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Hausdorff and packing dimensions of subsets of Moran fractals with prescribed mixed group frequency of their codings

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Summary. We compute the Hausdorff dimension and the packing dimension of subsets of Moran fractals with prescribed *mixed group* frequencies. For example, if E denotes the set of real numbers x in $[0,1]$ for which the group of digits $\{1,2,3,4\}$ in the decimal expansion of x occurs with relative frequency $t_1 \in [0,1]$ and the group of digits $\{0,1,2,8,9\}$ in the decimal expansion of x occurs with relative frequency $t_2 \in [0,1]$, then our results shows that

$$\dim_{\mathsf{H}} E = \dim_{\mathsf{P}} E = -\frac{1}{\log 10} \log \left(\frac{t_1^{t_1} t_2^{t_2} (1-t_1)^{1-t_1} (1-t_2)^{1-t_2}}{2^{t_1} 3^{1-t_1}} \right),$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension and $\dim_{\mathsf{P}}$ denotes the packing dimension. Observe that the two groups of digits with prescribed frequencies, namely $\{1,2,3,4\}$ and $\{0,1,2,8,9\}$, are *mixed*, i.e. they are not disjoint. Previous work [LD, O1, V] has investigated the non-mixed case. In this paper we investigate the more difficult problem of finding the Hausdorff dimension and packing dimension of subsets of Moran fractals with prescribed *mixed group* frequencies.

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1. Statement of results

This paper generalizes a result by Li & Dekking in [LD] on the Hausdorff dimension of subsets of Moran fractals with prescribed group frequency of their codings.

We first define the notion of a Moran fractal. Fix a positive integer N with $N \geq 2$. Write

$$\Sigma = \{1, 2, \dots, N\},$$

and

$$\Sigma^n = \{1, 2, \dots, N\}^n,$$

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$$\Sigma^* = \bigcup_{n\in\mathbb{N}} \{1,2,\ldots,N\}^n\,,$$
$$\Sigma^{\mathbb{N}} = \{1,2,\ldots,N\}^{\mathbb{N}}\,,$$

i.e. Σ^n denotes the family of all finite strings $\mathbf{i} = i_1 \ldots i_n$ of length n with entries i_j from $\{1,2,\ldots,N\}$, Σ^* denotes the family of all finite strings $\mathbf{i} = i_1 \ldots i_n$ with entries i_j from $\{1,2,\ldots,N\}$, and $\Sigma^{\mathbb{N}}$ denotes the family of all infinite strings $\mathbf{i} = i_1 i_2 \ldots$ with entries i_j from $\{1,2,\ldots,N\}$. For $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ and a positive integer n, let $\mathbf{i}|n = i_1 \ldots i_n$ denote the truncation of $\mathbf{i}$ to the n-th place. We can now define the notion of a Moran fractal, see [MU].

Definition (Moran constructions in $\mathbb{R}^d$)**.** A Moran construction in $\mathbb{R}^d$ is a list

$$\big((\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*}, (r_i)_{i=1,\ldots,N}\big),$$

where $(\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*}$ is a family of subsets of $\mathbb{R}^d$ and $(r_i)_{i=1,\ldots,N}$ is a family of positive numbers with $r_i \in (0,1)$ satisfying

(1) For each $\mathbf{i} \in \Sigma^*$, the set $\Delta_{\mathbf{i}}$ is a regular subset of $\mathbb{R}^d$ (here regular means that the closure of the interior of $\Delta_{\mathbf{i}}$ equals $\Delta_{\mathbf{i}}$).

(2) There are positive constants c' and c'' satisfying the following: for each $\mathbf{i} = i_1 \ldots i_n \in \Sigma^*$ there are balls $B'_{\mathbf{i}}$ and $B''_{\mathbf{i}}$ such that $B'_{\mathbf{i}} \subseteq \Delta_{\mathbf{i}} \subseteq B''_{\mathbf{i}}$ and

$$\operatorname{diam}(B'_{\mathbf{i}}) = c' \prod_{k=1}^{n} r_{i_k}\,, \quad \operatorname{diam}(B''_{\mathbf{i}}) = c'' \prod_{k=1}^{n} r_{i_k}\,.$$

(3) For each $\mathbf{i} \in \Sigma^*$ and for each $i \in \{1,2,\ldots,N\}$ we have

$$\Delta_{\mathbf{i}i} \subseteq \Delta_{\mathbf{i}}\,.$$

The limit set (or the Moran fractal set) K associated with the Moran construction is defined by

$$K = \bigcap_n \bigcup_{\mathbf{i}\in\Sigma^n} \Delta_{\mathbf{i}}\,.$$

An important class of Moran constructions is formed by constructions generated by iterated function systems, i.e. finite lists $(S_1,\ldots,S_N)$ of similarities $S_i : \mathbb{R}^d \to \mathbb{R}^d$. In this case $\Delta_{\mathbf{i}} = S_{i_1} \circ \cdots \circ S_{i_n}(\Delta)$ for some non-empty and compact set $\Delta \subseteq \mathbb{R}^d$ and $\mathbf{i} = i_1 \ldots i_n \in \Sigma^*$, and the limit set K is called the self-similar set associated with the list $(S_1,\ldots,S_N)$. Among the many examples of self-similar sets are the Cantor set and the Sierpinski triangle. We will always assume that the Moran construction satisfies the so-called Open Set Condition.

Definition (The Open Set Condition (OSC))**.** A Moran construction $\big((\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*}, (r_i)_{i=1,\ldots,N}\big)$ is said to satisfy the Open Set Condition if

$$\operatorname{int}(\Delta_{\mathbf{i}}) \cap \operatorname{int}(\Delta_{\mathbf{j}}) = \varnothing$$

for all positive integers n and all $\mathbf{i},\mathbf{j} \in \Sigma^n$ with $\mathbf{i} \neq \mathbf{j}$; here int denotes interior.

