in* (1.5)*) such that $f(\mathbf{p}^*)=f_{\max}$. Moreover, this point $\mathbf{p}^*=(p_d^*)_{d\in D}$ is uniquely determined by*

$$\begin{cases} p_d=\dfrac{q_{\sigma(d)}^{\theta}}{\sum_{d\in\Delta_j}q_{\sigma(d)}^{\theta}}\displaystyle\sum_{d\in\Delta_j}p_d, & \text{for } d\in\Delta_j,\ \ j=1,2,3,4,\\ \displaystyle\sum_{i=1}^{2}\log_m\frac{\sum_{d\in\Delta_i}q_{\sigma(d)}^{\theta}}{\sum_{d\in\Delta_i}p_d}-\sum_{i=3}^{4}\log_m\frac{\sum_{d\in\Delta_i}q_{\sigma(d)}^{\theta}}{\sum_{d\in\Delta_i}p_d}=0, & \\ \displaystyle\sum_{d\in\Delta_i}p_d+\sum_{d\in\Delta_3}p_d=c_i, & \text{for } i=1,2,\\ \displaystyle\sum_{i=1}^{4}\sum_{d\in\Delta_i}p_d=1, & \\ 0<p_d<1, & \text{for } d\in D. \end{cases}\tag{2.1}$$

where, as before $\theta=\frac{\alpha-1}{\alpha}$, and $(q_b)_{b\in B}$ is induced by $(p_d)_{d\in D}$.

Proof. Clearly, $f(\mathbf{p})$ can obtain its maximum on $\mathrm{cl}(\Xi)$, since $f(\mathbf{p})$ is continuous and $\mathrm{cl}(\Xi)$ is compact. We first show that the maximum point is unique. Note that

$f(\mathbf{p})$ is a strictly concave function in $\mathbf{p}$. In fact, the first summand of $f(\mathbf{p})$ is strictly concave and the second is concave. On the other hand, $\mathrm{cl}(\Xi)$ is convex, the constraint inequalities are both convex and concave and its constraint equalities are all linear. By a well-known property of convex programming , there exists a unique $\mathbf{p}^* = (p_d^*)_{d\in D} \in \mathrm{cl}(\Xi)$ such that $f(\mathbf{p})$ attains its maximum at the point $\mathbf{p}^*$.

We then show that the maximum of $f(\mathbf{p})$ is obtained in Ξ, equivalently, that $\mathbf{p}^* \in \Xi$. Let

$$Z_1(\mathbf{p}) = -\alpha \sum_{d\in D} p_d \log_m p_d \quad \text{and} \quad Z_2(\mathbf{p}) = (\alpha - 1)\sum_{b\in B} q_b \log_m q_b.$$

Then $f(\mathbf{p}) = Z_1(\mathbf{p}) + Z_2(\mathbf{p})$. Suppose $\mathbf{p}^* = (p_d^*)_{d\in D} \in \mathrm{cl}(\Xi)\backslash\Xi$. Let $D_1 = \{d \in D : p_d^* = 0\}$ and $D_2 = D\backslash D_1$. Then both D_1 and D_2 are nonempty. Take $\widetilde{\mathbf{p}} = (\widetilde{p}_d)_{d\in D} \in \Xi$. Let $\mathbf{p}_t = t\widetilde{\mathbf{p}} + (1-t)\mathbf{p}^* = (t\widetilde{p}_d + (1-t)p_d^*)_{d\in D},\ t \in [0,1]$. Then $\mathbf{p}_t \in \Xi$ for $t \in (0,1]$ and $\mathbf{p}_0 = \mathbf{p}^*$. Note that

$$\begin{aligned}
&\frac{d}{dt} Z_1(\mathbf{p}_t)\\
&\quad= -\alpha \frac{d}{dt}\left(\sum_{d\in D}(t\widetilde{p}_d + (1-t)p_d^*)\log_m(t\widetilde{p}_d + (1-t)p_d^*)\right)\\
&\quad= -\alpha \sum_{d\in D}(\widetilde{p}_d - p_d^*)\log_m(t\widetilde{p}_d + (1-t)p_d^*)\\
&\quad= -\alpha\left(\sum_{d\in D_1}\widetilde{p}_d \log_m(t\widetilde{p}_d + (1-t)p_d^*) + \sum_{d\in D_2}(\widetilde{p}_d - p_d^*)\log_m(t\widetilde{p}_d + (1-t)p_d^*)\right).
\end{aligned}$$

