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HOW SMOOTH IS A DEVIL'S STAIRCASE?

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Abstract

Let the self-similar set C in $\mathbf{R}$ be defined by $C = \bigcup_{j=0}^{r} h_j(C)$ with a disjoint union, where the h_j's are similitude mappings with ratios $0 < a_j < 1$. Let μ on C be the self-similar probability measure corresponding to the probability vector $(a_0^\xi, a_1^\xi, \ldots, a_r^\xi)$, where $\xi = \dim_H C$ is the Hausdorff dimension of C. Let S be the set of points at which the probability distribution function F of μ has no derivative, finite or infinite. We prove that $\dim_H S = (\dim_H C)^2$ and $\dim_P S = \dim_B S = \dim_H C$.

Keywords: Hausdorff Dimension; Packing Dimension; Non-Differentiability; Cantor Function; Self-Similar Measure.

1. INTRODUCTION

Let $h_i(x) = ax + i(1-a)$, $i = 0, 1$ with $x \in [0, 1]$ and $0 < a < \frac{1}{2}$. Then there exists a unique non-empty compact set $C \subset [0, 1]$ such that

$$C = h_0(C) \bigcup h_1(C) .$$

It is well-known that the Hausdorff dimension of C equals $\dim_H C = -\frac{\log 2}{\log a}$. Let μ be the uniform probability measure on C. Consider the distribution function which is often referred to as the Devil's staircase (for $a = \frac{1}{3}$):

$$F(x) = \mu([0, x]), \quad x \in [0, 1] .$$

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0, 1] \backslash C$ and the upper derivative of $F(x)$ is infinite on C. Let S be the set of points at which $F(x)$ is not differentiable, i.e. the set of points in

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C at which the lower derivative of $F(x)$ is finite. S can be decomposed into

$$S = N^+ \bigcup N^- \bigcup \{t : t \text{ is an endpoint of } C\} \quad (1)$$

where $N^+(N^-)$ is the set of non-end points of C at which the lower right (left) derivative of $F(x)$ is finite. Each $t \in C$ can be encoded by a unique $0-1$ sequence, denoted by $\tilde{t} = (t(1), t(2), \ldots)$, which satisfies $\{t\} = \bigcap_{n=1}^{\infty} h_{t(1)} \circ \cdots \circ h_{t(n)}([0, 1])$. Now let $z(t, n)$ denote the position of the nth zero in $\tilde{t}$. The set N^+ (symmetrically for N^-) is characterized by Darst[1] as follows:

(a) if $t \in N^+$, then $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \geq -\frac{\log a}{\log 2}$;
(b) if $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} > -\frac{\log a}{\log 2}$, then $t \in N^+$.

By means of the above (a) and (b), Darst[1] proves that

$$\dim_H S = \dim_H N^+ = \left[\frac{\log 2}{\log a}\right]^2 = (\dim_H C)^2 .$$

It is not difficult to show[1] that $\dim_H S = \dim_H N^+ = \left[\frac{\log(r+1)}{\log a}\right]^2 = (\dim_H C)^2$ still holds for a little bit more general Cantor set C with $C = \bigcup_{j=0}^{r} h_j(C)$, where $h_j(x) = ax + (1-a)\frac{j}{r}$, $j = 0, 1, \ldots, r$ and $0 < a < \frac{1}{r+1}$.

The Cantor sets C described above are all homogeneous in the sense that all similitude mappings $h_j(x)$ have the same scaling factor a and the gaps between the images $h_j([0, 1])$, $j = 0, 1, \ldots, r$, have the same length. In the following, let r be a positive integer and let the Cantor set C in $\mathbf{R}$ be defined by

$$C = \bigcup_{j=0}^{r} h_j(C)$$

where $h_j(x) = a_j x + b_j$, $j = 0, 1, \ldots, r$, with $0 < a_j < 1$. Without loss of generality, we shall assume that $b_0 = 0$ and $a_r + b_r = 1$. We furthermore assume that the images $h_j([0, 1])$, $j = 0, 1, \ldots, r$ are pairwise disjoint and are lined up from left to right. In this paper, we will determine the Hausdorff, box and packing dimensions of the set of non-differentiability points of the distribution function $F(x) = \mu([0, x])$ of the self-similar probability measure μ associated to the mappings $(h_j)_{j=0}^r$.