We note that the Hausdorff dimension and the packing dimension of the limit set of a Moran construction satisfying the OSC have been studied in detail during the past 10 years, see, for example, [Fa]. Define the natural projection $\pi : \Sigma^{\mathbb{N}} \to \mathbb{R}^d$ by

$$\{\pi(\mathbf{i})\} = \bigcap_n \Delta_{\mathbf{i}|n}$$

for $\mathbf{i} \in \Sigma^{\mathbb{N}}$. It is clear that K can be symbolically represented as $K = \pi(\Sigma^{\mathbb{N}})$. This motivates studying the frequencies of digits (or sets of digits) i_j in the "codings" $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ of points $x = \pi(\mathbf{i})$ in K. We therefore define frequencies as follows. For a subset Γ of Σ and a finite string $\mathbf{i} = i_1 i_2 \ldots i_n \in \Sigma^*$, let

$$\Pi_\Gamma(\mathbf{i}) = \frac{\big|\{1 \le j \le n \,|\, i_j \in \Gamma\}\big|}{n}$$

denote the frequency of the set Γ of digits in the string $\mathbf{i} = i_1 i_2 \ldots i_n$. Also, for a finite family $\mathbf{\Gamma}$ of subsets of Σ and a finite string $\mathbf{i} = i_1 i_2 \ldots i_n \in \Sigma^*$, let

$$\Pi_{\mathbf{\Gamma}}(\mathbf{i}) = \big(\Pi_\Gamma(\mathbf{i})\big)_{\Gamma \in \mathbf{\Gamma}}.$$

If $\mathbf{\Gamma}$ is a partition of Σ and $\mathbf{t} = (t_\Gamma)_{\Gamma \in \mathbf{\Gamma}}$ is a probability vector, Li & Dekking in [LD] (and later Olsen in [O1]; see also an earlier paper by Volkmann [V] for a special case) computed the Hausdorff dimension of the set of points $x = \pi(\mathbf{i})$ for which the frequency of the set Γ of digits among the first n entries of $\mathbf{i}$ approaches t_Γ as $n \to \infty$ for all $\Gamma \in \mathbf{\Gamma}$, i.e. the set

$$\pi\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \;\Big|\; \lim_n \Pi_{\mathbf{\Gamma}}(\mathbf{i}|n) = \mathbf{t}\Big\}.$$

Below we denote the Hausdorff dimension by $\dim_{\mathsf{H}}$ and we denote the packing dimension by $\dim_{\mathsf{P}}$.

Theorem A ([LD]). *Let $\big((\Delta_{\mathbf{i}})_{\mathbf{i} \in \Sigma^*}, (r_i)_{i=1,\ldots,N}\big)$ be a Moran construction satisfying the OSC. Let $\mathbf{\Gamma}$ be a partition of Σ. For a probability vector $\mathbf{t} = (t_\Gamma)_{\Gamma \in \mathbf{\Gamma}}$, let $s(\mathbf{t})$ be the unique real number such that*

$$\sum_{\Gamma \in \mathbf{\Gamma}} t_\Gamma \log t_\Gamma = \sum_{\Gamma \in \mathbf{\Gamma}} t_\Gamma \log \sum_{i \in \Gamma} r_i^{s(\mathbf{t})}.$$

For each probability vector $\mathbf{t} = (t_\Gamma)_{\Gamma \in \mathbf{\Gamma}}$, we have

$$\dim_{\mathsf{H}} \pi\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \;\Big|\; \lim_n \Pi_{\mathbf{\Gamma}}(\mathbf{i}|n) = \mathbf{t}\Big\} = s(\mathbf{t}).$$

In the present paper we generalize Theorem A to the case of arbitrary finite families $\mathbf{\Gamma}$ of subsets of Σ. In particular, we will not assume that $\mathbf{\Gamma}$ is a partition of Σ. Specifically, we prove the following generalization of Theorem A using results from [OW,O2]. Below $\mathcal{P}_S(\Sigma^{\mathbb{N}})$ denotes the family of shift invariant probability measures on $\Sigma^{\mathbb{N}}$. Also, for a sequence $(x_n)_n$ in a metric space M, we denote the set of accumulation points of $(x_n)_n$ by $\mathsf{A}(x_n)$, i.e.

$$\mathsf{A}(x_n) = \Big\{ x \in M \,\Big|\, \text{there is a subsequence } (x_{n_k})_k \text{ such that } x_{n_k} \to x \Big\}.$$

Theorem 1. *Let $\big((\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*}, (r_i)_{i=1,\ldots,N}\big)$ be a Moran construction satisfying the OSC. Let $\mathbf{\Gamma}$ be a finite family of subsets of Σ (observe that $\mathbf{\Gamma}$ is not necessarily a partition of Σ). Let*

$$\mathbf{\Pi} = \left\{ \bigcap_{\Gamma\in\mathbf{\Gamma}} \Xi_\Gamma \,\middle|\, \Xi_\Gamma \in \Big\{\Gamma, \Sigma\setminus\Gamma\Big\} \text{ for all } \Gamma \right\}.$$

For $\mathbf{t} = (t_\Gamma)_{\Gamma\in\mathbf{\Gamma}} \in \mathbb{R}^{\mathbf{\Gamma}}$, let

$$\mathcal{K}(\mathbf{t}) = \left\{ (q_\Pi)_{\Pi\in\mathbf{\Pi}} \,\middle|\, (q_\Pi)_{\Pi\in\mathbf{\Pi}} \text{ is a probability vector with } \sum_{\Pi\subseteq\Gamma} q_\Pi = t_\Gamma \text{ for all } \Gamma \right\}.$$

For $\mathbf{t} \in \mathbb{R}^{\mathbf{\Gamma}}$ and $\mathbf{q} \in \mathcal{K}(\mathbf{t})$, let $s(\mathbf{q},\mathbf{t})$ be the unique real number such that

$$\sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log q_\Pi = \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \sum_{i\in\Pi} r_i^{s(\mathbf{q},\mathbf{t})} .$$

Put

$$X = \left\{ \Big(\mu\big(\cup_{i\in\Gamma} [i] \big) \Big)_{\Gamma\in\mathbf{\Gamma}} \,\middle|\, \mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \right\} \subseteq \mathbb{R}^{\mathbf{\Gamma}},$$

where $[i]$ denotes the cylinder $[i] = \{i_1 i_2 \ldots \in \Sigma^{\mathbb{N}} \,|\, i_1 = i\}$.

