Received: 21 July 2025 | Revised: 16 November 2025 | Accepted: 3 December 2025

DOI: 10.1112/jlms.70408

RESEARCH ARTICLE

Journal of the London Mathematical Society

On the intersection of Cantor sets with the unit circle and some sequences

Kan Jiang[1] | **Derong Kong**[2] | **Wenxia Li**[3] | **Zhiqiang Wang**[4]

[1]Department of Mathematics, Ningbo University, Ningbo, People's Republic of China

[2]College of Mathematics and Statistics, Center of Mathematics, Chongqing University, Chongqing, People's Republic of China

[3]School of Mathematical Sciences, Key Laboratory of MEA (Ministry of Education) & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai, People's Republic of China

[4]College of Mathematics and Statistics, Center of Mathematics, Key Laboratory of Nonlinear Analysis and its Applications (Ministry of Education), Chongqing University, Chongqing, People's Republic of China

Correspondence
Kan Jiang, College of Mathematics and Statistics, Ningbo University, Ningbo, 315211, People's Republic of China.
Email: jiangkan@nbu.edu.cn

Funding information
Zhejiang Provincial Natural Science Foundation of China, Grant/Award Number: LMS25A010009; Chongqing NSF, Grant/Award Number: CQYC20220511052; Scientific Research Innovation Capacity Support Project for Young Faculty, Grant/Award Number: ZYGXQNISKYCXNLZCXM-P2P; NSFC, Grant/Award Numbers: 12471085, 12501110; Science and Technology Commission of Shanghai Municipality, Grant/Award Number: 22DZ2229014; China Postdoctoral Science Foundation, Grant/Award Number: 2024M763857

Abstract

For $\lambda \in (0, 1/2)$, let K_λ be the self-similar set in $\mathbb{R}$ generated by the iterated function system $\{f_0(x) = \lambda x, f_1(x) = \lambda x + 1 - \lambda\}$. In this paper, we investigate the intersection of the unit circle $\mathbb{S} \subset \mathbb{R}^2$ with the Cartesian product $K_\lambda \times K_\lambda$. We prove that for $\lambda \in (0, 2 - \sqrt{3}]$, the intersection is *trivial*, that is,

$$\mathbb{S} \cap (K_\lambda \times K_\lambda) = \{(0, 1), (1, 0)\}.$$

If $\lambda \in [0.330384, 1/2)$, then the intersection $\mathbb{S} \cap (K_\lambda \times K_\lambda)$ is nontrivial. In particular, if $\lambda \in [0.407493, 1/2)$, the intersection $\mathbb{S} \cap (K_\lambda \times K_\lambda)$ is of cardinality continuum. Furthermore, the bound $2 - \sqrt{3}$ is sharp: there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \searrow 2 - \sqrt{3}$ such that $\mathbb{S} \cap (K_{\lambda_n} \times K_{\lambda_n})$ is nontrivial for all $n \in \mathbb{N}$. This result provides a negative answer to a problem posed by Yu (2023). Our methods extend beyond the unit circle and remain effective for many nonlinear curves. We also characterize the intersection of missing digits Cantor sets with the sequence $\{1/n^2\}$ by utilizing the Legendre symbol.

MSC 2020
28A78, 28A80 (primary)

1 | INTRODUCTION

Given a continuous curve Γ and a fractal set $\mathcal{K}$ on the plane, determining the closed form of their intersection is in general a challenging problem. The classical Marstrand's slicing theorem [10] describes the Hausdorff dimension of the intersection $\Gamma \cap \mathcal{K}$ for a typical line Γ. However, the intersection is difficult to analyze if the curve Γ is nonlinear. Analogous problems can be considered in the discrete case. Given a sequence $\{x_n\}_{n=1}^{\infty}$ and a Cantor set K in $\mathbb{R}$, how can we describe the intersection $\{x_n : n \in \mathbb{N}\} \cap K$? For instance, we even do not know the structures of the following intersections:

$$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap C \quad \text{and} \quad \left\{\frac{1}{n!} : n \in \mathbb{N}\right\} \cap C.$$

Here and throughout the paper, C stands for the middle-third Cantor set.

This paper originates from a question posed by Yu [17] in his investigation of the ℓ^1-dimension for the distribution of missing digits points near manifolds. Yu posed the following general question [17, Question 1.5].

Question 1.1. Let M be a nondegenerate analytic manifold, and let $\mathcal{K}$ be a missing digits set in $\mathbb{R}^d$. To determine whether or not $M \cap \mathcal{K}$ is infinite.

For $\lambda \in (0, 1/2)$, let K_λ be the self-similar set generated by the iterated function system (IFS) (cf. [6])

$$\left\{f_0(x) = \lambda x,\ f_1(x) = \lambda x + 1 - \lambda\right\}.$$

Then, the convex hull of K_λ is the unit interval [0,1] for all $\lambda \in (0, 1/2)$. A particular special case of Question 1.1 was formulated by Yu [17, Question on page 8] as follows.

