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# THE POINTWISE DENSITIES OF NON-SYMMETRIC CANTOR SETS

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We consider a class of non-symmetric Cantor sets $C$ determined by $C = a_0C \cup (a_1C + 1 - a_1)$, $a_0, a_1 \in (0, 1)$. Under certain conditions on $a_0$ and $a_1$, the upper and lower densities are obtained explicitly for each $x \in C$. It extends the results for symmetric Cantor sets obtained by Feng, Hua and Wen (The pointwise densities of the Cantor measure, *J. Math. Anal. Appl.* **250** (2000) 692–705).

*Keywords*: Cantor measure; non-symmetric Cantor sets; upper and lower densities.

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## 1. Introduction

Let $C \subseteq \mathbb{R}$ be the unique nonempty compact set invariant under $h_0, h_1$:

$$C = h_0(C) \cup h_1(C), \tag{1}$$

where $h_j(x) = a_jx + b_j$, $j = 0, 1$, with $0 < a_j < 1$. The set $C$ is also termed as *Cantor set*, or the self-similar set determined by $h_0$ and $h_1$. $C$ is called a *symmetric Cantor set* if $a_0 = a_1$, otherwise, a *non-symmetric Cantor set*. Without loss of generality, we shall assume that $b_0 = 0, a_1 + b_1 = 1$, equivalently, $h_0(0) = 0$ and $h_1(1) = 1$. Furthermore, we assume that the images $h_j([0, 1]), j = 0, 1$ are pairwise disjoint, i.e. the $h_j$'s satisfy the strongly separated condition. Thus, the above assumptions indicate that

$$h_0(x) = a_0x,\ \ h_1(x) = a_1x + 1 - a_1 \quad \text{with } a_0 + a_1 < 1.$$

It is well known that (cf. [1, 7]) $\dim_H C = \dim_P C = \dim_B C = \xi$ and $\mathcal{H}^\xi(C) = 1$ (cf. [1]), where $\xi$ is given by

$$a_0^\xi + a_1^\xi = 1. \tag{2}$$

The set $C$ has a natural symbolic representation defined as follows. Let

$$\Sigma^* = \bigcup_{n=1}^{\infty} \{0, 1\}^n \quad \text{and} \quad \Sigma^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}},$$

i.e. $\Sigma^*$ is the family of all finite strings $j_1 \dots j_n$ with entries $j_i$ from $\{0,1\}$ and $\Sigma^{\mathbb{N}}$ denotes the family of all infinite strings $j_1 j_2 \dots$ with entries $j_i$ from $\{0,1\}$. For $\omega = j_1 j_2 \dots \in \Sigma^{\mathbb{N}}$ and a positive integer $n$, let $\omega|n = j_1 \dots j_n$ denote the truncation of $\omega$ to the $n$th place. Finally, we define $\pi : \Sigma^{\mathbb{N}} \to C$ by

$$\{\pi(\omega)\} = \bigcap_{n=1}^{\infty} h_{j_1} \circ \cdots \circ h_{j_n}([0,1])$$

for $\omega = j_1 j_2 \dots \in \Sigma^{\mathbb{N}}$. It is easy to check that $\pi$ is a bijection and for $\omega = j_1 j_2 \dots \in \Sigma^{\mathbb{N}}$,

$$\pi(\omega) = \lim_{n\to\infty} h_{\omega|n}(0) = b_{j_1} + \sum_{n=2}^{\infty} a_{j_1} \dots a_{j_{n-1}} b_{j_n}, \tag{3}$$

where $b_0 = 0$ and $b_1 = 1 - a_1$. It is well known that $C$ can be symbolically represented as $C = \pi(\Sigma^{\mathbb{N}})$. Thus, each point of $C$ can be encoded by a unique element from $\Sigma^{\mathbb{N}}$. Throughout this paper, by $\tilde{t} = t_1 t_2 \dots$, we denote the code of $t \in C$, i.e. $\tilde{t} = t_1 t_2 \dots \in \Sigma^{\mathbb{N}}$ with $\pi(\tilde{t}) = t$. Alternatively, we have

$$C = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in \{0,1\}^n} h_\omega([0,1]),$$

where $h_\omega := h_{j_1} \circ \cdots \circ h_{j_n}$ for $\omega = j_1 \dots j_n \in \{0,1\}^n$. The endpoints of $h_\omega([0,1])$ for all $\omega \in \Sigma^*$ will be called the *endpoints of* $C$.

Let $\mu$ be the restriction of the Hausdorff measure $\mathcal{H}^\xi$ over the set $C$, i.e.

$$\mu(A) := \mathcal{H}^\xi|_C(A) = \mathcal{H}^\xi(C \cap A) \quad \text{for any Borel set } A \subseteq \mathbb{R}.$$

Note that $\mathcal{H}^\xi(C) = 1$ (cf. [1, Corollary 4.5]). Thus, an alternative definition of $\mu$ is that $\mu$ is the unique probability measure satisfying

$$\mu(A) = a_0^\xi \mu(h_0^{-1}(A)) + a_1^\xi \mu(h_1^{-1}(A)) \quad \text{for any Borel set } A \subseteq \mathbb{R}, \tag{4}$$

where $\xi$ is given by (2). Obviously, $\mu$ is atomless and called the *Cantor measure.* Two well-known properties of $\mu$ are (cf. [5, Chap. 6] and [6])

(i) *there exist finite positive numbers* $a$ *and* $b$ *such that* $ar^\xi \le \mu([x-r, x+r]) \le br^\xi$ *for any* $x \in C$ *and* $0 < r \le 1$;

(ii) *there exist* $0 < d_* < d^* < \infty$ *such that for* $\mu$*-a.e.* $x \in C$

$$\Theta^{*\xi}(\mu, x) := \limsup_{r\downarrow 0} \frac{\mu([x-r, x+r])}{(2r)^\xi} = \limsup_{r\downarrow 0} \frac{\mathcal{H}^\xi(C \cap [x-r, x+r])}{(2r)^\xi} = d^*$$

*and*

$$\Theta_*^{\xi}(\mu, x) := \liminf_{r\downarrow 0} \frac{\mu([x-r, x+r])}{(2r)^\xi} = \liminf_{r\downarrow 0} \frac{\mathcal{H}^\xi(C \cap [x-r, x+r])}{(2r)^\xi} = d_*.$$