(1) *If C is not a subcontinuum of X, then*

$$\Big\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\big) = C \Big\} = \varnothing .$$

(2) *If C is a subcontinuum of X, then*

$$\dim_{\mathsf{H}} \pi \Big\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big) = C \Big\} = \inf_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}),$$

$$\dim_{\mathsf{P}} \pi \Big\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big) = C \Big\} = \sup_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}).$$

(3) *If C is closed and convex subset of $\mathbb{R}^{\mathbf{\Gamma}}$, then*

$$\dim_{\mathsf{H}} \pi \Big\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big) \subseteq C \Big\} = \sup_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}),$$

$$\dim_{\mathsf{P}} \pi \Big\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big) \subseteq C \Big\} = \sup_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}) .$$

Theorem 1 is proved in Section 3. Of course, by putting $C = \{\mathbf{t}\}$ in Theorem 1, we obtain the following result.

Corollary 2. *Let $\big((\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*},(r_i)_{i=1,\dots,N}\big)$ be a Moran construction satisfying the OSC. Let $\mathbf{\Gamma}$ be a finite family of subsets of Σ (observe that $\mathbf{\Gamma}$ is not necessarily a partition of Σ). Let $\mathbf{\Pi}$, $\mathcal{K}(\mathbf{t})$, $s(\mathbf{q},\mathbf{t})$ and X be as in Theorem 1. If $\mathbf{t}\in X$, then*

$$\dim_{\mathsf{H}}\pi\Big\{\mathbf{i}\in\Sigma^{\mathbb{N}}\ \Big|\ \lim_n \Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)=\mathbf{t}\Big\}=\sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}),$$
$$\dim_{\mathsf{P}}\pi\Big\{\mathbf{i}\in\Sigma^{\mathbb{N}}\ \Big|\ \lim_n \Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)=\mathbf{t}\Big\}=\sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}).$$

It is clear that Corollary 2 simplifies to Theorem A if $\mathbf{\Gamma}$ is a partition of Σ.

2. An example: mixed group frequencies of *N*-adic digits

As an application of Theorem 1 we will now compute the Hausdorff dimension and the packing dimension of various sets associated with mixed group frequencies of N-adic digits. For $x\in[0,1]$, write

$$x=\sum_{n=1}^{\infty}\frac{d_n(x)}{N^n}\,,$$

where $d_n(x)\in\Sigma=\{0,1,\dots,N-1\}$, for the unique non-terminating N-adic expansion of x. For a set $\Gamma\subseteq\Sigma$ of digits let

$$\Pi_\Gamma(x;n)=\frac{\big|\{1\le k\le n\,|\,d_k(x)\in\Gamma\}\big|}{n}$$

denote the frequency of the set Γ of digits among the first n of the N-adic digits of x. Also, for a finite family $\mathbf{\Gamma}$ of subsets of digits, write

$$\Pi_{\mathbf{\Gamma}}(x;n)=\big(\Pi_\Gamma(x;n)\big)_{\Gamma\in\mathbf{\Gamma}}\,.$$

Defining $S_1,\dots,S_N:\mathbb{R}\to\mathbb{R}$ by $S_i(x)=\frac{x+i-1}{N}$ and applying Theorem 1 to the Moran construction defined by $\Delta_{\mathbf{i}}=S_{i_1}\circ\cdots\circ S_{i_n}([0,1])$ for $\mathbf{i}=i_1\dots i_n\in\Sigma^*$ we immediately obtain the following corollary.

Corollary 3. *Let $\mathbf{\Gamma}$ be a finite family of subsets of $\Sigma=\{0,1,\dots,N-1\}$ (observe that $\mathbf{\Gamma}$ is not necessarily a partition of Σ). Let*

$$\mathbf{\Pi}=\left\{\bigcap_{\Gamma\in\mathbf{\Gamma}}\Xi_\Gamma\ \middle|\ \Xi_\Gamma\in\Big\{\Gamma,\Sigma\setminus\Gamma\Big\}\text{ for all }\Gamma\right\}.$$

For $\mathbf{t}=(t_\Gamma)_{\Gamma\in\mathbf{\Gamma}}\in\mathbb{R}^{\mathbf{\Gamma}}$, let

$$\mathcal{K}(\mathbf{t})=\left\{(q_\Pi)_{\Pi\in\mathbf{\Pi}}\ \middle|\ (q_\Pi)_{\Pi\in\mathbf{\Pi}}\text{ is a probability vector with }\sum_{\Pi\subseteq\Gamma}q_\Pi=t_\Gamma\text{ for all }\Gamma\right\}.$$

Put

$$X = \left\{ \left(\mu\left(\cup_{i\in\Gamma} [i] \right) \right)_{\Gamma\in\mathbf{\Gamma}} \,\middle|\, \mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \right\} \subseteq \mathbb{R}^{\mathbf{\Gamma}} ,$$

where $[i]$ *denotes the cylinder* $[i] = \{ i_1 i_2 \ldots \in \Sigma^{\mathbb{N}} \,|\, i_1 = i \}$.