Question 1.2. Consider the unit circle

$$\mathbb{S} := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}.$$

Is the intersection $\mathbb{S} \cap (K_{1/5} \times K_{1/5})$ infinite?

Yu remarked that the methods developed by Shmerkin [13] and Wu [16] may be used to deduce that the points under consideration form a set with zero Hausdorff dimension. But this is not sufficient to conclude the finiteness. Note that $K_{1/3} = C$ is the middle-third Cantor set. Recently, with the assistance of computers, Du, Jiang, and Yao [3] proved that the intersection $\mathbb{S}\cap(K_{1/3}\times K_{1/3})$ contains at least 10 000 000 points.

In this paper, we address two aspects. On one hand, for a Cantor set $K\subset\mathbb{R}$, we consider the intersection of $K\times K$ with a curve, and give a negative answer to Question 1.2 and present specific examples pertinent to Question 1.1. On the other hand, we investigate the intersection of K with a sequence of real numbers. Throughout the paper, let $\mathbb{N}$ denote the set of all positive integers, and for $k\in\mathbb{N}$, we define $\mathbb{N}_{\geqslant k}=[k,+\infty)\cap\mathbb{N}$. For $a,b\in\mathbb{N}$, we write $\gcd(a,b)$ for the greatest common divisor of a and b. We use $\#A$ to denote the cardinality of a set A.

Note that for each $\lambda\in(0,1/2)$, the self-similar set K_λ is a Cantor set in $\mathbb{R}$. Our first main result focuses on the intersection $\mathbb{S}\cap(K_\lambda\times K_\lambda)$. We say the intersection $\mathbb{S}\cap(K_\lambda\times K_\lambda)$ is *trivial* if it consists of exactly two trivial points (0,1) and (1,0); otherwise, we say $\mathbb{S}\cap(K_\lambda\times K_\lambda)$ is *nontrivial.*

Theorem 1.1.

(i) *For* $0<\lambda\leqslant 2-\sqrt{3}$, *the intersection* $\mathbb{S}\cap(K_\lambda\times K_\lambda)$ *is trivial.*
(ii) *For* $0.330384\leqslant\lambda<1/2$, *the intersection* $\mathbb{S}\cap(K_\lambda\times K_\lambda)$ *is nontrivial.*
(iii) *For* $0.407493\leqslant\lambda<1/2$, *the intersection* $\mathbb{S}\cap(K_\lambda\times K_\lambda)$ *is of cardinality continuum.*

Remark 1.2.

(a) Note that $0<1/5<2-\sqrt{3}$. So, by Theorem 1.1 (i), it follows that

$$\mathbb{S}\cap(K_{1/5}\times K_{1/5})=\{(0,1),(1,0)\}.$$

This provides a negative answer to Question 1.2.

(b) For $k\in\mathbb{N}$, take $x=\lambda^k(1-\lambda)$ and $y=1-\lambda^{2k+1}$ in K_λ. Define $\lambda_k\in(0,1/2)$ to be the appropriate root of the equation $\lambda^{2k}(1-\lambda)^2+(1-\lambda^{2k+1})^2=1$, that is, $\lambda^{2k+2}+\lambda^2-4\lambda+1=0$. Note that the intersection $\mathbb{S}\cap(K_{\lambda_k}\times K_{\lambda_k})$ is nontrivial, and the sequence $\{\lambda_k\}$ is decreasing to $2-\sqrt{3}$ as $k\to\infty$. Thus, the constant $2-\sqrt{3}$ in Theorem 1.1 (i) is optimal.

(c) The constants in Theorem 1.1 (ii) and (iii) are not optimal. We conjecture that the intersection $\mathbb{S}\cap(K_\lambda\times K_\lambda)$ is infinite for all $2-\sqrt{3}<\lambda<1/2$.

Note that if $\lambda>1/4$, then $\dim_H(K_\lambda\times K_\lambda)=2\dim_H K_\lambda>1$, where $\dim_H$ denotes the Hausdorff dimension. Comparable with Marstrand's slicing theorem [10], one might expect that $\dim_H((K_\lambda\times K_\lambda)\cap\mathbb{S})>0$ for typical $\lambda>1/4$. However, Theorem 1.1 (i) demonstrates that this is not true in general (note that $2-\sqrt{3}>1/4$). In terms of Theorem 1.1, one may expect that there exists a critical value $\lambda_0\in(0,1/2)$ such that $\dim_H((K_\lambda\times K_\lambda)\cap\mathbb{S})=0$ for $0<\lambda<\lambda_0$, and $\dim_H((K_\lambda\times K_\lambda)\cap\mathbb{S})>0$ for $\lambda_0<\lambda<1/2$. We believe that the latter case occurs when λ is close to $1/2$, but have no idea to determine the critical value λ_0. We cannot even determine the measurability of the dimension function $\lambda\mapsto\dim_H((K_\lambda\times K_\lambda)\cap\mathbb{S})$.