Thus we have $\lim_{t\to 0+} Z_1'(\mathbf{p}_t) = +\infty$. The same argument shows that $\lim_{t\to 0+} Z_2'(\mathbf{p}_t) = +\infty$ if $q_b^* = 0$ for some $b \in B$, or equals to a finite real number. Therefore, $\lim_{t\to 0+} f'(\mathbf{p}_t) = +\infty$. Note that $\lim_{t\to 0+} f(\mathbf{p}_t) = f(\mathbf{p}^*)$. Thus, $f(\mathbf{p}_t) > f(\mathbf{p}^*) = f_{\max}$ when t is small enough, leading a contradiction.

Now let

$$\begin{aligned}
G(\mathbf{p},\lambda_1,\lambda_2,u) = {}& -\alpha\sum_{d\in D} p_d \log_m p_d - (1-\alpha)\sum_{b\in B} q_b \log_m q_b\\
&+ \sum_{j=1}^{2}\frac{\lambda_j}{\log m}\left(\sum_{d\in\Gamma_j} p_d - c_j\right) + \frac{u}{\log m}\left(\sum_{d\in D} p_d - 1\right).
\end{aligned}$$

Since $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Xi$ is the unique point such that $f(\mathbf{p}^*) = \max_{\mathbf{p}\in\Xi} f(\mathbf{p})$ and $f(\mathbf{p})$ is a strictly concave function in $\mathbf{p}$, $\mathbf{p}^*$ is uniquely solved by method of Lagrange multipliers

$$\begin{cases}
\dfrac{\partial G}{\partial p_d} = 0, & d \in D,\\[2mm]
\dfrac{\partial G}{\partial \lambda_i} = 0, & i = 1,2,\\[2mm]
\dfrac{\partial G}{\partial u} = 0;
\end{cases}$$

i.e.

$$\begin{cases} -\alpha(\log p_d + 1) - (1-\alpha)(\log q_{\sigma(d)} + 1) + \lambda_1 + u = 0, & d \in \Delta_1, \\ -\alpha(\log p_d + 1) - (1-\alpha)(\log q_{\sigma(d)} + 1) + \lambda_2 + u = 0, & d \in \Delta_2, \\ -\alpha(\log p_d + 1) - (1-\alpha)(\log q_{\sigma(d)} + 1) + \lambda_1 + \lambda_2 + u = 0, & d \in \Delta_3, \\ -\alpha(\log p_d + 1) - (1-\alpha)(\log q_{\sigma(d)} + 1) + u = 0, & d \in \Delta_4, \\ \displaystyle\sum_{d\in\Delta_j} p_d + \sum_{d\in\Delta_3} p_d = c_j, & j = 1, 2, \\ \displaystyle\sum_{d\in D} p_d = 1. \end{cases} \tag{2.2}$$

From the first equality of (2.2) it follows that

$$p_d = q_{\sigma(d)}^{\theta} e^{\alpha^{-1}(\lambda_1+u-1)}, \quad d \in \Delta_1,$$

which gives

$$e^{\alpha^{-1}(\lambda_1+u-1)} = \frac{\sum_{d\in\Delta_1} p_d}{\sum_{d\in\Delta_1} q_{\sigma(d)}^{\theta}}.$$

Thus,

$$p_d = q_{\sigma(d)}^{\theta} e^{\alpha^{-1}(\lambda_1+u-1)} = \frac{q_{\sigma(d)}^{\theta}}{\sum_{d\in\Delta_1} q_{\sigma(d)}^{\theta}} \sum_{d\in\Delta_1} p_d, \quad d \in \Delta_1.$$