In order to encode the elements of C, we introduce some notations. Let $\Omega = \{0, 1, \ldots, r\}$. We will write

(i) $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \ldots) : \sigma(j) \in \Omega\}$;
(ii) $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k)) : \sigma(j) \in \Omega\}$ for $k \in \mathbf{N}$ and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$;
(iii) $|\cdot|$ is used to denote the length of word. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \ldots, \sigma(|\sigma|), \tau(1), \ldots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \ldots, \tau(|\tau|), \sigma(1), \sigma(2), \ldots)$ for any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$;
(iv) $\sigma|k = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ for $\sigma \in \Omega^\omega$ and $k \in \mathbf{N}$.

Denote $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \Omega^k$ and $x \in \mathbf{R}$. Then for $\sigma \in \Omega^k$, the intervals $h_{\sigma*0}([0, 1])$, $h_{\sigma*1}([0, 1]), \ldots, h_{\sigma*r}([0, 1])$ are contained in $h_\sigma([0, 1])$ in this order where the left endpoint of $h_{\sigma*0}([0, 1])$ coincides with the left endpoint of $h_\sigma([0, 1])$, and the right endpoint of $h_{\sigma*r}([0, 1])$ coincides with the right endpoint of $h_\sigma([0, 1])$. Moreover the length of interval $h_\sigma([0, 1])$ equals $\lambda(h_\sigma([0, 1])) = \prod_{j=1}^{k} a_{\sigma(j)} =: a_\sigma$ for $\sigma \in \Omega^k$. For $j = 1, 2, \ldots$, we define

$$C_j = \bigcup_{\sigma\in\Omega^j} h_\sigma([0, 1]) .$$

Then $C_j \downarrow C$ as $j \to \infty$ and $x \in C$ can be encoded by a unique $\sigma \in \Omega^\omega$ satisfying $\{x\} = \bigcap_{k=1}^{\infty} h_{\sigma|k}([0, 1])$. We usually denote this unique code of x by $\tilde{x}$ and use $x(k)$ to denote the kth component of $\tilde{x}$.

It is well-known that $\dim_H C = \dim_B C = \dim_P C = \xi$ where ξ is given by

$$\sum_{j=0}^{r} a_j^\xi = 1 . \quad (2)$$

Let μ be the self-similar probability measure on C corresponding to the probability vector $(a_0^\xi, a_1^\xi, \ldots, a_r^\xi)$, i.e. [see e.g. Hutchinson[2]], the measure satisfying

$$\mu(h_\sigma([0, 1])) = \prod_{j=1}^{k} a_{\sigma(j)}^\xi = a_\sigma^\xi, \text{ for any } \sigma \in \Omega^k,\ k \in \mathbf{N} .$$

Consider the distribution function of the probability measure μ, also called Cantor function:

$$F(x) = \mu([0, x]), \quad x \in [0, 1] .$$

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0, 1]\backslash C$. We will show that the upper derivative of $F(x)$ is infinite on C. Let S be the set of points at which $F(x)$ is not differentiable, i.e. the set of points in C at which the lower derivative of $F(x)$ is finite. The set

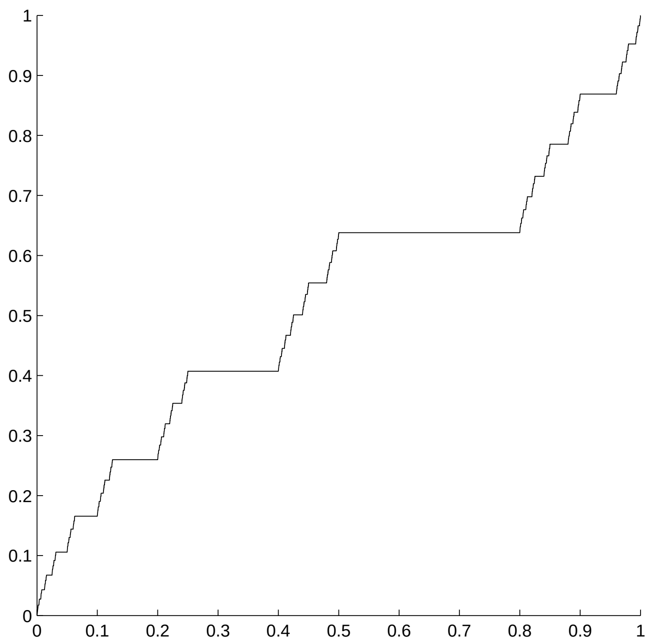

Fig. 1 The graph of $F(x)$ for the case $a_0 = 0.5$, $b_0 = 0$, $a_1 = 0.2$ and $b_1 = 0.8$.