In Theorem 1.1, we consider the intersection of the unit circle $\mathbb{S}$ with a class of Cantor sets $K_\lambda\times K_\lambda$ where $\lambda\in(0,1/2)$. The method can be adapted to study the intersection of some other algebraic curves with $K_\lambda\times K_\lambda$.

Theorem 1.3.

(i) *Let* $0<\lambda\leqslant 1/3$, *and let* q *be an integer with* $2\leqslant q\leqslant 1/\lambda-1$. *Then*

$$\{(x,y):y=x^q\}\cap(K_\lambda\times K_\lambda)=\left\{(\lambda^k,\lambda^{qk}):k\in\mathbb{N}\right\}\cup\{(0,0),(1,1)\}.$$

(ii) *For* $0 < \lambda \leqslant 0.187915$, *we have*

$$\{(x,y,z) : x^2 + y^2 = z^2\} \cap (K_\lambda \times K_\lambda \times K_\lambda) = \{(x,0,x),(0,x,x) : x \in K_\lambda\}.$$

Remark 1.4.

(a) A direct corollary of Theorem 1.3 (i) is that

$$\{(x,y) : y = x^2\} \cap (K_{1/3} \times K_{1/3}) = \left\{(3^{-k}, 9^{-k}) : k \in \mathbb{N}\right\} \cup \{(0,0),(1,1)\}.$$

(b) The constant in Theorem 1.3 (ii) is not optimal.

Note by the Weierstrass approximation theorem that any nonalgebraic curves can be approximated by algebraic curves. Then our method in Theorems 1.1 and 1.3 can be applied to study the intersection of nonalgebraic curves with $K_\lambda \times K_\lambda$. For instance, we can prove that

$$\{(x,y) : y = \cos(x)\} \cap \left(K_{1/4} \times K_{1/4}\right) = \{(0,1)\}.$$

Furthermore, instead of looking at the Cartesian product $K_\lambda \times K_\lambda$, we can consider self-similar sets. Our method may be applied to investigate the intersection of a curve or a manifold with a self-similar set.

Finally, we investigate the intersection of Cantor set with some sequences. We will employ some ideas from q-expansions and quadratic residues from number theory. Let $p \in \mathbb{N}$ be an odd prime, and let $a \in \mathbb{Z}$ with $p \nmid a$. We use

$$\left(\frac{a}{p}\right)_L$$

to denote the Legendre symbol, see Section 4 for more details.

Let $m \in \mathbb{N}_{\geqslant 3}$ and $D \subset \{0, 1, \dots, m-1\}$ with $1 < \#D < m$. We define

$$K_{m,D} = \left\{ \sum_{k=1}^{\infty} \frac{d_k}{m^k} : d_k \in D \ \forall k \in \mathbb{N} \right\}.$$

The set $K_{m,D}$ is a (homogeneous) self-similar set generated by the IFS $\{\varphi_d(x) = (x+d)/m : d \in D\}$. Note that $K_{3,\{0,2\}} = C$ is the middle-third Cantor set. The structure of rational points in Cantor set $K_{m,D}$ has been studied in [1, 7, 9, 11, 12, 14, 15]. For $p \in \mathbb{N}_{\geqslant 2}$, let $\mathcal{D}_p$ be the set of all rational numbers in [0,1] having a finite p-ary expansion. If $\gcd(p,m) = 1$, Schleischitz [12, Corollary 4.4] showed that the intersection $\mathcal{D}_p \cap K_{m,D}$ is a finite set. In the following, we give a complete description on the intersection of $K_{m,D}$ with the sequence $\left\{1/n^2 : n \in \mathbb{N}\right\}$.

Theorem 1.5. *Let* $m \in \mathbb{N}_{\geqslant 3}$ *be a prime, and* $D \subset \{0, 1, \dots, m-1\}$ *with* $0 \in D$ *and* $1 < \#D < m$. *Suppose that*

$$\left(\frac{-a}{m}\right)_L = -1 \quad \forall a \in D \setminus \{0\}.$$

Then, we have

(i) *if* $1 \notin D$,

$$\left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \cap K_{m,D} = \emptyset;$$

(ii) *if* $1 \in D$,

$$\left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \cap K_{m,D} = \left\{ \frac{1}{m^{2\ell}} : \ell \in \mathbb{N} \right\}.$$

Remark 1.6.

(a) By [5, Theorem 84], there are $(p-1)/2$ quadratic residues and $(p-1)/2$ quadratic non-residues of an odd prime p in the set $\{1, 2, \ldots, p-1\}$. So, the cardinality of the set D in Theorem 1.5 can be at most $(m+1)/2$. By taking a sufficiently large odd prime p, applying Theorem 1.5 (i), we can obtain that for any $\epsilon > 0$, there exists a self-similar set $K_\varepsilon \subset [0, 1]$ with $0 = \min K_\varepsilon$ such that $\dim_H K_\varepsilon > 1 - \epsilon$ and

$$\left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \cap K_\varepsilon = \emptyset.$$

(b) As a direct corollary of Theorem 1.5, we have

$$\left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \cap K_{3,\{0,1\}} = \left\{ \frac{1}{9^\ell} : \ell \in \mathbb{N} \right\}.$$

That is,

$$\left\{ \frac{2}{n^2} : n \in \mathbb{N} \right\} \cap C = \left\{ \frac{2}{9^\ell} : \ell \in \mathbb{N} \right\}.$$

(c) Using the almost identical argument in the proof of Theorem 1.5, we can deal with a cubic sequence in the following concrete example:

$$\left\{ \frac{1}{n^3} : n \in \mathbb{N} \right\} \cap K_{7,\{0,2,3,4,5\}} = \emptyset,$$

because one can easily check that $n^3 \equiv 1$ or $-1 \pmod 7$ for any $7 \nmid n$.