The quantities $\Theta^{*\xi}(\mu,x)$ and $\Theta_*^{\xi}(\mu,x)$ defined above are called the *upper and lower $\xi$-densities* of $C$ at $x$, respectively. Unfortunately, few results have been obtained for the exact values of $d^*$ and $d_*$, or more general, for the exact values of $\Theta^{*\xi}(\mu,x)$ and $\Theta_*^{\xi}(\mu,x)$ for each $x \in C$. When $a_0 = a_1 = \frac{1}{3}$ (so $C$ is the classical middle-third Cantor set), the exact values of $\Theta^{*\xi}(\mu,x)$ and $\Theta_*^{\xi}(\mu,x)$ were obtained for each $x \in C$ by Feng, Hua and Wen in [2]. As the authors pointed out, their approaches are effective for the case that $a_0 = a_1 \in (0,1/3)$, i.e. $C$ is a symmetric Cantor set. To our knowledge, for the case of non-symmetric Cantor set, the exact values of $\Theta^{*\xi}(\mu,x)$ and $\Theta_*^{\xi}(\mu,x)$ are still unknown.

In the present paper, we try to determine the exact values of $\Theta^{*\xi}(\mu,x)$ and $\Theta_*^{\xi}(\mu,x)$ for certain non-symmetric Cantor sets. We will focus on those Cantor sets which have "big gaps" in their geometrical structure. In other words, we assume

$$1 - a_0 - a_1 \geq \max\{a_0, a_1\}. \tag{5}$$

Thus, in each step of the process to geometrically generate the Cantor set $C$, one removes an open interval of length not less than that of each of the remaining two closed intervals. It is easy to see that

$$1 - a_0 - a_1 \geq \max\{a_0, a_1\} \quad \text{if and only if } a_0 + 2a_1 \leq 1 \quad \text{and} \quad 2a_0 + a_1 \leq 1.$$

Let $T : [0,a_0] \cup [1-a_1, 1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} \dfrac{1}{a_0}x & x \in [0,a_0], \\ \dfrac{x-(1-a_1)}{a_1} & x \in [1-a_1, 1]. \end{cases} \tag{6}$$

Then, $T$ is a two-to-one mapping satisfying $T(C) = T^{-1}(C) = C$. Its inverse consists of two branches: $h_0$ and $h_1$. Therefore,

$$T \circ h_0(x) = T \circ h_1(x) = x \quad \text{for each } x \in [0,1],$$

and

$$h_0 \circ T(x) = x \text{ for each } x \in [0,a_0], \quad h_1 \circ T(x) = x \text{ for each } x \in [1-a_1, 1].$$

Based on [2], we develop some techniques and obtain the following theorem in the present paper.

**Theorem 1.1.** *Let $C$, $\xi$, $\mu$ and $T$ be defined as in* (1), (2), (4) *and* (6). *Suppose that $a_0$ and $a_1$ satisfy* (5). *Then,*

(i) *for a non-endpoint $x$ of $C$,*

$$\Theta^{*\xi}(\mu,x) = \frac{1}{2^{\xi}(\min\{1 - a_0 \limsup_{k\to\infty} T^k(x), a_1 \liminf_{k\to\infty} T^k(x) + 1 - a_1\})^{\xi}};$$

(ii) *for an endpoint $x$ of $C$,*

$$\Theta^{*\xi}(\mu,x)=2^{-\xi};$$

(iii) *for $\mu$-a.e. $x\in C$,*

$$\Theta^{*\xi}(\mu,x)=\frac{1}{2^{\xi}(\min\{1-a_0,1-a_1\})^{\xi}}.$$

*Furthermore, if* $\left(1+\frac{a_1}{1-a_0-a_1}\right)^{\xi}\le 1+\left(\frac{a_1}{1-a_1}\right)^{\xi}$ *and* $\left(1+\frac{a_0}{1-a_0-a_1}\right)^{\xi}\le 1+\left(\frac{a_0}{1-a_0}\right)^{\xi}$, *then,*

(i′) *for a non-endpoint $x$ of $C$,*

$$\Theta_*^{\xi}(\mu,x)=\min\left\{\frac{a_0^{\xi}}{2^{\xi}(1-a_1-\liminf_{k\to\infty}T^k(x))^{\xi}},\right.$$
$$\left.\frac{a_1^{\xi}}{2^{\xi}(\limsup_{k\to\infty}T^k(x)-a_0)^{\xi}}\right\};$$

(ii′) *for a right endpoint $x$ of $C$,*

$$\Theta_*^{\xi}(\mu,x)=\frac{a_1^{\xi}}{2^{\xi}(1-a_0)^{\xi}};$$

*for a left endpoint $x$ of $C$,*

$$\Theta_*^{\xi}(\mu,x)=\frac{a_0^{\xi}}{2^{\xi}(1-a_1)^{\xi}};$$

(iii′) *for $\mu$-a.e. $x\in C$,*

$$\Theta_*^{\xi}(\mu,x)=\min\left\{\frac{a_0^{\xi}}{2^{\xi}(1-a_1)^{\xi}},\frac{a_1^{\xi}}{2^{\xi}(1-a_0)^{\xi}}\right\}.$$

**Remark 1.2.** (i) As we can see in Theorem 1.1, to determine the upper density $\Theta^{*\xi}(\mu,x)$, we require that $a_0$ and $a_1$ satisfy condition (5). However, in order to determine the lower density $\Theta_*^{\xi}(\mu,x)$, we need, besides (5), additional conditions that

$$\begin{cases}\left(1+\dfrac{a_1}{1-a_0-a_1}\right)^{\xi}\le 1+\left(\dfrac{a_1}{1-a_1}\right)^{\xi}\\[2ex] \left(1+\dfrac{a_0}{1-a_0-a_1}\right)^{\xi}\le 1+\left(\dfrac{a_0}{1-a_0}\right)^{\xi}.\end{cases}\tag{7}$$

Here on, we like to point out that (5) plus $\xi\le\frac{1}{2}$ (recall $\xi=\dim_H C$) implies (7). In fact, it is easy to verify that

$$(a+b)^p\ge a^p+b^p+pa^{p-1}b\quad\text{if } a,b>0\quad\text{and}\quad p\ge 2.$$

Thus, when $\xi \le \frac{1}{2}$,

$$\left(1+\left(\frac{a_1}{1-a_1}\right)^{\xi}\right)^{\frac{1}{\xi}} \ge 1+\frac{a_1}{1-a_1}+\frac{1}{\xi}\left(\frac{a_1}{1-a_1}\right)^{\xi}.$$