S can be decomposed in the same way as in (1). The endpoints of $h_\sigma([0, 1])$ for a $\sigma \in \Omega^*$ will be called the endpoints of C. Obviously, any endpoint e of C lies in C and except for a finite number of terms, its coding $\tilde{e}$ consists of either only the symbol 0 if e is the left endpoint of some $h_\sigma([0, 1])$, or only the symbol r if e is the right endpoint of some $h_\sigma([0, 1])$.

In this paper, we will prove that $\dim_H S = \dim_H N^+ = \dim_H N^- = (\dim_H C)^2 = \xi^2$ and $\dim_B S = \dim_P S = \dim_H C = \xi$.

2. CODES OF NON-DIFFERENTIABILITY POINTS

In this section, we characterize the set S of non-differentiability points by means of the behavior of their codings. We focus on N^+. Results on N^- can be obtained symmetrically.

Proposition 2.1. *The upper derivative of $F(x)$ is infinite for all $x \in C$.*

Proof. For any t in C, t not a right endpoint, let the code be $\tilde{t} = (t(1), t(2), \ldots)$. Then $\tilde{t}$ has infinitely many entries lying in $\Omega \backslash \{r\}$. Suppose $\tilde{t}$ has a entry from $\Omega\backslash\{r\}$ in position j. Then t lies in the interval $h_{\tilde{t}|(j-1)}([0, 1])$ but is not equal to the right endpoint u of $h_{\tilde{t}|(j-1)}([0, 1])$, where

$$\tilde{u} = (t(1), \ldots, t(j-1), r, r, \ldots)\,.$$

Note that u is also the right endpoint of $h_{\tilde{u}|j}([0, 1])$ and that $t \notin h_{\tilde{u}|j}([0, 1])$. Thus, we have that $t, u \in h_{\tilde{t}|(j-1)}([0, 1])$ and $(t, u] \supseteq h_{\tilde{u}|j}([0, 1])$. Consider the slope of the line segment from the point $P = (t, F(t))$ on the graph of F to the point $Q = (u, F(u))$. We have

$$\begin{aligned}\frac{F(u) - F(t)}{u - t} &= \frac{\mu((t, u])}{u - t} \geq \frac{\mu(h_{\tilde{u}|j}([0, 1]))}{|h_{\tilde{t}|(j-1)}([0, 1])|} \\ &= \frac{a^{\xi}_{\tilde{t}|(j-1)} a^{\xi}_{r}}{a_{\tilde{t}|(j-1)}} \\ &= a^{\xi}_{r} (a_{\tilde{t}|(j-1)})^{\xi - 1} \to \infty \text{ as } j \to \infty\,.\end{aligned}$$

Symmetrically, the upper left derivative of t at a non-left-end point of C is infinite. □

Proposition 2.2. *Let $\Gamma = \{0, 1, \ldots, r-1\}$. Let $\underline{a} = \min_{j\in\Gamma} a_j$ and $\bar{a} = \max_{j\in\Gamma} a_j$. Let $t \in C$ be not an endpoint of C and let $z(t, n)$ denote the position of the nth occurrence of elements of Γ in $\tilde{t}$, then*

(i) *if $t \in N^+$, then $\limsup_{n\to\infty} \left[\frac{n}{z(t, n)}(1 - \xi^{-1})\cdot \left(\frac{\log \bar{a}}{\log a_r} - 1\right) + \frac{z(t, n+1)}{z(t, n)} - \xi^{-1}\right] \geq 0$;*

(ii) *if $\limsup_{n\to\infty} \left[\frac{n}{z(t, n)}(1 - \xi^{-1}) \left(\frac{\log \underline{a}}{\log a_r} - 1\right) + \frac{z(t, n+1)}{z(t, n)} - \xi^{-1}\right] > 0$, then $t \in N^+$.*

Proof. We first prove statement (i), i.e. the lower-right derivative of $F(x)$ is infinite at a non-endpoint t of C when

$$\limsup_{n\to\infty} \left[\frac{n}{z(t, n)}(1 - \xi^{-1}) \left(\frac{\log \bar{a}}{\log a_r} - 1\right) + \frac{z(t, n+1)}{z(t, n)} - \xi^{-1}\right] < 0\,.$$

Consider such a point t with $\tilde{t} = (t(1), t(2), \ldots)$. Let k be a positive integer such that

$$\frac{n}{z(t, n)}(1-\xi^{-1}) \left(\frac{\log \bar{a}}{\log a_r} - 1\right) + \frac{z(t, n+1)}{z(t, n)} - \xi^{-1} < q \tag{3}$$

for some negative real number q whenever $n \geq k$. Let u be a positive number such that u is smaller than the distance between t and $[0, 1]\backslash C_l$ with