The paper is organized as follows. In Section 2, we focus on the intersection with the unit circle and prove Theorem 1.1. We will prove Theorem 1.3 in Section 3. Section 4 is dedicated to proving Theorem 1.5 and some corollaries. Finally, we list some questions in Section 5.

2 | INTERSECTION WITH THE UNIT CIRCLE

In this section, we will prove Theorem 1.1. Before the proof, we sketch the main idea. For part (i), when λ is small, we analyze the equation $x^2 + y^2 = 1$ with $x, y \in K_\lambda$. By symmetry, one can assume $x \leqslant y$, which implies $x \in [0,\lambda] \cap K_\lambda$. In fact, a more stringent restriction holds that $x \in [0,\lambda^k] \cap K_\lambda$ for all $k \in \mathbb{N}$. We prove this by induction, where the inductive step crucially relies on the nonlinearity of the circle equation, which forces x into arbitrarily small intervals. This approach, leveraging the curve's nonlinearity to constrain points in the Cantor set, is applicable to a broader class of nonlinear curves. The proofs of parts (ii) and (iii) rely on a different key observation. When λ is close to $1/2$, the continuous images of the basic intervals under the map $f(x,y) = x^2 + y^2$ exhibit significant overlap. Specifically, some points in the image are covered at least twice. This allows us to employ a bifurcation method (or a branching argument) to construct a continuum of points in $\mathbb{S} \cap (K_\lambda \times K_\lambda)$.

Recall that K_λ is a self-similar set generated by the IFS $\{f_0(x) = \lambda x,\ f_1(x) = \lambda x + 1 - \lambda\}$ for $\lambda \in (0,1/2)$. For $\mathbf{i} = i_1 i_2 \dots i_n \in \{0,1\}^n$, define

$$f_{\mathbf{i}} = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}.$$

For $n \in \mathbb{N}$, the n-level basic intervals of K_λ are defined by

$$\mathcal{F}_n := \{f_{\mathbf{i}}([0,1]) : \mathbf{i} \in \{0,1\}^n\}.$$

For $I \in \mathcal{F}_n$ and $k \geqslant n$, we also define

$$\mathcal{F}_k^I := \{J : J \in \mathcal{F}_k,\ J \subset I\}.$$

For a collection of sets $\mathcal{F}$, let $\bigcup \mathcal{F}$ denote the union of all sets in $\mathcal{F}$. Then, we have

$$K_\lambda = \bigcap_{n=1}^{\infty} \bigcup \mathcal{F}_n \quad \text{and} \quad K_\lambda \cap I = \bigcap_{k=n}^{\infty} \bigcup \mathcal{F}_k^I.$$

Recall that $\mathbb{S} = \{(x,y) : x^2 + y^2 = 1\}$ is the unit circle. The proof of Theorem 1.1 will be split into three propositions.

Proposition 2.1. *For $0 < \lambda \leqslant 2 - \sqrt{3}$, we have*

$$\mathbb{S} \cap (K_\lambda \times K_\lambda) = \{(0,1),(1,0)\}.$$

Proof. Take a point $(x,y) \in \mathbb{S} \cap (K_\lambda \times K_\lambda)$. By symmetry, we assume $x \leqslant y$. It suffices to show $x = 0$.

First, we show $x \in f_0([0,1])$. Observe that

$$x \in K_\lambda = f_0(K_\lambda) \cup f_1(K_\lambda) \subset f_0([0,1]) \cup f_1([0,1]).$$

Suppose on the contrary that $x \in f_1([0,1])$. Then $x \geqslant 1 - \lambda$, and hence, we have

$$x^2 + y^2 \geqslant 2x^2 \geqslant 2(1-\lambda)^2 \geqslant 2(\sqrt{3}-1)^2 > 1.$$

This leads to a contradiction with $x^2 + y^2 = 1$.

Next, assuming $x \in f_{0^k}([0,1])$ for some $k \in \mathbb{N}$, we show $x \in f_{0^{k+1}}([0,1])$. If this were done, then by induction, we would conclude that $x \in f_{0^\ell}([0,1])$ for all $\ell \in \mathbb{N}$, and thus, $x = 0$.