(1) *If* C *is not a subcontinuum of* X*, then*

$$\left\{ x\in[0,1] \,\middle|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(x;n)\Big) = C \right\} = \varnothing.$$

(2) *If* C *is a subcontinuum of* X*, then*

$$\dim_{\mathsf{H}} \left\{ x\in[0,1] \,\middle|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(x;n)\Big) = C \right\} = \inf_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N} \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \frac{q_\Pi}{|\Pi|} ,$$

$$\dim_{\mathsf{P}} \left\{ x\in[0,1] \,\middle|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(x;n)\Big) = C \right\} = \sup_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N} \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \frac{q_\Pi}{|\Pi|} .$$

(3) *If* C *is closed and convex subset of* $\mathbb{R}^{\mathbf{\Gamma}}$*, then*

$$\dim_{\mathsf{H}} \left\{ x\in[0,1] \,\middle|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(x;n)\Big) \subseteq C \right\} = \sup_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N} \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \frac{q_\Pi}{|\Pi|} ,$$

$$\dim_{\mathsf{P}} \left\{ x\in[0,1] \,\middle|\, \mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(x;n)\Big) \subseteq C \right\} = \sup_{\mathbf{t}\in C} \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N} \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \frac{q_\Pi}{|\Pi|} .$$

By putting $C = \{\mathbf{t}\}$ in Corollary 3, we obtain the following result.

Corollary 4. *Let* $\mathbf{\Gamma}$ *be a finite family of subsets of* $\Sigma = \{0, 1, \ldots, N-1\}$ *(observe that* $\mathbf{\Gamma}$ *is not necessarily a partition of* Σ*). Let* $\mathbf{\Pi}$, $\mathcal{K}(\mathbf{t})$ *and* X *be as in Corollary 3. If* $\mathbf{t} \in X$*, then*

$$\dim_{\mathsf{H}} \left\{ x\in[0,1] \,\middle|\, \lim_n \Pi_{\mathbf{\Gamma}}(x;n) = \mathbf{t} \right\} = \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N} \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \frac{q_\Pi}{|\Pi|} ,$$

$$\dim_{\mathsf{P}} \left\{ x\in[0,1] \,\middle|\, \lim_n \Pi_{\mathbf{\Gamma}}(x;n) = \mathbf{t} \right\} = \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N} \sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log \frac{q_\Pi}{|\Pi|} .$$

We now consider a concrete example of Corollary 3 and Corollary 4. In particular, we consider the following case: $N = 10$ and $\mathbf{\Gamma} = \{\Gamma_1, \Gamma_2\}$ where $\Gamma_1 = \{1,2,3,4\}$ and $\Gamma_2 = \{0,1,2,8,9\}$; observe that the sets $\Gamma_1 = \{1,2,3,4\}$ and $\Gamma_2 = \{0,1,2,8,9\}$ are not pairwise disjoint and do not form a partition of $\Sigma = \{0,1,\ldots,9\}$. For $\mathbf{t} = (t_{\Gamma_1}, t_{\Gamma_2})$, we will now compute the Hausdorff dimension and the packing dimension of the set $\{x \in [0,1] \,|\, \lim_n \Pi_{\mathbf{\Gamma}}(x;n) = \mathbf{t}\}$. Observe that it follows from Corollary 4 that

$$\dim_{\mathsf{H}} \left\{ x\in[0,1] \,\middle|\, \lim_n \Pi_{\mathbf{\Gamma}}(x;n) = \mathbf{t} \right\} = \gamma(\mathbf{t}),$$

$$\dim_{\mathsf{P}} \left\{ x\in[0,1] \,\middle|\, \lim_n \Pi_{\mathbf{\Gamma}}(x;n) = \mathbf{t} \right\} = \gamma(\mathbf{t}),$$

where

$$\gamma(\mathbf{t}) = \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} -\frac{1}{\log N}\sum_{\Pi\in\mathbf{\Pi}} q_\Pi \log\frac{q_\Pi}{|\Pi|}\,.$$

We will now compute $\gamma(\mathbf{t})$. However, we first consider a slightly more general problem. Fix $\Gamma_1,\Gamma_2\subseteq\Sigma$ and put $\mathbf{\Gamma}=\{\Gamma_1,\Gamma_2\}$. Also, let $\mathbf{t}=(t_{\Gamma_1},t_{\Gamma_2})$ and write $t_i=t_{\Gamma_i}$ for brevity. In this case we have $\mathbf{\Pi}=\{\Pi_1,\Pi_2,U,V\}$ where $\Pi_1=\Gamma_1\setminus\Gamma_2$, $\Pi_2=\Gamma_2\setminus\Gamma_1$, $U=\Gamma_1\cap\Gamma_2$ and $V=\Sigma\setminus(\Gamma_1\cup\Gamma_2)$. Finally, for $(q_{\Pi_1},q_{\Pi_2},q_U,q_V)\in\mathcal{K}(\mathbf{t})$ we write $q_i=q_{\Pi_i}$, $u=q_U$ and $v=q_V$ for brevity. Using this notation we clearly have

$$\begin{aligned}\mathcal{K}(\mathbf{t}) &= \Big\{(q_1,q_2,u,v)\in\mathbb{R}^4 \,\Big|\, (q_1,q_2,u,v) \text{ is a probability vector with } q_1+u=t_1,\ q_2+u=t_2\Big\}\\ &= \Big\{(t_1-u,t_2-u,u,1-t_1-t_2+u)\in\mathbb{R}^4 \,\Big|\, u\in[\,\max(0,t_1+t_2-1)\,,\,\min(t_1,t_2,1)\,]\Big\}\end{aligned}$$

and so

$$\begin{aligned}\gamma(\mathbf{t}) = \sup_{u\in[\,\max(0,t_1+t_2-1),\,\min(t_1,t_2,1)\,]} -\frac{1}{\log N}\Bigg((t_1-u)\log\frac{t_1-u}{|\Gamma_1\setminus\Gamma_2|}\\ +(t_2-u)\log\frac{t_2-u}{|\Gamma_2\setminus\Gamma_1|}+u\log\frac{u}{|\Gamma_1\cap\Gamma_2|}\\ +(1-t_1-t_2+u)\log\frac{1-t_1-t_2+u}{|\Sigma\setminus(\Gamma_1\cup\Gamma_2)|}\Bigg)\,.\end{aligned}$$