So, we have

$$\begin{aligned}
&\left(1+\left(\frac{a_1}{1-a_1}\right)^{\xi}\right)^{\frac{1}{\xi}} - \left(1+\frac{a_1}{1-a_0-a_1}\right)\\
&\quad \ge \frac{1}{\xi}\left(\frac{a_1}{1-a_1}\right)^{\xi} - \frac{a_0a_1}{(1-a_1)(1-a_0-a_1)}\\
&\quad = \frac{a_0a_1}{\xi(1-a_1)(1-a_0-a_1)}\left(\left(\frac{1-a_1}{a_1}\right)^{1-\xi}\frac{(1-a_0-a_1)}{a_0}-\xi\right)\\
&\quad \ge \frac{a_0a_1}{\xi(1-a_1)(1-a_0-a_1)}\left(\left(\frac{1-a_1}{a_1}\right)^{1-\xi}-\xi\right)\\
&\quad \ge \frac{a_0a_1}{\xi(1-a_1)(1-a_0-a_1)}(1-\xi) \ge 0,
\end{aligned}$$

by (5). This gives the first inequality in (7) and the second inequality can be proved in the same way. As an application, we know the conditions (5) and (7) hold when $a_0 = \frac{1}{9}, a_1 = \frac{4}{9}$.

(ii) One can check that the conditions (5) and (7) hold when $a_0 = a_1 \in (0, 1/3]$. Thus, Theorem 1.1 extends those given in [2].

The proof of Theorem 1.1 is given in the next section, Theorem 2.5 for the results of upper density and Theorem 2.7 for the results of lower density.

## 2. Proof of Theorem

As stated in Sec. 1, for $t \in C$, we use $\tilde{t} = t_1t_2\ldots \in \Sigma^{\mathbb{N}}$ to denote its (unique) code (so $\pi(\tilde{t}) = t$ by (3)). In addition, we denote by $y(t,n)$ the position of the $n$th occurrence of digit 1 in $\tilde{t}$ and by $z(t,n)$ the position of the $n$th occurrence of digit 0 in $\tilde{t}$, e.g. for $\tilde{t} = 00101101110\ldots$ we have $y(t,1) = 3, y(t,5) = 9$ and $z(t,3) = 4$. We adopt the convention that

$$y(t,n) = +\infty \quad \text{if } \#\{k \in \mathbb{N} : t_k = 1\} < n$$

and

$$z(t,n) = +\infty \quad \text{if } \#\{k \in \mathbb{N} : t_k = 0\} < n.$$

Therefore, for a $t \in C$, we have

$$t = \sum_{k=1}^{\infty} a_1^{k-1} a_0^{y(t,\,k)-k}(1-a_1) \quad \text{and} \quad \mu([0,t]) = \sum_{k=1}^{\infty} a_0^{\xi(y(t,\,k)-(k-1))} a_1^{\xi(k-1)}, \tag{8}$$

where we adopt the convention $a_0^{+\infty} = a_1^{+\infty} = 0$. The first identity follows directly from (3). To verify the second identity, let $x_n = \pi(t_1 t_2 \ldots t_{y(t,n)} 00 \ldots)$. If $t = x_n$ for some $n \in \mathbb{N}$, then

$$\mu([0,t]) = \mu([0,x_n]) = \sum_{k=1}^{n} a_0^{\xi(y(t,\,k)-(k-1))} a_1^{\xi(k-1)},$$

by (4). Otherwise, we have

$$\begin{aligned}\mu([0,t]) &= \lim_{n\to\infty} \mu([0,x_n]) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} a_0^{\xi(y(t,\,k)-(k-1))} a_1^{\xi(k-1)} \\ &= \sum_{k=1}^{\infty} a_0^{\xi(y(t,\,k)-(k-1))} a_1^{\xi(k-1)}.\end{aligned}$$

The following lemma was obtained in a more general setting in [3]. For readers' convenience, we give a distinct proof here.

**Lemma 2.1.** *Let $\mu$ be defined as in* (4). *Then, for any $t \in [0,1]$, we have*

$$\frac{a_0^\xi}{(1-a_1)^\xi} t^\xi \le \mu([0,t]) \le t^\xi \quad \textit{and} \quad \frac{a_1^\xi}{(1-a_0)^\xi}(1-t)^\xi \le \mu([t,1]) \le (1-t)^\xi.$$

**Proof.** First, the inequalities $\mu([0,t]) \le t^\xi$ and $\mu([t,1]) \le (1-t)^\xi$ simply come from a well-known result that $\mathcal{H}^\xi(C \cap U) \le |U|^\xi$ for any $U \subseteq \mathbb{R}$ (cf. [4]). We now prove the left part of the first inequality. When $t \in C$,

$$\frac{\mu([0,t])}{t^\xi} = \frac{a_0^\xi a_1^{-\xi} \sum_{k=1}^{\infty} \left(a_0^{y(t,\,k)-k} a_1^k\right)^\xi}{(1-a_1)^\xi a_1^{-\xi} \left(\sum_{k=1}^{\infty} a_0^{y(t,\,k)-k} a_1^k\right)^\xi} \ge \frac{a_0^\xi}{(1-a_1)^\xi},$$

by (8), where we have used an elementary inequality that $\left(\sum_{k=1}^{\infty} c_k\right)^s \le \sum_{k=1}^{\infty} c_k^s$ if $0 < s < 1$ and all $c_k \ge 0$. When $t \in [0,1]\backslash C$, let $t_* = \inf\{x \in C : x > t\}$. Then, $t_*$ is a left endpoint of $C$ and

$$\mu([0,t]) = \mu([0,t_*]) \ge \frac{a_0^\xi}{(1-a_1)^\xi} t_*^\xi > \frac{a_0^\xi}{(1-a_1)^\xi} t^\xi.$$

In the following, we prove the left part of the second inequality. Put

$$f_0(x) = a_1 x \quad \text{and} \quad f_1(x) = a_0 x + 1 - a_0.$$

It is easy to check that $1 - C := \{1 - x : x \in C\}$ is the (compact) attractor of IFS $\{f_0, f_1\}$, i.e.