Notice that

$$x \in f_{0^k}([0,1]) \cap K_\lambda \subset f_{0^{k+1}}([0,1]) \cup f_{0^k 1}([0,1]).$$

Suppose on the contrary that $x \in f_{0^k 1}([0,1])$. Then, we have

$$\lambda^k(1-\lambda) \leqslant x \leqslant \lambda^k.$$

If $y \in f_{1^{2k+1}}([0,1])$, then we have $y \geqslant 1 - \lambda^{2k+1}$, and

$$x^2 + y^2 \geqslant \lambda^{2k}(1-\lambda)^2 + (1-\lambda^{2k+1})^2 = 1 + \lambda^{4k+2} + (\lambda^2 - 4\lambda + 1)\lambda^{2k} > 1,$$

where the last inequality follows from $0 < \lambda \leqslant 2 - \sqrt{3}$. If $y \notin f_{1^{2k+1}}([0,1])$, then we have $y \leqslant 1 - \lambda^{2k} + \lambda^{2k+1}$ because $y \in K_\lambda$. Using $0 < \lambda \leqslant 2 - \sqrt{3}$, we get

$$\begin{aligned} x^2 + y^2 &\leqslant \lambda^{2k} + (1 - \lambda^{2k} + \lambda^{2k+1})^2 \\ &= 1 - (1-2\lambda)\lambda^{2k} + (1-\lambda)^2\lambda^{4k} \\ &< 1 - (1-2\lambda)\lambda^{2k} + (1-\lambda)^2\lambda^{2k+1} \\ &< 1. \end{aligned}$$

Thus, we obtain $x^2 + y^2 \neq 1$, a contradiction. Hence, we conclude

$$x \in f_{0^{k+1}}([0,1]),$$

completing the proof. □

Note that $\mathcal{F}_n$ consists of 2^n pairwise disjoint intervals, and each of them is of the form $[a, a+\lambda^n]$ with

$$a \in \left\{ (1-\lambda)\sum_{j=1}^{n} d_j \lambda^{j-1} \,:\, d_j \in \{0,1\}\, \forall 1 \leqslant j \leqslant n \right\}.$$

In the remainder of this section, we always assume that $2 - \sqrt{3} < \lambda < 1/2$, and write $g(x,y) = x^2 + y^2$.

Lemma 2.2. *Let $I, J \in \mathcal{F}_n$ with $I = [a, a+\lambda^n]$, $J = [b, b+\lambda^n]$, and $a \leqslant b$. Suppose that*

$$\frac{1-2\lambda}{\lambda}(a+\lambda^n) \leqslant b \leqslant \frac{a}{1-2\lambda}.$$

Then we have

$$g(I \times J) = g((I_1 \cup I_2) \times (J_1 \cup J_2)),$$

where

$$I_1 = [a, a + \lambda^{n+1}], \quad I_2 = [a + (1-\lambda)\lambda^n, a + \lambda^n],$$

and

$$J_1 = [b, b + \lambda^{n+1}], \quad J_2 = [b + (1-\lambda)\lambda^n, b + \lambda^n].$$

Before the proof, we point out that $\mathcal{F}^I_{n+1} = \{I_1, I_2\}$ and $\mathcal{F}^J_{n+1} = \{J_1, J_2\}$.

Proof. It is easy to calculate that

$$\begin{aligned} g(I_1 \times J_1) &= [a^2 + b^2, a^2 + b^2 + 2(a+b)\lambda^{n+1} + 2\lambda^{2n+2}], \\ g(I_2 \times J_1) &= [a^2 + b^2 + 2a(1-\lambda)\lambda^n + (1-\lambda)^2\lambda^{2n}, \\ &\qquad a^2 + b^2 + 2(a + b\lambda)\lambda^n + (1+\lambda^2)\lambda^{2n}], \\ g(I_1 \times J_2) &= [a^2 + b^2 + 2b(1-\lambda)\lambda^n + (1-\lambda)^2\lambda^{2n}, \\ &\qquad a^2 + b^2 + 2(a\lambda + b)\lambda^n + (1+\lambda^2)\lambda^{2n}], \\ g(I_2 \times J_2) &= [a^2 + b^2 + 2(a+b)(1-\lambda)\lambda^n + 2(1-\lambda)^2\lambda^{2n}, \\ &\qquad a^2 + b^2 + 2(a+b)\lambda^n + 2\lambda^{2n}]. \end{aligned}$$

It remains to verify that the above four intervals are overlapping, which is equivalent to

$$\begin{cases} -(\lambda-1)^2\lambda^n \leqslant 2b\lambda - 2(a+\lambda^n)(1-2\lambda), \\ -\lambda^{n+1} \leqslant a - b(1-2\lambda), \\ (1 - 4\lambda + \lambda^2)\lambda^n \leqslant 2b\lambda - 2a(1-2\lambda). \end{cases} \tag{2.1}$$

For $2 - \sqrt{3} < \lambda < 1/2$, the left-hand sides of all three inequalities in (2.1) are negative. Thus, we only need

$$b\lambda - (a + \lambda^n)(1-2\lambda) \geqslant 0 \quad \text{and} \quad a - b(1-2\lambda) \geqslant 0.$$

That is,

$$\frac{1-2\lambda}{\lambda}(a + \lambda^n) \leqslant b \leqslant \frac{a}{1-2\lambda},$$

as desired. □

The following lemma is an easy exercise in real analysis, and we only state it without a detailed proof.