The number $\gamma(\mathbf{t})$ can now be found by a simple calculus argument. For example, if

$$|\Gamma_1\setminus\Gamma_2|=|\Gamma_1\cap\Gamma_2|\,,\ |\Gamma_2\setminus\Gamma_1|=|\Sigma\setminus(\Gamma_1\cup\Gamma_2)|\,, \tag{2.1}$$

then

$$\begin{aligned}\gamma(\mathbf{t}) = -\frac{1}{\log N}\Bigg((t_1-t_1t_2)\log\frac{t_1-t_1t_2}{|\Gamma_1\setminus\Gamma_2|}+(t_2-t_1t_2)\log\frac{t_2-t_1t_2}{|\Gamma_2\setminus\Gamma_1|}\\ +t_1t_2\log\frac{t_1t_2}{|\Gamma_1\cap\Gamma_2|}+(1-t_1-t_2+t_1t_2)\log\frac{1-t_1-t_2+t_1t_2}{|\Sigma\setminus(\Gamma_1\cup\Gamma_2)|}\Bigg)\end{aligned}$$

for $\mathbf{t}=(t_1,t_2)\in[0,1]^2$, and

$$\gamma(\mathbf{t})=0$$

for $\mathbf{t}=(t_1,t_2)\not\in[0,1]^2$. Condition (2.1) is, for example, satisfied if $N=10$ and $\Gamma_1=\{1,2,3,4\}$ and $\Gamma_2=\{0,1,2,8,9\}$; observe that the sets $\Gamma_1=\{1,2,3,4\}$ and $\Gamma_2=\{0,1,2,8,9\}$ are not pairwise disjoint and do not form a partition of

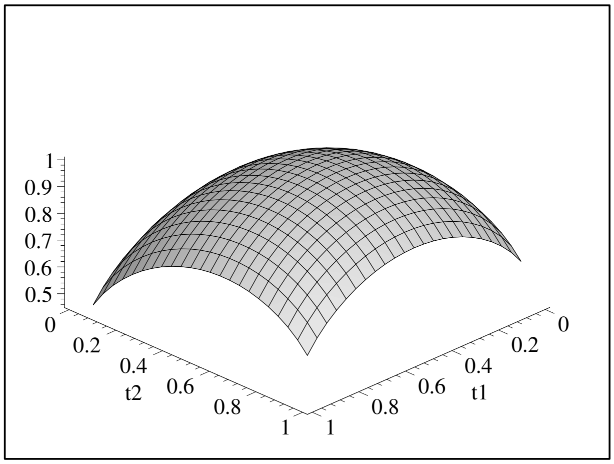

Fig. 1. The graph of the function $\gamma(\mathbf{t}) = -\frac{1}{\log 10}\log\left(\frac{t_1^{t_1}t_2^{t_2}(1-t_1)^{1-t_1}(1-t_2)^{1-t_2}}{2^{t_1}3^{1-t_1}}\right)$ for $\mathbf{t} = (t_1, t_2) \in [0,1]^2$.

$\Sigma = \{0, 1, \ldots, 9\}$. In this case we clearly have $|\Gamma_1 \setminus \Gamma_2| = |\Gamma_1 \cap \Gamma_2| = 2$ and $|\Gamma_2 \setminus \Gamma_1| = |\Sigma \setminus (\Gamma_1 \cup \Gamma_2)| = 3$, whence

$$\begin{aligned}\gamma(\mathbf{t}) &= -\frac{1}{\log 10}\Bigg((t_1 - t_1t_2)\log\frac{t_1 - t_1t_2}{2} + (t_2 - t_1t_2)\log\frac{t_2 - t_1t_2}{3} \\ &\qquad + t_1t_2\log\frac{t_1t_2}{2} + (1 - t_1 - t_2 + t_1t_2)\log\frac{1 - t_1 - t_2 + t_1t_2}{3}\Bigg) \\ &= -\frac{1}{\log 10}\log\left(\frac{t_1^{t_1}t_2^{t_2}(1-t_1)^{1-t_1}(1-t_2)^{1-t_2}}{2^{t_1}3^{1-t_1}}\right) \qquad (2.2)\end{aligned}$$

for $\mathbf{t} = (t_1, t_2) \in [0,1]^2$. Figure 1 shows the graph of the function $\gamma(\mathbf{t})$ in (2.2). If condition (2.1) is not satisfied it is possible to obtain a rather more complicated formula for $\gamma(\mathbf{t})$. However, since this formula is easily derived but fairly long we are omitting it.

3. Proof of Theorem 1

In [OW, O2] (see also [BOS]) Theorem 3.1 below is proved. We note that Theorem 3.1 is proved in [OW, O2] for so-called graph directed self-conformal sets. However, the proofs in [OW, O2] only depend on the contracting ratios of the maps, and the results in [OW, O2] therefore also hold in the Moran case. For a probability measure μ on $\Sigma^{\mathbb{N}}$, we let $h(\mu)$ denote the entropy of μ. We also use the following notation. For a positive integer n and $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$, let $L_n(\mathbf{i})$

denote the n-th empirical measure, i.e.

$$L_n(\mathbf{i}) = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{S^k(\mathbf{i})}$$

where $S : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ denotes the shift map. For $i \in \{1, 2, \dots, N\}$, we define the cylinder $[i]$ by $[i] = \{i_1 i_2 \ldots \in \Sigma^{\mathbb{N}} \mid i_1 = i\}$. Finally, recall that $\mathcal{P}_S(\Sigma^{\mathbb{N}})$ denotes the family of shift invariant probability measures on $\Sigma^{\mathbb{N}}$ and let $\mathcal{P}(\Sigma^{\mathbb{N}})$ denote the family of all probability measures on $\Sigma^{\mathbb{N}}$.