$$1 - C = f_0(1-C) \cup f_1(1-C).$$

Thus, each element of $1 - C$ can be encoded uniquely by an element from $\Sigma^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ in the same way as the elements of $C$ do via (3). Note that for each $t \in C$ with code $\tilde{t} = t_1t_2\ldots \in \Sigma^{\mathbb{N}}$, $1-t \in 1-C$ is encoded by $(1-t_1)(1-t_2)\ldots := \omega \in \Sigma^{\mathbb{N}}$. So

$$1-t = \lim_{n\to\infty} f_{\omega|n}(0) = \sum_{k=1}^{\infty} a_0^{k-1} a_1^{y(1-t,\ k)-k}(1-a_0) = \sum_{k=1}^{\infty} a_0^{k-1} a_1^{z(t,\ k)-k}(1-a_0),$$

where, if no confusion, $y(1-t,\ k)$ is the position of the $k$th digit 1 in the code $(1-t_1)(1-t_2)\ldots$ of $1-t \in 1-C$, whereas $z(t,\ k)$ is the position of the $k$th digit 0 in the code $t_1t_2\ldots$ of $t \in C$. Let $\mu^*$ be the restriction of the Hausdorff measure $\mathcal{H}^\xi$ over the set $1-C$, i.e. $\mu^*(\cdot) = \mathcal{H}^\xi((1-C)\cap\cdot)$ which satisfies

$$\mu^*(A) = a_1^\xi \mu^*(f_0^{-1}(A)) + a_0^\xi \mu^*(f_1^{-1}(A)) \quad \text{for any Borel set } A \subseteq \mathbb{R}.$$

Then

$$\mu([t,1]) = \mu^*[0,1-t] = \sum_{k=1}^{\infty} a_1^{\xi(y(1-t,\ k)-(k-1))} a_0^{\xi(k-1)} = \sum_{k=1}^{\infty} a_1^{\xi(z(t,\ k)-(k-1))} a_0^{\xi(k-1)}.$$

The desired result can be proved in the same way as above. □

**Lemma 2.2.** *Let $\mu$ be defined as in* (4). *Let $a_0, a_1$ satisfy* (5). *Then, for any Borel set $A \subseteq (-1,2)$ and any $i_1i_2\ldots i_k \in \{0,1\}^k$ with $k \in \mathbb{N}$, we have*

$$\mu(h_{i_1}\circ\cdots\circ h_{i_k}(A)) = a_{i_1}^\xi \cdots a_{i_k}^\xi \mu(A).$$

**Proof.** Note that (5) implies that

$$h_0(A)\cap h_1(C) = \emptyset \quad \text{and} \quad h_1(A)\cap h_0(C) = \emptyset.$$

Therefore, the desired result can be verified by (4). □

Using the same method as that in [2, Proposition 2.10], we get the following proposition.

**Proposition 2.3.** *Let $\mu$ be defined as in* (4). *Then* $\liminf_{k\to\infty} T^k(t) = 0$ *and* $\limsup_{k\to\infty} T^k(t) = 1$ *for $\mu$-a.e. $t \in C$.*

**Proof.** We only prove that $\liminf_{k\to\infty} T^k(t) = 0$ for $\mu$-a.e. $t \in C$. The other one can be proved in the same way. For any $\mathbf{i} = i_1\ldots i_\ell \in \{0,1\}^\ell$, denote $h_{\mathbf{i}} = h_{i_1}\circ\cdots\circ h_{i_\ell}$. Then

$$C = \bigcup_{\mathbf{i}\in\{0,1\}^\ell} h_{\mathbf{i}}(C).$$

Let $\mathbf{0} \in \{0,1\}^\ell$ be the element consisting only digit 0. Let $B_\ell$ be the attractor of IFS $\{h_{\mathbf{i}} : \mathbf{i} \in \{0,1\}^\ell, \mathbf{i} \neq \mathbf{0}\}$. Then $B_\ell$ is a proper subset of $C$ and

$$B_\ell = \{t \in C : t_{m\ell+1}\ldots t_{(m+1)\ell} \neq \mathbf{0} \text{ for all } m \geq 0\},$$

where, as before, $t_1t_2\ldots \in \{0,1\}^{\mathbb{N}}$ is the code of $t$. In addition, $\mu(B_\ell) = \mathcal{H}^\xi(B_\ell \cap C) = 0$ since $\dim_H B_\ell < \xi$. Consequently, $\mu(\cup_{\ell\geq 1} B_\ell) = 0$ and so $\mu(C\backslash\cup_{\ell\geq 1} B_\ell) = 1$.

Fix $t \in C\backslash\cup_{\ell\geq 1}B_\ell$. Then, for any $\ell \in \mathbb{N}$, there exists an $m \geq 0$ such that $\tilde{y}|\ell = t_{m\ell+1}\dots t_{(m+1)\ell} = \mathbf{0}$ where $y = T^{m\ell}(t)$. This implies $T^{m\ell}(t) \leq a_0^\ell$, leading to $\liminf_{k\to\infty} T^k(t) = 0$. $\square$

**Lemma 2.4.** *Let $\xi$ and $\mu$ be defined as in* (2) *and* (4). *Suppose that $a_0$ and $a_1$ satisfy* (5). *Then,*

(i) *for any $x \in [0, a_0]$ and any $\max\{x, a_0 - x\} \leq r \leq 1 - x$,*

$$\frac{\mu([x-r, x+r])}{(2r)^\xi} \leq \max\left\{\frac{a_0^\xi}{2^\xi(\max\{x, a_0-x\})^\xi}, \frac{1}{2^\xi(1-x)^\xi}\right\},$$

*with the equality holding either at $r = \max\{x, a_0 - x\}$ or at $r = 1 - x$;*

(ii) *for any $x \in [1 - a_1, 1]$ and any $\max\{1 - x, x - (1 - a_1)\} \leq r \leq x$,*

$$\frac{\mu([x-r, x+r])}{(2r)^\xi} \leq \max\left\{\frac{a_1^\xi}{2^\xi(\max\{1-x, x-(1-a_1)\})^\xi}, \frac{1}{2^\xi x^\xi}\right\},$$

*with the equality holding either at $r = \max\{1 - x, x - (1 - a_1)\}$ or at $r = x$.*