Lemma 2.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, and let $\{F_n\}_{n=1}^\infty$ be a decreasing sequence of nonempty compact subsets of $\mathbb{R}^2$. Then we have*

$$f\left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{n=1}^{\infty} f(F_n).$$

Lemma 2.4. *Let $I, J \in \mathcal{F}_n$ with $I = [a, a+\lambda^n]$, $J = [b, b+\lambda^n]$, and $a \leqslant b$. Suppose that*

$$\frac{1-2\lambda}{\lambda}(a+\lambda^n) \leqslant b < b+\lambda^n \leqslant \frac{a}{1-2\lambda}.$$

Then, we have

$$g((K_\lambda \cap I) \times (K_\lambda \cap J)) = g(I \times J).$$

Proof. Fix $k \geqslant n$ and take $I' = [a', a'+\lambda^k] \in \mathcal{F}_k^I$ and $J' = [b', b'+\lambda^k] \in \mathcal{F}_k^J$ with $a' \leqslant b'$. Note that $a \leqslant a' \leqslant a+\lambda^n-\lambda^k$ and $b \leqslant b' \leqslant b+\lambda^n-\lambda^k$. Using our assumptions, we clearly have

$$\frac{1-2\lambda}{\lambda}(a'+\lambda^k) \leqslant \frac{1-2\lambda}{\lambda}(a+\lambda^n) \leqslant b \leqslant b',$$

and

$$b' < b+\lambda^n \leqslant \frac{a}{1-2\lambda} \leqslant \frac{a'}{1-2\lambda}.$$

Thus, by Lemma 2.2, we obtain

$$g(I' \times J') = g\left(\left(\bigcup \mathcal{F}_{k+1}^{I'}\right) \times \left(\bigcup \mathcal{F}_{k+1}^{J'}\right)\right). \tag{2.2}$$

If $a < b$, then for any $I' = [a', a'+\lambda^k] \in \mathcal{F}_k^I$, $J' = [b', b'+\lambda^k] \in \mathcal{F}_k^J$, we have $a' < b'$. Thus, by (2.2), we obtain

$$\begin{aligned} g\left(\left(\bigcup \mathcal{F}_k^I\right) \times \left(\bigcup \mathcal{F}_k^J\right)\right) &= \bigcup_{I' \in \mathcal{F}_k^I} \bigcup_{J' \in \mathcal{F}_k^J} g(I' \times J') \\ &= \bigcup_{I' \in \mathcal{F}_k^I} \bigcup_{J' \in \mathcal{F}_k^J} g\left(\left(\bigcup \mathcal{F}_{k+1}^{I'}\right) \times \left(\bigcup \mathcal{F}_{k+1}^{J'}\right)\right) \\ &= g\left(\left(\bigcup \mathcal{F}_{k+1}^{I}\right) \times \left(\bigcup \mathcal{F}_{k+1}^{J}\right)\right). \end{aligned}$$

If $a = b$, then $I = J$ and let $\mathscr{F}_k$ be the collection of pairs $(I', J') \in \mathcal{F}_k^I \times \mathcal{F}_k^J$ with $I' = [a', a'+\lambda^k]$, $J' = [b', b'+\lambda^k]$, and $a' \leqslant b'$. Note that $g(x, y) = g(y, x)$. By (2.2), we also have

$$\begin{aligned} g\left(\left(\bigcup \mathcal{F}_k^I\right) \times \left(\bigcup \mathcal{F}_k^J\right)\right) &= \bigcup_{(I', J') \in \mathscr{F}_k} g(I' \times J') \\ &= \bigcup_{(I', J') \in \mathscr{F}_k} g\left(\left(\bigcup \mathcal{F}_{k+1}^{I'}\right) \times \left(\bigcup \mathcal{F}_{k+1}^{J'}\right)\right) \\ &= g\left(\left(\bigcup \mathcal{F}_{k+1}^{I}\right) \times \left(\bigcup \mathcal{F}_{k+1}^{J}\right)\right). \end{aligned}$$

Finally, by Lemma 2.3, we conclude that

$$\begin{aligned} g(I\times J) &= \bigcap_{k=n}^{\infty} g\Big(\Big(\bigcup \mathcal{F}_k^I\Big)\times\Big(\bigcup \mathcal{F}_k^J\Big)\Big) \\ &= g\left(\bigcap_{k=n}^{\infty}\Big(\bigcup \mathcal{F}_k^I\Big)\times\Big(\bigcup \mathcal{F}_k^J\Big)\right) \\ &= g((K_\lambda\cap I)\times(K_\lambda\cap J)), \end{aligned}$$

as desired. □

Proposition 2.5. *For* $0.330384 \leqslant \lambda < 1/2$*, we have*

$$\#(\mathbb{S}\cap(K_\lambda\times K_\lambda)) > 2.$$

Proof. Suppose that $I, J \in \mathcal{F}_n$ satisfy all the conditions in Lemma 2.4, and moreover, we have $1 \in g(I, J)$. Then, by Lemma 2.4, we have $1 \in g\big((K_\lambda\cap I)\times(K_\lambda\cap J)\big)$. This means that there exists $(x, y) \in (K_\lambda\cap I)\times(K_\lambda\cap J)$ such that $x^2 + y^2 = 1$. Thus, we only need to find desired n-level basic intervals $I, J \in \mathcal{F}_n$ for some $n \in \mathbb{N}$.