$l = z(t, k)$. Let x be a point in the segment $(t, t+u)$. Then $t, x \in h_{\tilde{t}|l}([0, 1])$. We will see that $(F(x) - F(t))/(x-t)$ is large relative to k, so t is not in N^+. Let i denote the level at which $x \notin h_{\tilde{t}|i}([0, 1])$ but $x \in h_{\tilde{t}|(i-1)}([0, 1])$. Note that also $t \in h_{\tilde{t}|(i-1)}([0, 1])$. Thus $x - t \le |h_{\tilde{t}|(i-1)}([0, 1])| = a_{\tilde{t}|(i-1)}$; also $i = z(t, n)$ for some $n > k$. Put $j = z(t, n+1) - 1$, and by v we denote the right endpoint of $h_{\tilde{t}|j}([0, 1])$, which implies that $\tilde{v} = (t(1), \ldots, t(j), r, r, \ldots)$ and $(t, v] \supseteq h_{\tilde{v}|(j+1)}([0, 1])$. Then we have $t < v < x$ and $F(v) - F(t) = \mu((t, v]) \ge \mu(h_{\tilde{v}|(j+1)}([0, 1])) = (a_{t|j})^{\xi} a_r^{\xi}$. Therefore, defining

$$\widetilde{\Pi}_n = \prod_{i=1, t(i)\neq r}^{z(t,n)-1} a_{t(i)}$$

we have

$$\begin{aligned}
&\frac{F(x) - F(t)}{x - t} \\
&\ge \frac{(a_{\tilde{t}|j})^{\xi} a_r^{\xi}}{a_{\tilde{t}|(i-1)}} = \frac{a_r^{\xi}\left(\prod_{i=1}^{z(t,n+1)-1} a_{t(i)}\right)^{\xi}}{\prod_{i=1}^{z(t,n)-1} a_{t(i)}} \\
&= \frac{a_r^{\xi}\left[a_r^{z(t,n+1)-1-n} a_{t(z(t,n))}\widetilde{\Pi}_n\right]^{\xi}}{a_r^{z(t,n)-1-(n-1)}\widetilde{\Pi}_n} \\
&= a_{t(z(t,n))}^{\xi}\widetilde{\Pi}_n^{\xi-1} a_r^{(z(t,n+1)-n)\xi - z(t,n)+n} \\
&= a_{t(z(t,n))}^{\xi}\left[\widetilde{\Pi}_n^{\frac{\xi-1}{z(t,n)}} a_r^{\left(\frac{z(t,n+1)}{z(t,n)} - \frac{n}{z(t,n)}\right)\xi - 1 + \frac{n}{z(t,n)}}\right]^{z(t,n)} \\
&\ge \underline{a}^{\xi}\left[\bar{a}^{\frac{(n-1)(\xi-1)}{z(t,n)}} a_r^{\left(\frac{z(t,n+1)}{z(t,n)} - \frac{n}{z(t,n)}\right)\xi - 1 + \frac{n}{z(t,n)}}\right]^{z(t,n)} \\
&= \underline{a}^{\xi}\bar{a}^{1-\xi}\left[\left(\frac{\bar{a}}{a_r}\right)^{\frac{n(\xi-1)}{z(t,n)}} a_r^{\frac{z(t,n+1)}{z(t,n)}\xi - 1}\right]^{z(t,n)}. \qquad (4)
\end{aligned}$$

Let

$$Q = \left(\frac{\bar{a}}{a_r}\right)^{\frac{n(\xi-1)}{z(t,n)}} a_r^{\frac{z(t,n+1)}{z(t,n)}\xi - 1}.$$

Taking logs and by (3), we have

$$\begin{aligned}
\log Q = \xi \log a_r &\left[\frac{n}{z(t, n)}(1 - \xi^{-1})\left(\frac{\log \bar{a}}{\log a_r} - 1\right)\right. \\
&\left. + \frac{z(t, n+1)}{z(t, n)} - \xi^{-1}\right] \\
&\ge \xi q \log a_r > 0. \qquad (5)
\end{aligned}$$

Since t is a non-end point, $z(t, n) \to \infty$ and the lower-right derivative of $F(x)$ is infinite at t by (4) and (5).