**Proof.** (i) If $\max\{x, a_0 - x\} \leq r \leq 1 - a_1 - x$, then

$$\frac{\mu([x-r, x+r])}{(2r)^\xi} = \frac{a_0^\xi}{(2r)^\xi} \leq \frac{a_0^\xi}{2^\xi(\max\{x, a_0-x\})^\xi}$$

with equality holding at $r = \max\{x, a_0 - x\}$. If $1 - a_1 - x \leq r \leq 1 - x$, then by Lemma 2.1 and (4),

$$\begin{aligned}\frac{\mu([x-r, x+r])}{(2r)^\xi} &= \frac{a_0^\xi + \mu[1-a_1, x+r]}{(2r)^\xi} = \frac{a_0^\xi + a_1^\xi\mu\left[0, \frac{x+r-(1-a_1)}{a_1}\right]}{(2r)^\xi}\\ &\leq \frac{a_0^\xi + (x+r-1+a_1)^\xi}{(2r)^\xi} = \frac{1}{2^\xi}\frac{u^\xi + \left(\frac{a_0}{a_1}\right)^\xi}{\left(u + \frac{1-a_1-x}{a_1}\right)^\xi},\end{aligned}$$

where $u = \frac{x+r-1+a_1}{a_1} \in [0, 1]$. Consider a function in $u$:

$$m_x(u) = \frac{u^\xi + \left(\frac{a_0}{a_1}\right)^\xi}{\left(u + \frac{1-a_1-x}{a_1}\right)^\xi}, \quad u \in [0, 1].$$

Then

$$\begin{aligned}m_x'(u) &= \frac{\xi\left[\frac{1-a_1-x}{a_1}u^{\xi-1} - \left(\frac{a_0}{a_1}\right)^\xi\right]}{\left(u + \frac{1-a_1-x}{a_1}\right)^{\xi+1}}\\ &\geq \frac{\xi\left[\frac{1-a_1-x}{a_1} - \left(\frac{a_0}{a_1}\right)^\xi\right]}{\left(u + \frac{1-a_1-x}{a_1}\right)^{\xi+1}} \geq 0, \quad u \in (0, 1],\end{aligned}$$

since we have $\frac{1-a_0-a_1}{a_1} \geq \left(\frac{a_0}{a_1}\right)^\xi$ by (5). So

$$\frac{\mu([x-r,x+r])}{(2r)^\xi} \leq \frac{1}{2^\xi} m_x(1) \leq \frac{a_1^\xi + a_0^\xi}{2^\xi(1-x)^\xi},$$

with equality holding at $r = 1-x$.

(ii) If $\max\{1-x, x-(1-a_1)\} \leq r \leq x - a_0$, then

$$\frac{\mu([x-r,x+r])}{(2r)^\xi} = \frac{a_1^\xi}{(2r)^\xi} \leq \frac{a_1^\xi}{2^\xi(\max\{1-x,x-(1-a_1)\})^\xi}$$

with the equality holding at $r = \max\{1-x, x-(1-a_1)\}$. If $x - a_0 \leq r \leq x$, then by Lemma 2.1 and (4),

$$\begin{aligned}\frac{\mu([x-r,x+r])}{(2r)^\xi} &= \frac{a_1^\xi + \mu[x-r,a_0]}{(2r)^\xi} = \frac{a_1^\xi + a_0^\xi \mu[\frac{x-r}{a_0},1]}{(2r)^\xi} \leq \frac{a_1^\xi + (a_0 - x + r)^\xi}{(2r)^\xi} \\ &= \frac{1}{2^\xi} \frac{\left(\frac{a_0+r-x}{a_0}\right)^\xi + \left(\frac{a_1}{a_0}\right)^\xi}{\left(\frac{a_0+r-x}{a_0} + \frac{x-a_0}{a_0}\right)^\xi} \leq \frac{1}{2^\xi} \frac{1^\xi + \left(\frac{a_1}{a_0}\right)^\xi}{\left(1 + \frac{x-a_0}{a_0}\right)^\xi} = \frac{1}{2^\xi x^\xi},\end{aligned}$$

with equality holding at $r = x$. □

**Theorem 2.5.** *Let* $C$, $\xi$, $\mu$ *and* $T$ *be defined as in* (1), (2), (4) *and* (6). *Suppose that* $a_0$ *and* $a_1$ *satisfy* (5). *Then,*

(i) *for a non-endpoint* $x$ *of* $C$,

$$\Theta^{*\xi}(\mu,x) = \frac{1}{2^\xi(\min\{1 - a_0 \limsup_{k\to\infty} T^k(x), a_1 \liminf_{k\to\infty} T^k(x) + 1 - a_1\})^\xi};$$

(ii) *for an endpoint* $x$ *of* $C$,

$$\Theta^{*\xi}(\mu,x) = 2^{-\xi};$$

(iii) *for* $\mu$*-a.e.* $x \in C$,

$$\Theta^{*\xi}(\mu,x) = \frac{1}{2^\xi(\min\{1-a_0, 1-a_1\})^\xi}.$$

**Proof.** (i) Fix a non-endpoint $x$ of $C$. Let $i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ be the code of $x$. Then $h_{i_1} \circ \cdots \circ h_{i_k}([0,1]) \downarrow \{x\}$ as $k \to \infty$. For a given $0 < r < \min\{a_0, a_1\}$, there exists a $k \in \mathbb{N}$ such that $[x-r,x+r]$ contains the interval $h_{i_1} \circ \cdots \circ h_{i_k}([0,1])$, but does not contain the interval $h_{i_1} \circ \cdots \circ h_{i_{k-1}}([0,1])$. Hence,

$$h_{i_k}[0,1] \subseteq \left(h_{i_1} \circ \cdots \circ h_{i_{k-1}}\right)^{-1}([x-r,x+r]) \subseteq (-1,2).$$

With $y = (h_{i_1} \circ \cdots \circ h_{i_{k-1}})^{-1}(x)$ (then $y = T^{k-1}(x)$) and $r' = r(a_{i_1} \cdots a_{i_{k-1}})^{-1}$, we have

$$\frac{\mu[x-r,x+r]}{(2r)^\xi} = \frac{(a_{i_1} \cdots a_{i_{k-1}})^\xi \mu[y-r',y+r']}{(a_{i_1} \cdots a_{i_{k-1}})^\xi (2r')^\xi} = \frac{\mu[y-r',y+r']}{(2r')^\xi}, \tag{9}$$

by Lemma 2.2.