(i) For $9/25 \leqslant \lambda < 1/2$, choose $I = J = [1-\lambda, 1-\lambda+\lambda^2]$. That is, $n = 2$ and $a = b = 1-\lambda$ in Lemma 2.4. Note that $2(1-\lambda)^2 \leqslant 512/625 < 1$ and $2(1-\lambda+\lambda^2)^2 \geqslant 9/8 > 1$. This implies that $1 \in g(I\times J)$. We clearly have $b+\lambda^2 < 1 < a/(1-2\lambda)$. By Lemma 2.4, it remains to verify

$$\frac{1-2\lambda}{\lambda}(1-\lambda+\lambda^2) \leqslant 1-\lambda,$$

that is, $2\lambda^3 - 4\lambda^2 + 4\lambda - 1 \geqslant 0$, which can be easily checked.

(ii) For $0.330384 \leqslant \lambda < 9/25$, choose $I = [1-\lambda, 1-\lambda+\lambda^3]$, and $J = [1-\lambda+\lambda^2-\lambda^3, 1-\lambda+\lambda^2]$. That is, $n = 3$, $a = 1-\lambda$, and $b = 1-\lambda+\lambda^2-\lambda^3$ in Lemma 2.4. We clearly have $b+\lambda^3 < 1 < a/(1-2\lambda)$. By Lemma 2.4, we need

$$\frac{1-2\lambda}{\lambda}(1-\lambda+\lambda^3) \leqslant 1-\lambda+\lambda^2-\lambda^3,$$

that is, $\lambda^4 - 3\lambda^2 + 4\lambda - 1 \geqslant 0$, which can be checked directly. Note that $(1-\lambda+\lambda^3)^2 + (1-\lambda+\lambda^2)^2 > (1-\lambda)^2 + (1-\lambda+\lambda^2)^2 > 1$ for $0 < \lambda \leqslant 9/25$. In order to show $1 \in g(I\times J)$, it remains to check

$$(1-\lambda)^2 + (1-\lambda+\lambda^2-\lambda^3)^2 \leqslant 1. \tag{2.3}$$

Observe that the left-hand side in (2.3) is decreasing for $0 < \lambda < 1/2$. Thus, one can easily verify (2.3) by plugging $\lambda = 0.330384$ into the inequality. □

Finally, we will prove Theorem 1.1 (iii) by using the binary branching argument. For $I, J \in \mathcal{F}_n$, we define *the set of double covering points* $\mathcal{G}(I, J)$ to be the set of $z \in g(I\times J)$ satisfying that there

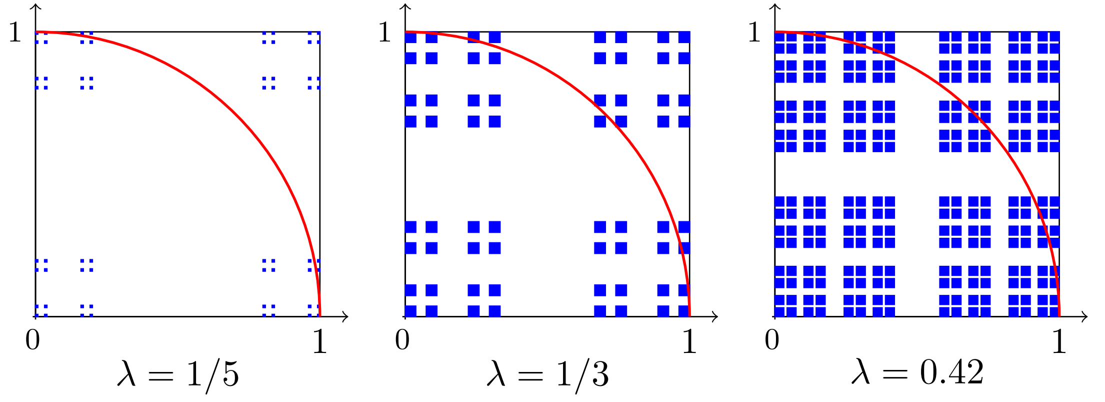

FIGURE 1 The intersection $\mathbb{S} \cap (K_\lambda \times K_\lambda)$ with $\lambda = 1/5, 1/3$, and 0.42.

exists $k > n$ and two distinct pairs $(I_1, J_1), (I_2, J_2) \in \mathcal{F}_k^I \times \mathcal{F}_k^J$ such that z is an interior point of both $g(I_1 \times J_1)$ and $g(I_2 \times J_2)$. This means that the point z can be covered at least twice by the images of basic subintervals. Under some conditions, we will show that the set of double covering points $\mathcal{G}(I, J)$ is the whole interval $g(I \times J)$ except for endpoints.