Now we turn to the proof of statement (ii). Let t be such that

$$\limsup_{n\to\infty}\left[\frac{n}{z(t, n)}(1 - \xi^{-1})\left(\frac{\log \underline{a}}{\log a_r} - 1\right) + \frac{z(t, n+1)}{z(t, n)} - \xi^{-1}\right] > 0.$$

Then there exists a sequence $\{n_k\}$ of positive integers such that

$$\frac{n_k}{z(t, n_k)}(1 - \xi^{-1})\left(\frac{\log \underline{a}}{\log a_r} - 1\right) + \frac{z(t, n_k+1)}{z(t, n_k)} - \xi^{-1} > c \qquad (6)$$

for some positive constant c. Let x_k be the left endpoint of $h_{(\tilde{t}|j_k)*(t(j_k+1)+1)}([0, 1])$, where $j_k = z(t, n_k) - 1$. Thus we have $\tilde{x}_k = (t(1), \ldots, t(j_k), t(j_k+1)+1, 0, \ldots, 0, \ldots)$. Let u_k be the right endpoint of $h_{\tilde{t}|(j_k+1)}([0, 1]$. Then $\tilde{u}_k = (t(1), \ldots, t(j_k), t(j_k+1), r, r, r, \ldots)$. Thus, $[u_k, x_k]$ is the gap on the right side of $h_{\tilde{t}|(j_k+1)}([0, 1])$ and $\lambda([u_k, x_k]) = x_k - u_k = a_{\tilde{t}|j_k}\beta_{j_k+1}$ where by β_j, $j = 0, 1, \ldots, r-1$, we denote length of the gap between images $h_j([0, 1])$ and $h_{j+1}([0, 1])$. Note that $[t, x_k] \supseteq [u_k, x_k]$ and $\mu((t, x_k]) = \mu((t, u_k]) + \mu((u_k, x_k]) = \mu((t, u_k]) \le \mu(h_{\tilde{t}|(z(t,n_k+1)-1)}([0, 1]))$ since $\tilde{t}|(z(t, n_k+1)-1) = \tilde{u}_k|(z(t, n_k+1)-1)$. Therefore we have

$$\begin{aligned}
F(x_k) - F(t) &= \mu((t, x_k]) \\
&\le \mu(h_{\tilde{t}|(z(t,n_k+1)-1)}([0, 1])) \\
&= a_{\tilde{t}|(z(t,n_k+1)-1)}^{\xi}
\end{aligned}$$

and

$$x_k - t \ge \lambda([u_k, x_k]) = a_{\tilde{t}|(z(t, n_k)-1)}\beta_{t(z(t,n_k))}.$$

Denote $\underline{\beta} = \min_{j\in\{0,1,\ldots,r-1\}}\beta_j$. Then we obtain with a similar reasoning which led to (4)

$$\begin{aligned}
&\frac{F(x_k) - F(t)}{x_k - t} \\
&\le \frac{a_{\tilde{t}|(z(t,n_k+1)-1)}^{\xi}}{a_{\tilde{t}|(z(t,n_k)-1)}\beta_{\tilde{t}(z(t,n_k))}} \le \frac{\left(\prod_{i=1}^{z(t,n_k+1)-1} a_{t(i)}\right)^{\xi}}{\underline{\beta}\prod_{i=1}^{z(t,n_k)-1} a_{t(i)}} \\
&\le \frac{\bar{a}^{\xi}\underline{a}^{1-\xi}}{a_r^{\xi}\underline{\beta}}\left[\left(\frac{\underline{a}}{a_r}\right)^{\frac{n_k(\xi-1)}{z(t,n_k)}} a_r^{\frac{z(t,n_k+1)}{z(t,n_k)}\xi - 1}\right]^{z(t,n_k)}. \qquad (7)
\end{aligned}$$

Let

$$Q = \left(\frac{\underline{a}}{a_r}\right)^{\frac{n_k(\xi-1)}{z(t,n_k)}} a_r^{\frac{z(t,n_k+1)}{z(t,n_k)}\xi-1}.$$

Taking logs and using (6), we obtain

$$\log Q = \xi \log a_r \left[\frac{n_k}{z(t,\, n_k)}(1-\xi^{-1})\left(\frac{\log \underline{a}}{\log a_r} - 1\right) + \frac{z(t,\, n_k+1)}{z(t,\, n_k)} - \xi^{-1}\right] \le \xi c \log a_r < 0\,. \tag{8}$$

Equations (7) and (8) imply that the right lower derivative at t is finite by letting $k \to \infty$. □

3. DIMENSIONS OF THE SET OF NON-DIFFERENTIABILITY POINTS

In this section, we determine the dimensions of S. The proof uses the following lemma on dimensions of subsets of Moran sets which is a special case of the main result in Li *et al.*[3]