Therefore, we have (the first equality in (2.1))

$$p_d = \frac{q_{\sigma(d)}^{\theta}}{\sum_{d\in\Delta_j} q_{\sigma(d)}^{\theta}} \sum_{d\in\Delta_j} p_d, \quad \text{for } d \in \Delta_j, \ \ j = 1, 2, 3, 4,$$

in the same way. Note that the probability vector $(p_d)_{d\in D}$ described above (i.e. the probability vector $\mathbf{p}^* = (p_d^*)_{d\in D}$) has a uniform distribution on each horizontal fibre of Δ_j, i.e. $p_d = p_{d'}$ for $d, d' \in \Delta_j$ with $\sigma(d) = \sigma(d')$.

Taking logarithm of above expression we have

$$\log p_d = \theta \log q_{\sigma(d)} + \log \sum_{d\in\Delta_j} p_d - \log \sum_{d\in\Delta_j} q_{\sigma(d)}^{\theta}, \quad \text{for } d \in \Delta_j, \ \ j = 1, 2, 3, 4.$$

It follows that for $d \in \Delta_j, j = 1, 2, 3, 4$,

$$\alpha \log p_d + (1-\alpha) \log q_{\sigma(d)} = \alpha \left(\log \sum_{d\in\Delta_j} p_d - \log \sum_{d\in\Delta_j} q_{\sigma(d)}^{\theta} \right). \tag{2.3}$$

Combining this with (2.2) we have (the second equality in (2.1))

$$\sum_{j=1}^{2} \left(\log \sum_{d\in\Delta_j} p_d - \log \sum_{d\in\Delta_j} q_{\sigma(d)}^{\theta} \right) - \sum_{j=3}^{4} \left(\log \sum_{d\in\Delta_j} p_d - \log \sum_{d\in\Delta_j} q_{\sigma(d)}^{\theta} \right) = 0.$$

Therefore, $\mathbf{p}^* = (p_d^*)_{d\in D}$ is uniquely determined by (2.1). $\square$

By means of (2.1) we can rewrite $f(\mathbf{p}^*)$ as

$$f(\mathbf{p}^*) = \alpha \sum_{j=1}^{2} c_j \log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*} + \alpha(1-c_1-c_2)\log_m \frac{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_4} p_d^*}. \tag{2.4}$$

In fact, from the second equality in (2.1) it follows that

$$\log_m \frac{\sum_{d\in\Delta_3} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_3} p_d^*} = \sum_{i=1}^{2} \log_m \frac{\sum_{d\in\Delta_i} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_i} p_d^*} - \log_m \frac{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_4} p_d^*}. \tag{2.5}$$

Thus we have

$$\begin{aligned}
f(\mathbf{p}^*) &= -\alpha \sum_{d\in D} p_d^* \log_m p_d^* - (1-\alpha)\sum_{d\in D} p_d^* \log_m q_{\sigma(d)}^* \\
&= -\sum_{d\in D} p_d^*(\alpha \log p_d^* + (1-\alpha)\log q_{\sigma(d)}^*) \\
&= -\sum_{j=1}^{4}\sum_{d\in\Delta_j} p_d^*(\alpha \log p_d^* + (1-\alpha)\log q_{\sigma(d)}^*) \\
&= \alpha \sum_{j=1}^{4}\sum_{d\in\Delta_j} p_d^* \left(\log_m \sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta} - \log_m \sum_{d\in\Delta_j} p_d^*\right) \\
&= \alpha \sum_{j\in\{1,2,4\}}\sum_{d\in\Delta_j} p_d^* \log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*} + \alpha \sum_{d\in\Delta_3} p_d^* \log_m \frac{\sum_{d\in\Delta_3} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_3} p_d^*} \\
&= \alpha \sum_{j\in\{1,2,4\}}\sum_{d\in\Delta_j} p_d^* \log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*} \\
&\quad + \alpha \sum_{d\in\Delta_3} p_d^* \left(\sum_{j=1}^{2} \log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*} - \log_m \frac{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_4} p_d^*}\right) \\
&= \alpha \sum_{j=1}^{2} c_j \log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*} + \alpha(1-c_1-c_2)\log_m \frac{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_4} p_d^*},
\end{aligned}$$

where the fourth equality follows from (2.3), the sixth equality from (2.5) and the last equality from the third and fourth equalities in (2.1).