Theorem 3.1 ([OW, O2]). *Let $\big((\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*}, (r_i)_{i=1,\dots,N}\big)$ be a Moran construction satisfying the OSC. Let V be a normed vector space and let $\Xi : \mathcal{P}(\Sigma^{\mathbb{N}}) \to V$ be an affine and continuous map. For $\mathbf{t} \in V$, let*

$$\Lambda(\mathbf{t}) = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ \Xi(\mu)=\mathbf{t}}} -\frac{h(\mu)}{\sum_i \mu([i])\log r_i}\,.$$

Put

$$X = \Big\{\, \Xi(\mu) \,\Big|\, \mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}) \Big\} \subseteq V.$$

(1) *If C is not a subcontinuum of X, then*

$$\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Xi(L_n(\mathbf{i}))\Big) = C\Big\} = \varnothing.$$

(2) *If C is a subcontinuum of X, then*

$$\dim_{\mathsf{H}} \pi\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Xi(L_n(\mathbf{i}))\Big) = C\Big\} = \inf_{\mathbf{t}\in C} \Lambda(\mathbf{t})\,,$$
$$\dim_{\mathsf{P}} \pi\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Xi(L_n(\mathbf{i}))\Big) = C\Big\} = \sup_{\mathbf{t}\in C} \Lambda(\mathbf{t}).$$

(3) *If C is closed and convex subset of $\mathbb{R}^{\mathbf{\Gamma}}$, then*

$$\dim_{\mathsf{H}} \pi\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Xi(L_n(\mathbf{i}))\Big) \subseteq C\Big\} = \sup_{\mathbf{t}\in C} \Lambda(\mathbf{t})\,,$$
$$\dim_{\mathsf{P}} \pi\Big\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\Big|\, \mathsf{A}\Big(\Xi(L_n(\mathbf{i}))\Big) \subseteq C\Big\} = \sup_{\mathbf{t}\in C} \Lambda(\mathbf{t}).$$

The following result follows immediately by applying Theorem 3.1 to the map $\Xi : \mathcal{P}(\Sigma^{\mathbb{N}}) \to \mathbb{R}^{\mathbf{\Gamma}}$ defined by

$$\Xi(\mu) = \Big(\mu\big(\cup_{i\in\Gamma}[i]\big)\Big)_{\Gamma\in\mathbf{\Gamma}}\,,$$

and noticing that in this case

$$\Xi(L_n(\mathbf{i})) = \Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\,.$$

Theorem 3.2. *Let $\big((\Delta_{\mathbf{i}})_{\mathbf{i}\in\Sigma^*},(r_i)_{i=1,\dots,N}\big)$ be a Moran construction satisfying the OSC. Let $\mathbf{\Gamma}$ be a finite family of subsets of Σ (observe that $\mathbf{\Gamma}$ is not necessarily a partition of Σ). For $\mathbf{t}=(t_\Gamma)_{\Gamma\in\mathbf{\Gamma}}\in\mathbb{R}^{\mathbf{\Gamma}}$, let*

$$\Lambda(\mathbf{t})=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ \mu(\cup_{i\in\Gamma}[i])=t_\Gamma \text{ for all } \Gamma}} -\frac{h(\mu)}{\sum_i\mu([i])\log r_i}\,.$$

Put

$$X=\left\{\big(\mu\big(\cup_{i\in\Gamma}[i]\big)\big)_{\Gamma\in\mathbf{\Gamma}}\,\middle|\,\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\right\}\subseteq\mathbb{R}^{\mathbf{\Gamma}}.$$

(1) *If C is not a subcontinuum of X, then*

$$\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big)=C\right\}=\varnothing.$$

(2) *If C is a subcontinuum of X, then*

$$\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big)=C\right\}=\inf_{\mathbf{t}\in C}\Lambda(\mathbf{t})\,,$$

$$\dim_{\mathsf{P}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big)=C\right\}=\sup_{\mathbf{t}\in C}\Lambda(\mathbf{t}).$$

(3) *If C is closed and convex subset of $\mathbb{R}^{\mathbf{\Gamma}}$, then*

$$\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big)\subseteq C\right\}=\sup_{\mathbf{t}\in C}\Lambda(\mathbf{t}),$$

$$\dim_{\mathsf{P}}\pi\left\{\mathbf{i}\in\Sigma^{\mathbb{N}}\,\middle|\,\mathsf{A}\Big(\Pi_{\mathbf{\Gamma}}(\mathbf{i}|n)\Big)\subseteq C\right\}=\sup_{\mathbf{t}\in C}\Lambda(\mathbf{t}).$$

We see that Theorem 1 follows from Theorem 3.2 provided we can prove that $\Lambda(\mathbf{t})=L(\mathbf{t})$. This equality is proved as follows. For a probability vector $\mathbf{p}=(p_i)_{i\in\Sigma}$, define $H(\mathbf{p})$ and $R(\mathbf{p})$ by

$$H(\mathbf{p})=-\sum_j p_j\log p_j\,,$$

$$R(\mathbf{p})=-\sum_j p_j\log r_j\,.$$

Next, for $\mathbf{t}=(t_\Gamma)_{\Gamma\in\mathbf{\Gamma}}\in\mathbb{R}^{\mathbf{\Gamma}}$, define $L(\mathbf{t})$ by

$$L(\mathbf{t})=\sup_{\substack{\mathbf{p}=(p_i)_{i\in\Sigma}\text{ is a probability vector}\\ \sum_{i\in\Gamma}p_i=t_\Gamma\text{ for all }\Gamma}}\frac{H(\mathbf{p})}{R(\mathbf{p})}\,.$$

In Lemma 3.3 and Lemma 3.4 below we prove the following two equalities

$$\Lambda(\mathbf{t})=L(\mathbf{t})\qquad\text{[this is proved in Lemma 3.3 below]},\tag{3.3}$$

$$L(\mathbf{t}) = \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t}) \qquad \text{[this is proved in Lemma 3.4 below].} \tag{3.4}$$

Combining (3.1) and (3.2) we conclude that

$$\Lambda(\mathbf{t}) = \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t})\,. \tag{3.3}$$

Theorem 1 follows immediately from Theorem 3.2 and (3.3).

Lemma 3.3. *For* $\mathbf{t} \in \mathbb{R}^{\mathbf{\Gamma}}$, *we have*

$$\Lambda(\mathbf{t}) = L(\mathbf{t}).$$

Proof. Write $\mathbf{t} = (t_\Gamma)_{\Gamma\in\mathbf{\Gamma}}$.