Lemma 3.1. *Let* $\Gamma = \{0, 1, \ldots, r-1\}$ *and* $z(t, n)$ *denote the position of the* n*th occurrence of elements of* Γ *in* $\tilde{t}$. *For given* $0 < p \le 1$, *let*

$$C(p) = \left\{t \in C\backslash\{\text{right endpoints of } C\} : \limsup_{n\to\infty} \frac{z(t,\, n+1)}{z(t,\, n)} \ge p^{-1}\right\}. \tag{9}$$

Let $\eta = \eta(p)$ *be such that*

$$p \log \sum_{j\in\Omega} a_j^{\eta} + (1-p)\log a_r^{\eta} = 0\,. \tag{10}$$

Then we have $\dim_H C(p) = \eta$ *and* $\dim_P C(p) = \dim_B C(p) = \dim_H C = \xi$ *where* ξ *is defined in* (2).

It is easy to verify that $\eta(p)$ is strictly increasing and continuous and that $\eta(0) \le \eta(p) \le \eta(1) = \xi$. We also consider for $0 < p \le 1$ and with the same Γ

$$C^*(p) = \left\{t \in C\backslash\{\text{right endpoints of } C\} : \limsup_{n\to\infty} \frac{z(t,\, n+1)}{z(t,\, n)} > p^{-1}\right\}. \tag{11}$$

Directly from Lemma 3.1, it follows that $\dim_P C^*(p) = \dim_B C^*(p) = \dim_H C = \xi$. Moreover, $\dim_H C^*(p) = \eta(p)$. To see that this follows from Lemma 3.1, approximate $C^*(p)$ by a union of $C(p_k)$'s, where $p_k \uparrow p$.

Lemma 3.2. *Let* $\Gamma = \{0, 1, \ldots, r-1\}$, $\underline{a} = \min_{j\in\Gamma} a_j$, $\bar{a} = \max_{j\in\Gamma} a_j$, *and let*

$$p_1^{-1} = \max\left\{\xi^{-1},\, (\xi^{-1}-1)\frac{\log \underline{a}}{\log a_r} + 1\right\};$$

$$p_2^{-1} = \min\left\{\xi^{-1},\, (\xi^{-1}-1)\frac{\log \bar{a}}{\log a_r} + 1\right\}.$$

Then $C^*(p_1) \subseteq N^+ \subseteq C(p_2)$ *and* $\eta(p_1) \le \dim_H(N^+) \le \eta(p_2)$.

Proof. By Lemma 3.1, it suffices to prove $C^*(p_1) \subseteq N^+ \subseteq C(p_2)$. Let $t \in N^+$. By Proposition 2.2, we have

$$\limsup_{n\to\infty}\left[\frac{n}{z(t,\, n)}(1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r} - 1\right) + \frac{z(t,\, n+1)}{z(t,\, n)} - \xi^{-1}\right] \ge 0\,.$$

Now if $\frac{\log \bar{a}}{\log a_r} > 1$, then $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \ge \xi^{-1}$. If not, we use that $\frac{n}{z(t,n)} \le 1$ and so $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \ge \xi^{-1} - (1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r} - 1\right)$. So we find that

$$\limsup_{n\to\infty} \frac{z(t,\, n+1)}{z(t,\, n)} \ge p_2^{-1}$$

i.e. $t \in C(p_2)$. On the other hand, $t \in C^*(p_1)$ implies in a similar way that $t \in N^+$. □

Theorem 3.3. *Let* $C = \bigcup_{j=0}^{r} h_j(C)$ *be the Cantor set determined by* $\{h_j(x) = a_j x + b_j : 0 \le j \le r\}$, *and let* ξ *be its Hausdorff dimension. If* S *is the set of non-differentiability points of the Cantor distribution determined by* $a_0, \ldots, a_r$ *on* C, *then* $\dim_H S = \xi^2 = (\dim_H C)^2$ *and* $\dim_B S = \dim_P S = \dim_H C = \xi$.