3. Proofs

We first prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] Now let $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Xi$ be the unique point such that $f(\mathbf{p}^*) = f_{\max}$, as described in Proposition 2.1. As discussed in Sec. 1, we only need to prove

$$\dim_H K_T(L(D,\Gamma_1,\Gamma_2,\mathbf{c})) \le f(\mathbf{p}^*) = \alpha \sum_{j=1}^{2} c_j \log_m \frac{\sum_{d\in\Delta_j} q^{*\theta}_{\sigma(d)}}{\sum_{d\in\Delta_j} p^*_d} + \alpha(1-c_1-c_2)\log_m \frac{\sum_{d\in\Delta_4} q^{*\theta}_{\sigma(d)}}{\sum_{d\in\Delta_4} p^*_d}.$$

By (2.1), we have

$$\log p^*_d - \log q^*_{\sigma(d)} = (\theta-1)\log q^*_{\sigma(d)} + \log\sum_{d\in\Delta_j} p^*_d - \log\sum_{d\in\Delta_j} q^{*\theta}_{\sigma(d)}, \tag{3.1}$$

for $d \in \Delta_j,\ j = 1,2,3,4$. By the definitions of $N_k(x,d)$ and $N_k(x,\Gamma_j)$ in (1.1) and (1.4) respectively, we have that for any $x = (x_i)_{i=1}^\infty \in D^{\mathbb{N}}$ and any $k \in \mathbb{N}$

$$N_k(x,\Gamma_j) = \sum_{d\in\Gamma_j} N_k(x,d), \quad j = 1,2.$$

Let

$$S_j(k,x) = \sum_{d\in\Delta_j} N_k(x,d)\log_m q^*_{\sigma(d)}, \quad j = 1,2,3,4. \tag{3.2}$$

For any $x = (x_i)_{i=1}^\infty \in L(D,\Gamma_1,\Gamma_2,\mathbf{c})$, we have

$$\begin{aligned}
&\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x))\\
&= \sum_{i=1}^{[\alpha k]} \log_m p^*_{x_i} + \sum_{i=[\alpha k]+1}^{k} \log_m q^*_{\sigma(x_i)}\\
&= \sum_{j=1}^{4}\sum_{d\in\Delta_j} N_{[\alpha k]}(x,d)\log_m p^*_d + \sum_{j=1}^{4}\sum_{d\in\Delta_j} (N_k(x,d) - N_{[\alpha k]}(x,d))\log_m q^*_{\sigma(d)}\\
&= \sum_{j=1}^{4}\sum_{d\in\Delta_j} N_{[\alpha k]}(x,d)(\log_m p^*_d - \log_m q^*_{\sigma(d)}) + \sum_{j=1}^{4}\sum_{d\in\Delta_j} N_k(x,d)\log_m q^*_{\sigma(d)}\\
&= \sum_{j=1}^{4}\sum_{d\in\Delta_j} N_{[\alpha k]}(x,d)\left((\theta-1)\log_m q^*_{\sigma(d)} + \log_m\sum_{d\in\Delta_j} p^*_d - \log_m\sum_{d\in\Delta_j} q^{*\theta}_{\sigma(d)}\right)\\
&\quad + \sum_{j=1}^{4}\sum_{d\in\Delta_j} N_k(x,d)\log_m q^*_{\sigma(d)}\\
&= \sum_{j=1}^{4}\sum_{d\in\Delta_j} N_{[\alpha k]}(x,d)\left(\log_m\sum_{d\in\Delta_j} p^*_d - \log_m\sum_{d\in\Delta_j} q^{*\theta}_{\sigma(d)}\right)\\
&\quad + \sum_{j=1}^{4}\sum_{d\in\Delta_j} (N_k(x,d) - (1-\theta)N_{[\alpha k]}(x,d))\log_m q^*_{\sigma(d)},
\end{aligned} \tag{3.3}$$