If $k = z(x,\ n)$ (i.e. $i_k = 0$) for some $n \in \mathbb{N}$, then $y \in [0, a_0]$ and $[y - r', y + r']$ contains $[0, a_0]$, but does not contain $[0, 1]$. So, $\max\{y, a_0 - y\} \le r' \le 1 - y$ and by Lemma 2.4, we have

$$\begin{aligned}
&\frac{\mu([y-r', y+r'])}{(2r')^\xi} \\
&\quad \le \max\left\{\frac{a_0^\xi}{2^\xi(\max\{T^{z(x,\ n)-1}(x), a_0 - T^{z(x,\ n)-1}(x)\})^\xi}, \frac{1}{2^\xi(1 - T^{z(x,\ n)-1}(x))^\xi}\right\} \\
&\quad = \max\left\{\frac{1}{2^\xi(\max\{T^{z(x,\ n)}(x), 1 - T^{z(x,\ n)}(x)\})^\xi}, \frac{1}{2^\xi(1 - T^{z(x,\ n)-1}(x))^\xi}\right\}, \qquad (10)
\end{aligned}$$

where we have used the fact that $T^{z(x,\ n)-1}(x) = a_0 T^{z(x,\ n)}(x)$ by (6) since $T^{z(x,\ n)-1}(x) \in [0, a_0]$.

If $k = y(x,\ n)$ (i.e. $i_k = 1$) for some $n \in \mathbb{N}$, then $y \in [1 - a_1, 1]$ and $[y - r', y + r']$ contains $[1 - a_1, 1]$ but does not contain $[0, 1]$. So, $\max\{1 - y, y - (1 - a_1)\} \le r' \le y$ and by Lemma 2.4, we have

$$\begin{aligned}
&\frac{\mu([y-r', y+r'])}{(2r')^\xi} \\
&\quad \le \max\left\{\frac{a_1^\xi}{2^\xi(\max\{1 - T^{y(x,n)-1}(x), T^{y(x,n)-1}(x) - (1 - a_1)\})^\xi}, \frac{1}{2^\xi(T^{y(x,n)-1}(x))^\xi}\right\} \\
&\quad = \max\left\{\frac{1}{2^\xi(\max\{1 - T^{y(x,\ n)}(x), T^{y(x,\ n)}(x)\})^\xi}, \frac{1}{2^\xi(T^{y(x,\ n)-1}(x))^\xi}\right\}, \qquad (11)
\end{aligned}$$

where $T^{y(x,\ n)-1}(x) - (1 - a_1) = a_1 T^{y(x,\ n)}(x)$ by (6) since $T^{y(x,\ n)-1}(x) \in [1 - a_1, 1]$. Note that

$$1 - T^{z(x,\ n)-1}(x) = \max\{T^{z(x,\ n)-1}(x), 1 - T^{z(x,\ n)-1}(x)\}$$

and

$$T^{y(x,\ n)-1}(x) = \max\{T^{y(x,\ n)-1}(x), 1 - T^{y(x,\ n)-1}(x)\}.$$

Hence, by (9)–(11),

$$\limsup_{r \downarrow 0} \frac{\mu([x - r, x + r])}{(2r)^\xi} \le \frac{1}{2^\xi(\liminf_{k\to\infty} \max\{T^k(x), 1 - T^k(x)\})^\xi}.$$

In fact, we have

$$\Theta^{*\xi}(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu([x - r, x + r])}{(2r)^\xi} = \frac{1}{2^\xi(\liminf_{k\to\infty} \max\{T^k(x), 1 - T^k(x)\})^\xi}.$$

This is because the upper bounds in (10) and (11) can be reached at certain $r$'s as $r \downarrow 0$ by Lemma 2.4. More exactly, the upper bounds in (10) and (11) are reached when

$$\begin{aligned} r \in & \bigcup_{k=z(x,n), n\in\mathbb{N}} \{\max\{x - h_{i_1} \circ \cdots \circ h_{i_{k-1}}(0), h_{i_1} \circ \cdots \circ h_{i_{k-1}}(a_0) - x\}, \\ & h_{i_1} \circ \cdots \circ h_{i_{k-1}}(1) - x\} \\ \cup & \bigcup_{k=y(x,n), n\in\mathbb{N}} \{\max\{x - h_{i_1} \circ \cdots \circ h_{i_{k-1}}(1-a_1), h_{i_1} \circ \cdots \circ h_{i_{k-1}}(1) - x\}, \\ & x - h_{i_1} \circ \cdots \circ h_{i_{k-1}}(0)\}. \end{aligned}$$

Finally, the left we need to show is

$$\begin{aligned} & \liminf_{k\to\infty} \max\left\{T^k(x), 1 - T^k(x)\right\} \\ & \quad = \min\left\{1 - a_0 \limsup_{k\to\infty} T^k(x),\ a_1 \liminf_{k\to\infty} T^k(x) + 1 - a_1\right\}. \end{aligned}$$

Let $\Omega_1 = \{k \in \mathbb{N} : T^k(x) \in [0, a_0]\}$ and $\Omega_2 = \{k \in \mathbb{N} : T^k(x) \in [1 - a_1, 1]\}$. Then

$$\begin{aligned} & \liminf_{k\to\infty} \max\{T^k(x), 1 - T^k(x)\} \\ & \quad = \min\left\{\liminf_{k\in\Omega_1,\ k\to\infty} (1 - T^k(x)), \liminf_{k\in\Omega_2,\ k\to\infty} T^k(x)\right\}. \end{aligned}$$

Note that $1 - T^k(x) = 1 - a_0 T^{k+1}(x)$ for $k \in \Omega_1$, $T^k(x) = 1 - a_1 + a_1 T^{k+1}(x)$ for $k \in \Omega_2$. Thus

$$\liminf_{k\in\Omega_1,\ k\to\infty} (1 - T^k(x)) = 1 - a_0 \limsup_{k\in\Omega_1,\ k\to\infty} T^{k+1}(x) = 1 - a_0 \limsup_{k\to\infty} T^k(x),$$

and

$$\liminf_{k\in\Omega_2,\ k\to\infty} T^k(x) = 1 - a_1 + a_1 \liminf_{k\in\Omega_2,\ k\to\infty} T^{k+1}(x) = a_1 \liminf_{k\to\infty} T^k(x) + 1 - a_1.$$

(ii) From the proof of (i), it follows that if $x$ is a left endpoint of $C$, then

$$\begin{aligned} & \Theta^{*\xi}(\mu, x) \\ & \quad = \limsup_{n\to\infty} \max\left\{\frac{1}{2^\xi(\max\{T^{z(x,\,n)}(x), 1 - T^{z(x,\,n)}(x)\})^\xi}, \frac{1}{2^\xi(1 - T^{z(x,\,n)-1}(x))^\xi}\right\} \\ & \quad = 2^{-\xi}; \end{aligned}$$

if $x$ is a right endpoint of $C$, then

$$\begin{aligned} & \Theta^{*\xi}(\mu, x) \\ & \quad = \limsup_{n\to\infty} \max\left\{\frac{1}{2^\xi(\max\{1 - T^{y(x,\,n)}(x), T^{y(x,\,n)}(x)\})^\xi}, \frac{1}{2^\xi(T^{y(x,\,n)-1}(x))^\xi}\right\} \\ & \quad = 2^{-\xi}. \end{aligned}$$

(iii) It follows directly from (i) and Proposition 2.3. □

For the results of lower density we need the following lemma.