Part 1. We prove that $\Lambda(\mathbf{t}) \le L(\mathbf{t})$. Let $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$ with $\mu(\cup_{i\in\Gamma}[i]) = t_\Gamma$ for all Γ. Now define $\mathbf{p} = (p_i)_{i\in\Sigma}$ by $p_i = \mu([i])$ for all i. We have

$$-\frac{h(\mu)}{\sum_i \mu([i])\log r_i} \le \frac{\sum_i \mu([i])\log\mu([i])}{\sum_i \mu([i])\log r_i} = \frac{\sum_i p_i\log p_i}{\sum_i p_i \log r_i} = \frac{H(\mathbf{p})}{R(\mathbf{p})}\,. \tag{3.4}$$

Next, we observe that $\mathbf{p}$ is a probability vector with $\sum_{i\in\Gamma} p_i = \sum_{i\in\Gamma}\mu([i]) = \mu(\cup_{i\in\Gamma}[i]) = t_\Gamma$ for all Γ, and we therefore conclude from (3.4) that

$$-\frac{h(\mu)}{\sum_i \mu([i])\log r_i} \le \frac{H(\mathbf{p})}{R(\mathbf{p})} \le L(\mathbf{t})\,.$$

Finally, taking supremum over all measures $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$ with $\mu(\cup_{i\in\Gamma}[i]) = t_\Gamma$ shows that $\Lambda(\mathbf{t}) \le L(\mathbf{t})$.

Part 2. Next we prove that $L(\mathbf{t}) \le \Lambda(\mathbf{t})$. Let $\mathbf{p} = (p_i)_{i\in\Sigma}$ be a probability vector with $\sum_{i\in\Gamma} p_i = t_\Gamma$ for all Γ. Next, define the Bernoulli measure ν on $\Sigma^{\mathbb{N}}$ by $\nu = \prod_{\mathbb{N}}(\sum_i p_i\delta_i)$ (here δ_i denotes the Dirac measure concentrated at i). Since ν is a Bernoulli measure, we conclude that $h(\nu) = -\sum_i \nu([i])\log\nu([i])$, whence

$$\frac{H(\mathbf{p})}{R(\mathbf{p})} = \frac{\sum_i p_i\log p_i}{\sum_i p_i\log r_i} = \frac{\sum_i \nu([i])\log\nu([i])}{\sum_i \nu([i])\log r_i} = -\frac{h(\nu)}{\sum_i \nu([i])\log r_i}\,. \tag{3.5}$$

Next, we observe that $\nu(\cup_{i\in\Gamma}[i]) = \sum_{i\in\Gamma}\nu([i]) = \sum_{i\in\Gamma} p_i = t_\Gamma$ for all Γ, and we therefore conclude from (3.5) that

$$\frac{H(\mathbf{p})}{R(\mathbf{p})} = -\frac{h(\nu)}{\sum_i \nu([i])\log r_i} \le \Lambda(\mathbf{t})\,.$$

Finally, taking supremum over all probability vectors $\mathbf{p} = (p_i)_{i\in\Sigma}$ with $\sum_{i\in\Gamma} p_i = t_\Gamma$ shows that $L(\mathbf{t}) \le \Lambda(\mathbf{t})$. □

Lemma 3.4. *For* $\mathbf{t} \in \mathbb{R}^{\mathbf{\Gamma}}$, *we have*

$$L(\mathbf{t}) = \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t})\,.$$

Proof. Write $\mathbf{t} = (t_\Gamma)_{\Gamma\in\mathbf{\Gamma}}$.

Part 1. We prove that $L(\mathbf{t}) \le \sup_{\mathbf{q}\in\mathcal{K}(\mathbf{t})} s(\mathbf{q},\mathbf{t})$. Let $\mathbb{R}_+ = [0,\infty)$. Define $G, G_\Gamma : \mathbb{R}_+^\Sigma \to \mathbb{R}$ for $\Gamma \in \mathbf{\Gamma}$ by

$$G(\mathbf{u}) = \sum_i u_i\,, \quad G_\Gamma(\mathbf{u}) = \sum_{i\in\Gamma} u_i\,,$$

for $\mathbf{u} = (u_i)_{i\in\Sigma}$, and let

$$C = \left\{\mathbf{u}\in\mathbb{R}_+^\Sigma \,\middle|\, G(\mathbf{u}) = 1\,,\quad G_\Gamma(\mathbf{u}) = t_\Gamma \text{ for all } \Gamma \right\}.$$

Since C is compact and $\frac{H}{R}$ is continuous, the function $\frac{H}{R}$ restricted to the set C attains its maximum at some point $\mathbf{p}\in C$, whence

$$L(\mathbf{t}) = \frac{H(\mathbf{p})}{R(\mathbf{p})}\,.$$

Now let $\mathbf{q} = (q_\Pi)_{\Pi\in\mathbf{\Pi}}$ be defined by

$$q_\Pi = \sum_{i\in\Pi} p_i\,.$$

Since $\mathbf{p}\in C$, it is easily seen that $\mathbf{q}\in\mathcal{K}(\mathbf{t})$. We now claim that

$$\frac{H(\mathbf{p})}{R(\mathbf{p})} = s(\mathbf{q},\mathbf{t})\,. \tag{3.6}$$

In order to prove (3.6) we use Lagrange multipliers. Indeed, it follows from an appropriate version of Lagrange multiplier theorem (cf., for example, [O1]) that there exist real numbers λ, λ_Γ for $\Gamma\in\mathbf{\Gamma}$ such that

$$\nabla\frac{H}{R}(\mathbf{p}) - \lambda\nabla G(\mathbf{p}) - \sum_\Gamma \lambda_\Gamma \nabla G_\Gamma(\mathbf{p}) = 0\,. \tag{3.7}$$

Performing the differentiation in (3.7) shows that for all $i\in\Sigma$ we have

$$\frac{(-1-\log p_i)R(\mathbf{p}) + H(\mathbf{p})\log r_i}{R(\mathbf{p})^2} - \lambda - \sum_{i\in\Gamma}\lambda_\Gamma = 0\,. \tag{3.8}$$