by (3.1). Note that

$$\sum_{d\in\Gamma_j} N_{[\alpha k]}(x,d) = \sum_{d\in\Delta_j} N_{[\alpha k]}(x,d) + \sum_{d\in\Delta_3} N_{[\alpha k]}(x,d), \quad \text{for } j=1,2,$$

and

$$\sum_{d\in\Delta_4} N_{[\alpha k]}(x,d) = [\alpha k] - \sum_{d\in\Gamma_1} N_{[\alpha k]}(x,d) - \sum_{d\in\Gamma_2} N_{[\alpha k]}(x,d) + \sum_{d\in\Delta_3} N_{[\alpha k]}(x,d).$$

Thus we can rewrite (3.3) to be

$$\begin{aligned}
&\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \\
&\quad = \sum_{j=1}^{2}\left(\sum_{d\in\Gamma_j} N_{[\alpha k]}(x,d) - \sum_{d\in\Delta_3} N_{[\alpha k]}(x,d)\right)\log_m \frac{\sum_{d\in\Delta_j} p_d^*}{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}} \\
&\qquad + \sum_{d\in\Delta_3} N_{[\alpha k]}(x,d)\log_m \frac{\sum_{d\in\Delta_3} p_d^*}{\sum_{d\in\Delta_3} q_{\sigma(d)}^{*\theta}} \\
&\qquad + \left([\alpha k] - \sum_{d\in\Gamma_1} N_{[\alpha k]}(x,d) - \sum_{d\in\Gamma_2} N_{[\alpha k]}(x,d) + \sum_{d\in\Delta_3} N_{[\alpha k]}(x,d)\right) \\
&\qquad \times \log_m \frac{\sum_{d\in\Delta_4} p_d^*}{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}} + \sum_{j=1}^{4}\left(S_j(k,x) - \frac{1}{\alpha}S_j([\alpha k],x)\right) \\
&\quad = \sum_{j=1}^{2} N_{[\alpha k]}(x,\Gamma_j)\log_m \frac{\sum_{d\in\Delta_j} p_d^*}{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}} \\
&\qquad + \left([\alpha k] - \sum_{j=1}^{2} N_{[\alpha k]}(x,\Gamma_j)\right)\log_m \frac{\sum_{d\in\Delta_4} p_d^*}{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}} \\
&\qquad + \sum_{d\in\Delta_3} N_{[\alpha k]}(x,d)\left(\sum_{j=1}^{2}\log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*} - \sum_{j=3}^{4}\log_m \frac{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}}{\sum_{d\in\Delta_j} p_d^*}\right) \\
&\qquad + \sum_{j=1}^{4}\left(S_j(k,x) - \frac{1}{\alpha}S_j([\alpha k],x)\right) \\
&\quad = \sum_{j=1}^{2} N_{[\alpha k]}(x,\Gamma_j)\log_m \frac{\sum_{d\in\Delta_j} p_d^*}{\sum_{d\in\Delta_j} q_{\sigma(d)}^{*\theta}} + \left([\alpha k] - \sum_{j=1}^{2} N_{[\alpha k]}(x,\Gamma_j)\right) \\
&\qquad \times \log_m \frac{\sum_{d\in\Delta_4} p_d^*}{\sum_{d\in\Delta_4} q_{\sigma(d)}^{*\theta}} + \sum_{j=1}^{4}\left(S_j(k,x) - \frac{1}{\alpha}S_j([\alpha k],x)\right).
\end{aligned}$$

Therefore, for each $x = (x_i)_{i=1}^{\infty} \in L(D, \Gamma_1, \Gamma_2, \mathbf{c})$, we have

$$\limsup_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) = -f(\mathbf{p}^*) + \limsup_{k\to\infty} \sum_{j=1}^{4} \left(\frac{S_j(k,x)}{k} - \frac{S_j([\alpha k], x)}{\alpha k} \right),$$

where $f(\mathbf{p}^*)$ is given in (2.4).