**Lemma 2.6.** *Let* $\xi$ *and* $\mu$ *be defined as in* (2) *and* (4). *Suppose that* $a_0$ *and* $a_1$ *satisfy* (5) *and* (7). *Then,*

(i) *for any* $x \in [0, a_0]$ *and any* $\max\{x, a_0 - x\} \le r \le 1 - x$

$$\frac{\mu([x-r, x+r])}{(2r)^\xi} \ge \frac{a_0^\xi}{2^\xi(1 - a_1 - x)^\xi}$$

*with the equality holding at* $r = 1 - a_1 - x$;

(ii) *for any* $x \in [1 - a_1, 1]$ *and for* $\max\{1 - x, x - (1 - a_1)\} \le r \le x$

$$\frac{\mu([x-r, x+r])}{(2r)^\xi} \ge \frac{a_1^\xi}{2^\xi(x - a_0)^\xi},$$

*with the equality holding at* $r = x - a_0$.

**Proof.** (i) Fix an $x \in [0, a_0]$. If $\max\{x, a_0 - x\} \le r \le 1 - a_1 - x$, then

$$\frac{\mu([x-r, x+r])}{(2r)^\xi} = \frac{a_0^\xi}{(2r)^\xi} \ge \frac{a_0^\xi}{2^\xi(1 - a_1 - x)^\xi},$$

with the equality holding at $r = 1 - a_1 - x$. If $1 - a_1 - x \le r \le 1 - x$, then by (4) and Lemma 2.1,

$$\begin{aligned}\frac{\mu([x-r, x+r])}{(2r)^\xi} &= \frac{a_0^\xi + \mu[1 - a_1, x + r]}{(2r)^\xi} = \frac{a_0^\xi + a_1^\xi \mu\big[0, \frac{x+r-(1-a_1)}{a_1}\big]}{(2r)^\xi} \\ &\ge \frac{a_0^\xi + a_1^\xi\big(\frac{a_0}{1-a_1}\big)^\xi\big(\frac{x+r-1+a_1}{a_1}\big)^\xi}{(2r)^\xi} = \left(\frac{a_0}{2(1-a_1)}\right)^\xi \frac{u^\xi + \big(\frac{1-a_1}{a_1}\big)^\xi}{\big(u + \frac{1-a_1-x}{a_1}\big)^\xi},\end{aligned}$$

where $u = \frac{x+r-(1-a_1)}{a_1} \in [0, 1]$. Let

$$g(u) = \frac{u^\xi + \big(\frac{1-a_1}{a_1}\big)^\xi}{\big(u + \frac{1-a_1-x}{a_1}\big)^\xi}, \quad u \in [0, 1].$$

Then

$$g'(u) = \frac{\xi\big[u^{\xi-1}\big(\frac{1-a_1-x}{a_1}\big) - \big(\frac{1-a_1}{a_1}\big)^\xi\big]}{\big(u + \frac{1-a_1-x}{a_1}\big)^{\xi+1}}, \quad u \in (0, 1].$$

So, $g(u)$ is either increasing in $[0, 1]$ or first increasing and then decreasing in $[0, 1]$ (i.e. $g(u)$ is unimodal in $[0, 1]$). Note that the first inequality in (7) is equivalent to

$$(1 - a_1)^\xi \left(\frac{1 - a_0}{1 - a_0 - a_1}\right)^\xi \le a_1^\xi + (1 - a_1)^\xi.$$

Thus,

$$g(1)=\frac{a_1^\xi+(1-a_1)^\xi}{(1-x)^\xi}\geq\frac{(1-a_1)^\xi}{(1-x)^\xi}\left(\frac{1-a_0}{1-a_0-a_1}\right)^\xi$$
$$\geq\frac{(1-a_1)^\xi}{(1-x)^\xi}\left(\frac{1-x}{1-a_1-x}\right)^\xi=g(0),$$

leading to

$$\frac{\mu([x-r,x+r])}{(2r)^\xi}\geq\left(\frac{a_0}{2(1-a_1)}\right)^\xi g(0)=\frac{a_0^\xi}{2^\xi(1-a_1-x)^\xi},$$

with the equality holding at $r=1-a_1-x$.

(ii) Fix an $x\in[1-a_1,1]$. If $\max\{1-x,x-(1-a_1)\}\leq r\leq x-a_0$, then

$$\frac{\mu([x-r,x+r])}{(2r)^\xi}=\frac{a_1^\xi}{(2r)^\xi}\geq\frac{a_1^\xi}{2^\xi(x-a_0)^\xi},$$

with the equality holding at $r=x-a_0$. If $x-a_0\leq r\leq x$, then by (4) and Lemma 2.1,

$$\begin{aligned}\frac{\mu([x-r,x+r])}{(2r)^\xi}&=\frac{a_1^\xi+\mu[x-r,a_0]}{(2r)^\xi}=\frac{a_1^\xi+a_0^\xi\mu[\frac{x-r}{a_0},1]}{(2r)^\xi}\\&\geq\frac{a_1^\xi+\left(\frac{a_1}{1-a_0}\right)^\xi(a_0-x+r)^\xi}{(2r)^\xi}\\&=\left(\frac{a_1}{2(1-a_0)}\right)^\xi\frac{\left(\frac{a_0-x+r}{a_0}\right)^\xi+\left(\frac{1-a_0}{a_0}\right)^\xi}{\left(\frac{a_0-x+r}{a_0}+\frac{x-a_0}{a_0}\right)^\xi}\\&\geq\left(\frac{a_1}{2(1-a_0)}\right)^\xi\frac{0^\xi+\left(\frac{1-a_0}{a_0}\right)^\xi}{\left(0+\frac{x-a_0}{a_0}\right)^\xi}\\&=\frac{a_1^\xi}{2^\xi(x-a_0)^\xi},\end{aligned}$$

with the equality holding at $r=x-a_0$, where the last inequality is obtained in the same way as above by means of the second inequality in (7). □