Proof. Since $S = N^+ \bigcup N^- \bigcup\{$ the endpoints of $C\}$ and because of the symmetry between N^+ and N^-, it suffices to determine the dimensions of N^+. By Lemmas 3.2 and 3.1, $\dim_B N^+ = \dim_P N^+ = \dim_H C = \xi$ is trivial. We define $a = \min_{j\in\Omega} a_j$. Let $0 < \delta < a^2$, and let

$$\Omega_\delta = \{\sigma \in \Omega^* : a_\sigma \le \delta \text{ and } a_{\sigma|(|\sigma|-1)} > \delta\}\,.$$

Note that for each $\sigma \in \Omega_\delta$, we have $a\delta < a_\sigma \le \delta$. Thus for any $\sigma, \tau \in \Omega_\delta$

$$\frac{\log \delta}{\log a + \log \delta} \le \frac{\log a_\sigma}{\log a_\tau} \le \frac{\log a + \log \delta}{\log \delta}. \qquad (12)$$

Here we would like to relate Ω_δ to ξ by showing that

$$\lim_{\delta\to 0} \frac{\log \#\Omega_\delta}{-\log \delta} = \xi \qquad (13)$$

where ξ is defined in Eq. (2) and $\#\Omega_\delta$ denotes the number of elements of Ω_δ. By $N_\delta(C)$ we denote the smallest number of sets of diameter at most δ that cover C. By Lemma 9.2 in Falconer,[4] there exists a positive constant q independent of δ such that

$$q\#\Omega_\delta \le N_\delta(C) \le \#\Omega_\delta. \qquad (14)$$

Hence (13) holds by (14) and the fact $\lim_{\delta\to 0} \frac{\log N_\delta(C)}{-\log \delta} = \xi$. Now let $\mathcal{H}_\delta = \{h_\sigma : \sigma \in \Omega_\delta\}$. Note that h_σ is a similitude mapping with ratio $0 < a_\sigma < 1$ for each $\sigma \in \mathcal{H}_\delta$, that the family $\mathcal{H}_\delta$ of similitude mappings still satisfies the open set condition, and that the unique self-similar set determined by $\mathcal{H}_\delta$ equals C:

$$C = \bigcup_{\sigma\in\mathcal{H}_\delta} h_\sigma(C).$$

We also have that ξ defined in (2) satisfies $\sum_{\sigma\in\Omega_\delta} a_\sigma^\xi = 1$. If we denote μ_δ the self-similar probability measure on C corresponding to the probability vector $(a_\sigma^\xi : \sigma \in \Omega_\delta)$, then $\mu_\delta = \mu$ since $\mu_\delta(h_\tau([0,1])) = \mu(h_\tau([0,1]))$ for any $\tau \in \Omega_\delta^k$ and $k \in \mathbf{N}$. Hence for the corresponding non-differentiability points N_δ^+, we have $N_\delta^+ = N^+$ for all $\delta > 0$.

There exists a unique $\sigma \in \Omega_\delta$ with $\sigma(j) = r$ for $j = 1, 2, \ldots, |\sigma|$. We denote this special element by σ_δ. Note that σ_δ plays the same role in Ω_δ as r in Ω, in the sense that $h_{\sigma_\delta}([0,1])$ is the right-most interval in $[0,1]$ of the intervals $(h_\sigma([0,1]))_{\sigma\in\Omega_\delta}$. Let

$$\Gamma_\delta := \Omega_\delta \backslash \{\sigma_\delta\} \text{ and } a_\delta := a_{\sigma_\delta} = a_r^{|\sigma_\delta|}.$$

We will use the notations $\underline{a}_\delta = \min_{\sigma\in\Gamma_\delta} a_\sigma$, $\bar{a}_\delta = \max_{\sigma\in\Gamma_\delta} a_\sigma$,

$$p_1(\delta) = \left(\max\left\{\xi^{-1}, (\xi^{-1}-1)\frac{\log \underline{a}_\delta}{\log a_\delta} + 1\right\}\right)^{-1} \qquad (15)$$

and

$$p_2(\delta) = \left(\min\left\{\xi^{-1}, (\xi^{-1}-1)\frac{\log \bar{a}_\delta}{\log a_\delta} + 1\right\}\right)^{-1}.$$

Then by Lemmas 3.1 and 3.2 with Ω replaced by Ω_δ, Γ replaced by Γ_δ, a_r replaced by a_δ and p_1, p_2 by $p_1(\delta)$ respectively $p_2(\delta)$, we have

$$\eta(p_1(\delta)) \le \dim_H N_\delta^+ \le \eta(p_2(\delta)) \qquad (16)$$

where $\eta(p_1(\delta))$ and $\eta(p_2(\delta))$ are defined by formula (10):

$$p_1(\delta)\log \sum_{\sigma\in\Omega_\delta} a_\sigma^{\eta(p_1(\delta))} + (1-p_1(\delta))\log a_\delta^{\eta(p_1(\delta))} = 0 \qquad (17)$$

and

$$p_2(\delta)\log \sum_{\sigma\in\Omega_\delta} a_\sigma^{\eta(p_2(\delta))} + (1-p_2(\delta))\log a_\delta^{\eta(p_2(\delta))} = 0.$$