In the following, we will show that for each $x \in L(D, \Gamma_1, \Gamma_2, \mathbf{c})$,

$$\limsup_{k\to\infty} \sum_{j=1}^{4} \left(\frac{S_j(k,x)}{k} - \frac{S_j([\alpha k], x)}{\alpha k} \right) \geq 0. \tag{3.4}$$

Essentially this can be derived from [5, Lemma 4.1]. Note that for each point $x \in L(D, \Gamma_1, \Gamma_2, \mathbf{c})$ and any $j = 1, 2, 3, 4$

$$\sup_k |S_j(k+1, x) - S_j(k, x)| < \infty \tag{3.5}$$

by (3.2). For a fixed $x = (x_j)_{j=1}^{\infty} \in L(D, \Gamma_1, \Gamma_2, \mathbf{c})$, let $T_j(k) = S_j(k, x), j = 1, 2, 3, 4$. We extend each $T_j, j = 1, 2, 3, 4$, to $[1, +\infty)$ by piecewise linear interpolation. Then each $T_j,\ j = 1, 2, 3, 4$, is a Lipschitz function by (3.5). Now define $g_j : [0, \infty) \to \mathbb{R}$, $j = 1, 2, 3, 4$, by

$$g_j(z) = e^{-z} T_j(e^z).$$

We claim that each $g_j(z),\ j = 1, 2, 3, 4$, is bounded and uniformly continuous on $[0, \infty)$. In fact,

$$\begin{aligned} |g_j(z)| &\leq |g_j(0)|e^{-z} + |g_j(z) - g_j(0)e^{-z}| \\ &\leq |T_j(1)| + e^{-z}|T_j(e^z) - T_j(1)| \\ &\leq |T_j(1)| + \mathrm{Lip}T_j, \end{aligned}$$

and for any $\delta > 0$

$$\begin{aligned} |g_j(z+\delta) - g_j(z)| &= |e^{-(z+\delta)}T_j(e^{z+\delta}) - e^{-z}T_j(e^z)| \\ &\leq e^{-(z+\delta)}|T_j(e^{z+\delta}) - T_j(e^z)| + |g_j(z)|(1 - e^{-\delta}) \\ &\leq (1 - e^{-\delta})\mathrm{Lip}T_j + (1 - e^{-\delta})(|T_j(1)| + \mathrm{Lip}T_j). \end{aligned}$$

Now for any $v > -\log\alpha$,

$$\begin{aligned} &\left| \int_{-\log\alpha}^{v} \sum_{j=1}^{4} (g_j(z) - g_j(z + \log\alpha)) dz \right| \\ &\quad = \left| \sum_{j=1}^{4} \left(\int_{-\log\alpha}^{v} g_j(z) dz - \int_{-\log\alpha}^{v} g_j(z + \log\alpha) dz \right) \right| \\ &\quad = \left| \sum_{j=1}^{4} \left(\int_{-\log\alpha}^{v} g_j(z) dz - \int_{0}^{v+\log\alpha} g_j(z) dz \right) \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{j=1}^{4} \left(\int_0^{-\log \alpha} g_j(z)dz + \int_v^{v+\log \alpha} g_j(z)dz \right) \right| \\
&\leq \sum_{j=1}^{4} \left(\left| \int_0^{-\log \alpha} g_j(z)dz \right| + \left| \int_v^{v+\log \alpha} g_j(z)dz \right| \right) \\
&< \xi,
\end{aligned}
$$

for some positive number ξ, since each g_j is bounded on $[0, +\infty)$. Therefore,

$$\limsup_{z \to +\infty} \sum_{j=1}^{4} (g_j(z) - g_j(z + \log \alpha)) \geq 0.$$

By letting $z = \log t$, this gives

$$\limsup_{t \to +\infty} \sum_{j=1}^{4} \left(\frac{T_j(t)}{t} - \frac{T_j(\alpha t)}{\alpha t} \right) \geq 0.$$