**Theorem 2.7.** *Let* $C$, $\xi$, $\mu$ *and* $T$ *be defined as in* (1), (2), (4) *and* (6). *Suppose that* $a_0$ *and* $a_1$ *satisfy* (5) *and* (7). *Then*

(i′) *for a non-endpoint* $x$ *of* $C$,

$$\Theta_*^\xi(\mu,x)=\min\left\{\frac{a_0^\xi}{2^\xi(1-a_1-\liminf_{k\to\infty}T^k(x))^\xi},\right.$$
$$\left.\frac{a_1^\xi}{2^\xi(\limsup_{k\to\infty}T^k(x)-a_0)^\xi}\right\};$$

(ii$'$) *for a right endpoint $x$ of $C$,*

$$\Theta_*^\xi(\mu, x) = \frac{a_1^\xi}{2^\xi(1-a_0)^\xi};$$

*for a left endpoint $x$ of $C$,*

$$\Theta_*^\xi(\mu, x) = \frac{a_0^\xi}{2^\xi(1-a_1)^\xi};$$

(iii$'$) *for $\mu$-a.e. $x \in C$,*

$$\Theta_*^\xi(\mu, x) = \min\left\{\frac{a_0^\xi}{2^\xi(1-a_1)^\xi}, \frac{a_1^\xi}{2^\xi(1-a_0)^\xi}\right\}.$$

**Proof.** (i$'$) Fix a non-endpoint $x$ of $C$. Let $i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ be the code of $x$. Then $h_{i_1} \circ \cdots \circ h_{i_k}([0,1]) \downarrow \{x\}$ as $k \to \infty$. For a given $0 < r < \min\{a_0, a_1\}$, there exists a $k \in \mathbb{N}$ such that $[x-r, x+r]$ contains the interval $h_{i_1} \circ \cdots \circ h_{i_k}([0,1])$ but does not contain the interval $h_{i_1} \circ \cdots \circ h_{i_{k-1}}([0,1])$. Hence,

$$h_{i_k}[0,1] \subseteq (h_{i_1} \circ \cdots \circ h_{i_{k-1}})^{-1}([x-r, x+r]) \subseteq (-1, 2).$$

With $y = (h_{i_1} \circ \cdots \circ h_{i_{k-1}})^{-1}(x)$ (then $y = T^{k-1}(x)$) and $r' = r(a_{i_1} \cdots a_{i_{k-1}})^{-1}$, we have

$$\frac{\mu[x-r, x+r]}{(2r)^\xi} = \frac{(a_{i_1} \cdots a_{i_{k-1}})^\xi \mu[y-r', y+r']}{(a_{i_1} \cdots a_{i_{k-1}})^\xi (2r')^\xi} = \frac{\mu[y-r', y+r']}{(2r')^\xi},$$

by Lemma 2.2.

If $k = z(x,\ n)$ for some $n \in \mathbb{N}$, then $y \in [0, a_0]$ and $[y-r', y+r']$ contains $[0, a_0]$, but does not contain $[0,1]$. So, $\max\{y, a_0 - y\} \le r' \le 1 - y$ and by Lemma 2.6,

$$\frac{\mu([y-r', y+r'])}{(2r')^\xi} \ge \frac{a_0^\xi}{2^\xi(1 - a_1 - T^{z(x,\ n)-1}(x))^\xi}.$$

If $k = y(x,\ n)$ for some $n \in \mathbb{N}$, then $y \in [1-a_1, 1]$ and $[y-r', y+r']$ contains $[1-a_1, 1]$, but does not contain $[0,1]$. So, $\max\{1-y, y-(1-a_1)\} \le r' \le y$ and by Lemma 2.6,

$$\frac{\mu([y-r', y+r'])}{(2r')^\xi} \ge \frac{a_1^\xi}{2^\xi(T^{y(x,\ n)-1}(x) - a_0)^\xi}.$$

Hence

$$\begin{aligned}\Theta_*^\xi(\mu, x) &= \liminf_{r \downarrow 0} \frac{\mu([x-r, x+r])}{(2r)^\xi} \\ &\ge \min\left\{\liminf_{n\to\infty} \frac{a_0^\xi}{2^\xi(1-a_1-T^{z(x,\ n)-1}(x))^\xi}, \liminf_{n\to\infty} \frac{a_1^\xi}{2^\xi(T^{y(x,\ n)-1}(x) - a_0)^\xi}\right\}\end{aligned}$$

$$= \min\left\{\frac{a_0^\xi}{2^\xi(1-a_1-\liminf_{n\to\infty} T^{z(x,\,n)-1}(x))^\xi},\right.$$

$$\left.\frac{a_1^\xi}{2^\xi(\limsup_{n\to\infty} T^{y(x,\,n)-1}(x)-a_0)^\xi}\right\}$$

$$= \min\left\{\frac{a_0^\xi}{2^\xi(1-a_1-\liminf_{k\to\infty} T^{k}(x))^\xi}, \frac{a_1^\xi}{2^\xi(\limsup_{k\to\infty} T^{k}(x)-a_0)^\xi}\right\}.$$

By the same argument as that used in the proof of Theorem 2.5, the lower bound can be reached, i.e.

$$\Theta_*^\xi(\mu, x) = \min\left\{\frac{a_0^\xi}{2^\xi(1-a_1-\liminf_{n\to\infty} T^{k}(x))^\xi}, \frac{a_1^\xi}{2^\xi(\limsup_{n\to\infty} T^{k}(x)-a_0)^\xi}\right\}.$$

(ii′) From the proof of (i′), it follows that if $x$ is a right endpoint of $C$, then

$$\Theta_*^\xi(\mu, x) = \liminf_{n\to\infty} \frac{a_1^\xi}{2^\xi(T^{y(x,\,n)-1}(x)-a_0)^\xi} = \frac{a_1^\xi}{2^\xi(1-a_0)^\xi};$$

if $x$ is a left endpoint of $C$, then

$$\Theta_*^\xi(\mu, x) = \liminf_{n\to\infty} \frac{a_0^\xi}{2^\xi(1-a_1-T^{z(x,\,n)-1}(x))^\xi} = \frac{a_0^\xi}{2^\xi(1-a_1)^\xi}.$$

(iii′) This follows directly from (i′) and Proposition 2.3. □

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