Since $N_\delta^+ = N^+$ for all $\delta > 0$, it follows from (16) that $\dim_H N^+ = \xi^2$ holds if for any $\varepsilon > 0$, there exists a $\delta^* > 0$ such that when $0 < \delta < \delta^*$ we have $|\eta(p_1(\delta)) - \xi^2| < \varepsilon$ and $|\eta(p_2(\delta)) - \xi^2| < \varepsilon$. Verification of this claim will be given in the following only for $\eta(p_1(\delta))$, since the same argument can be employed for $\eta(p_2(\delta))$. To alleviate the notation we will write $p_\delta := p_1(\delta)$ and $\eta_\delta := \eta(p_1(\delta))$. For $x \in (0, \xi]$, let

$$T_\delta(x) = p_\delta \frac{\log \sum_{\sigma\in\Omega_\delta} a_\sigma^x}{x \log a_\delta} + 1 - p_\delta.$$

Note that

$$a^x\delta^x \#\Omega_\delta \le \sum_{\sigma\in\Omega_\delta} a_\sigma^x \le \delta^x \#\Omega_\delta$$

which implies

$$\begin{aligned}\frac{x\log\delta + \log\#\Omega_\delta}{x\log a_\delta} &\le \frac{\log \sum_{\sigma\in\Omega_\delta} a_\delta^x}{x\log a_\delta} \\ &\le \frac{x\log a + x\log\delta + \log\#\Omega_\delta}{x\log a_\delta}. \qquad (18)\end{aligned}$$

Therefore by (18), we have

$$0 \le T_\delta(x) - \left[1 + p_\delta\left(\frac{\log\delta}{\log a_\delta} - 1\right) + \frac{1}{x}\frac{p_\delta \log\#\Omega_\delta}{\log a_\delta}\right] \le \frac{p_\delta \log a}{\log a_\delta}. \qquad (19)$$

Note that by (12) and (15), we have

$$\lim_{\delta \downarrow 0} p_\delta = \xi \,. \tag{20}$$

Since $a\delta < a_\delta \le \delta$, and by (13) and (20), we have

$$\lim_{\delta \downarrow 0} p_\delta \left(\frac{\log \delta}{\log a_\delta} - 1 \right) = \lim_{\delta \downarrow 0} \frac{p_\delta \log a}{\log a_\delta} = 0$$

and

$$\lim_{\delta \downarrow 0} \frac{p_\delta \log \#\Omega_\delta}{\log a_\delta} = -\xi^2 \,.$$

Thus by (19) we have for all $x \in (0, \xi]$

$$\lim_{\delta \downarrow 0} T_\delta(x) = 1 - \frac{\xi^2}{x} =: T_0(x) \,. \tag{21}$$

Now for any given $\varepsilon > 0$ satisfying $0 < \xi^2 - \varepsilon < \xi^2 + \varepsilon < \xi < 1$, we see that

$$T_0(\xi^2 - \varepsilon) = \frac{-\varepsilon}{\xi^2 - \varepsilon} < -\varepsilon$$

and

$$T_0(\xi^2 + \varepsilon) = \frac{\varepsilon}{\xi^2 + \varepsilon} > \varepsilon \,.$$

By (21) we can take $\delta^* > 0$, so that for $0 < \delta < \delta^*$

$$|T_\delta(\xi^2 - \varepsilon) - T_0(\xi^2 - \varepsilon)| < \frac{\varepsilon}{2}$$

and

$$|T_\delta(\xi^2 + \varepsilon) - T_0(\xi^2 + \varepsilon)| < \frac{\varepsilon}{2} \,.$$

Then necessarily for these δ,

$$T_\delta(\xi^2 - \varepsilon) < -\frac{\varepsilon}{2} \text{ and } T_\delta(\xi^2 + \varepsilon) > \frac{\varepsilon}{2} \,.$$

Then for $0 < \delta < \delta^*$

$$\xi^2 - \varepsilon < \eta_\delta < \xi^2 + \varepsilon$$

since $T_\delta(x)$ is strictly increasing in x and $T_\delta(\eta_\delta) = 0$. □

Note added to proofs

Recently, K. J. Falconer has given a general analysis of the phenomenon discussed in this paper in a manuscript titled "Multifractal analysis of Ahlfors regular measures and devil's staircases."

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