Note that

$$
\begin{aligned}
\frac{T_j(t)}{t} - \frac{T_j(\alpha t)}{\alpha t} &= \left(\frac{T_j(t) - T_j([t])}{t} - \frac{T_j(\alpha t) - T_j([\alpha t])}{\alpha t} \right) + \frac{T_j([t])}{[t]} \left(\frac{[t]}{t} - 1 \right) \\
&\quad + \left(\frac{T_j([\alpha[t]])}{\alpha[t]} - \frac{T_j([\alpha t])}{\alpha t} \right) + \left(\frac{T_j([t])}{[t]} - \frac{T_j([\alpha[t]])}{\alpha[t]} \right), \qquad (3.6)
\end{aligned}
$$

where, as before, $[t]$ with $t \in \mathbb{R}$ denote the greatest integer function. However, the first three terms in the right side of (3.6) tend to zero as $t \to +\infty$ by the facts that both functions $|T_j(t) - T_j([t])|$ and $g_j(z)$ are bounded, and that $g_j(z)$ is uniformly continuous for all $j = 1, 2, 3, 4$. Hence (3.4) holds. Therefore, for every $x = (x_j)_{j=1}^{\infty} \in L(D, \Gamma_1, \Gamma_2, \mathbf{c})$ we have

$$\limsup_{k \to \infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \geq -f(\mathbf{p}^*),$$

which leads to

$$\limsup_{k \to \infty} (k\delta + \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x))) = \limsup_{k \to \infty} k \left(\delta + \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \right) = +\infty,$$

for any $\delta > f(\mathbf{p}^*)$. Now Lemma 2.1, (2) implies that $\dim_H K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c})) \leq f(\mathbf{p}^*)$. □

Proof. [Proof of Theorem 1.2] (I) If $c_j = \frac{|\Gamma_j|}{|D|}$ for all $1 \leq j \leq 2$, and D has uniform horizontal fibers, we first claim that $\mathbf{p}^* = (p_d^*)_{d \in D}$ determined by (2.1), is uniformly distributed on D, i.e. $\mathbf{p}^* = (\frac{1}{|D|}, \frac{1}{|D|}, \ldots, \frac{1}{|D|})$. This is done by simply checking that the probability vector $(\frac{1}{|D|}, \frac{1}{|D|}, \ldots, \frac{1}{|D|}) \in \Xi$ satisfies (2.1) because of its uniqueness. At this moment, we have

$$f(\mathbf{p}^*) = (1 - \alpha) \log_m |B| + \alpha \log_m |D|.$$

Therefore,

$$\begin{aligned} kf(\mathbf{p}^*) + \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) &= k((1-\alpha)\log_m |B| + \alpha \log_m |D|) \\ &\quad + [\alpha k] \log_m \frac{1}{|D|} + (k - [\alpha k]) \log_m \frac{1}{|B|} \\ &= (\alpha k - [\alpha k]) \log_m \frac{|D|}{|B|} \end{aligned}$$

for all $x \in L(D, \Gamma_1, \Gamma_2, \mathbf{c})$ and all $k \in \mathbb{N}$. Then (I) is justified by Lemma 2.1, (3).

(II) As shown in Theorem 1.1, we have

$$\gamma = \dim_H K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c})) = \dim_H K_T(L(D, \mathbf{p}^*)) = f(\mathbf{p}^*),$$

where $\mathbf{p}^*$, as described in Proposition 2.1, is such that $f(\mathbf{p}^*) = f_{\max}$. Note that

$$\mathcal{H}^\gamma(K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c}))) \geq \mathcal{H}^\gamma(K_T(L(D, \mathbf{p}^*))),$$

since $K_T(L(D, \Gamma_1, \Gamma_2, \mathbf{c})) \supset K_T(L(D, \mathbf{p}^*))$. As a result, (b) in (R3) shows that $\mathcal{H}^\gamma(K_T(L(D, \mathbf{p}^*))) = \infty$ under the given assumptions. □

Acknowledgments

The authors would like to thank the anonymous referees for their careful reading, valuable comments and suggestions that lead to the improvement of the manuscript. The first author is supported by the National Natural Science Foundation of China #10771071 and the Educational Office of Hubei province #Q20082802. The second author is supported by the National Natural Science Foundation of China #10571058 and Shanghai Leading Academic Discipline Project #B407.

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