Rearranging (3.8) shows that for all $i\in\Sigma$ we have

$$p_i = e^{-(\lambda+\sum_{i\in\Gamma}\lambda_\Gamma)R(\mathbf{p})-1}\, r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}}\,.$$

Fix $\Pi\in\mathbf{\Pi}$ and write $\mu_\Pi = \sum_{\Pi\subseteq\Gamma}\lambda_\Gamma$. Observe that for all $i\in\Pi$ we have

$$\begin{aligned} p_i &= e^{-(\lambda+\sum_{i\in\Gamma}\lambda_\Gamma)R(\mathbf{p})-1}\, r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}} \\ &= e^{-(\lambda+\mu_\Pi)R(\mathbf{p})-1}\, r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}}\,. \end{aligned} \tag{3.9}$$

Summing (3.9) over $i \in \Pi$ and then taking logarithms gives

$$\begin{aligned} \log q_\Pi &= \log \sum_{i\in\Pi} p_i \\ &= \log \sum_{i\in\Pi} e^{-(\lambda+\mu_\Pi)R(\mathbf{p})-1}\, r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}} \\ &= -(\lambda+\mu_\Pi)R(\mathbf{p}) - 1 + \log \sum_{i\in\Pi} r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}}\,. \end{aligned}$$

Finally, multiplying both sides by q_Π and then summing over $\Pi \in \mathbf{\Pi}$ yields

$$\sum_\Pi q_\Pi \log q_\Pi = -R(\mathbf{p}) \sum_\Pi (\lambda+\mu_\Pi) q_\Pi - 1 + \sum_\Pi q_\Pi \log \sum_{i\in\Pi} r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}}\,. \tag{3.10}$$

Rearranging (3.8) once more shows that for all $i \in \Sigma$ we have

$$(-1-\log p_i)R(\mathbf{p}) + H(\mathbf{p})\log r_i = \Bigl(\lambda + \sum_{i\in\Gamma}\lambda_\Gamma\Bigr) R(\mathbf{p})^2\,.$$

Fix $\Pi \in \mathbf{\Pi}$ and recall that $\mu_\Pi = \sum_{\Pi\subseteq\Gamma}\lambda_\Gamma$. Observe that for all $i \in \Pi$ we have

$$\begin{aligned} (-1-\log p_i)R(\mathbf{p}) + H(\mathbf{p})\log r_i &= \Bigl(\lambda + \sum_{i\in\Gamma}\lambda_\Gamma\Bigr) R(\mathbf{p})^2 \\ &= (\lambda+\mu_\Pi)R(\mathbf{p})^2\,. \end{aligned}$$

Multiplying this equality by p_i for $i \in \Pi$, summing over all $i \in \Pi$ and finally summing over all $\Pi \in \mathbf{\Pi}$, gives

$$\begin{aligned} \Bigl(-\sum_\Pi\sum_{i\in\Pi} p_i - \sum_\Pi\sum_{i\in\Pi} p_i\log p_i\Bigr) R(\mathbf{p}) + H(\mathbf{p}) \sum_\Pi\sum_{i\in\Pi} p_i \log r_i \\ = R(\mathbf{p})^2 \sum_\Pi\sum_{i\in\Pi} (\lambda+\mu_\Pi)p_i\,. \end{aligned}$$

This clearly simplifies to

$$(-1+H(\mathbf{p}))R(\mathbf{p}) - H(\mathbf{p})R(\mathbf{p}) = R(\mathbf{p})^2 \sum_\Pi (\lambda+\mu_\Pi) q_\Pi\,.$$

It follows from this that

$$-1 = R(\mathbf{p}) \sum_{\Pi} (\lambda + \mu_{\Pi}) q_{\Pi} \,. \tag{3.11}$$

Finally, combining (3.10) and (3.11) shows that

$$\sum_{\Pi} q_{\Pi} \log q_{\Pi} = \sum_{\Pi} q_{\Pi} \log \sum_{i \in \Pi} r_i^{\frac{H(\mathbf{p})}{R(\mathbf{p})}} \,,$$

and since $\mathbf{q} \in \mathcal{K}(\mathbf{t})$ this implies that $\frac{H(\mathbf{p})}{R(\mathbf{p})} = s(\mathbf{q}, \mathbf{t})$. This completes the proof of (3.6).

Since $\mathbf{q} \in \mathcal{K}(\mathbf{t})$, (3.6) implies that $L(\mathbf{t}) = \frac{H(\mathbf{p})}{R(\mathbf{p})} = s(\mathbf{q}, \mathbf{t}) \leq \sup_{\mathbf{v} \in \mathcal{K}(\mathbf{t})} s(\mathbf{v}, \mathbf{t})$.

Part 2. Next we prove that $\sup_{\mathbf{q} \in \mathcal{K}(\mathbf{t})} s(\mathbf{q}, \mathbf{t}) \leq L(\mathbf{t})$. Fix $\mathbf{q} \in \mathcal{K}(\mathbf{t})$, and note that

$$\left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \Pi_{\boldsymbol{\Pi}}(\mathbf{i}|n) = \mathbf{q} \right\} \subseteq \left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \Pi_{\boldsymbol{\Gamma}}(\mathbf{i}|n) = \mathbf{t} \right\}. \tag{3.12}$$

Hence

$$\begin{aligned}
s(\mathbf{q}, \mathbf{t}) &= \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \Pi_{\boldsymbol{\Pi}}(\mathbf{i}|n) = \mathbf{q} \right\} && \text{[by Theorem A]} \\
&\leq \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \Pi_{\boldsymbol{\Gamma}}(\mathbf{i}|n) = \mathbf{t} \right\} && \text{[by (3.12)]} \\
&= \Lambda(\mathbf{t}) && \text{[by Theorem 3.2]} \\
&= L(\mathbf{t}) && \text{[by Lemma 3.3].}
\end{aligned}$$

Taking supremum over $\mathbf{q} \in \mathcal{K}(\mathbf{t})$ gives the desired result. □

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