Lemma 2.6. *Let $I, J \in \mathcal{F}_n$ with $I = [a, a + \lambda^n]$, $J = [b, b + \lambda^n]$, and $a \leqslant b$. Suppose that*

$$\frac{1 - 2\lambda}{\lambda}(a + \lambda^n) \leqslant b < b + \lambda^n \leqslant \frac{\lambda a}{1 - 2\lambda}. \tag{2.4}$$

Then we have

$$(\alpha_n(a, b), \beta_n(a, b)) \subset \mathcal{G}(I, J),$$

where

$$\alpha_n(a, b) := a^2 + b^2 + 2a(1 - \lambda)\lambda^n + (1 - \lambda)^2\lambda^{2n},$$

and

$$\beta_n(a, b) := a^2 + b^2 + 2(a\lambda + b)\lambda^n + (1 + \lambda^2)\lambda^{2n}.$$

Proof. Let I_1, I_2, J_1, J_2 be defined as in Lemma 2.2. By Lemma 2.2 and (2.4), the four intervals $g(I_1 \times J_1)$, $g(I_2 \times J_1)$, $g(I_1 \times J_2)$, $g(I_2 \times J_2)$ are overlapping. Moreover, we want to show that

$$g(I_1 \times J_1) \cap g(I_1 \times J_2) \neq \emptyset \quad \text{and} \quad g(I_2 \times J_1) \cap g(I_2 \times J_2) \neq \emptyset,$$

which is equivalent to

$$\begin{cases} -(\lambda - 1)^2\lambda^n \leqslant 2a\lambda - 2(b + \lambda^n)(1 - 2\lambda), \\ (1 - 4\lambda + \lambda^2)\lambda^n \leqslant 2a\lambda - 2b(1 - 2\lambda). \end{cases} \tag{2.5}$$

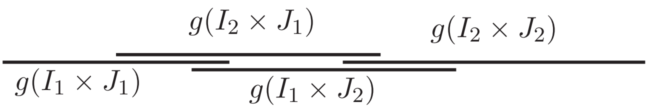

FIGURE 2 The relative position of intervals $g(I_1 \times J_1)$, $g(I_2 \times J_1)$, $g(I_1 \times J_2)$, and $g(I_2 \times J_2)$.

Note that the left-hand sides of both two inequalities in (2.5) are negative for $2 - \sqrt{3} < \lambda < 1/2$. Thus, we only need

$$a\lambda - (b + \lambda^n)(1 - 2\lambda) \geqslant 0,$$

which follows directly from (2.4).

Now, the relative position of intervals $g(I_1 \times J_1), g(I_2 \times J_1), g(I_1 \times J_2)$ and $g(I_2 \times J_2)$ is illustrated as in Figure 2. By the definition of the set of double covering points, we conclude that

$$(\alpha_n(a,b), \beta_n(a,b)) = \mathrm{int}(g(I_2 \times J_1) \cup g(I_1 \times J_2)) \subset \mathcal{G}(I,J),$$

completing the proof. □

Lemma 2.7. *Let $I, J \in \mathcal{F}_n$ with $I = [a, a + \lambda^n]$, $J = [b, b + \lambda^n]$, and $a \leqslant b$. Suppose that*

$$\frac{1 - \lambda - \lambda^2}{\lambda}(a + \lambda^n) \leqslant b < b + \lambda^n \leqslant \frac{\lambda a}{1 - 2\lambda}. \tag{2.6}$$

Then, we have

$$\left(a^2 + b^2, \beta_n(a,b)\right) \subset \mathcal{G}(I,J),$$

where $\beta_n(a,b)$ is defined as in Lemma 2.6.

Proof. Note that for any $I' \in \mathcal{F}_k^I, J' \in \mathcal{F}_k^J$ with $k \geqslant n$, we have $\mathcal{G}(I',J') \subset \mathcal{G}(I,J)$. For $k \geqslant n$, let $I_k' = [a, a + \lambda^k]$ and $J_k' = [b, b + \lambda^k]$. Then $I_k' \in \mathcal{F}_k^I$ and $J_k' \in \mathcal{F}_k^J$. Thus, we have

$$\bigcup_{k=n}^{\infty} \mathcal{G}(I_k', J_k') \subset \mathcal{G}(I,J).$$

Note that (2.6) implies (2.4). For each $k \geqslant n$, by Lemma 2.6, we obtain $\left(\alpha_k(a,b), \beta_k(a,b)\right) \subset \mathcal{G}(I_k', J_k')$. It follows that

$$\bigcup_{k=n}^{\infty} (\alpha_k(a,b), \beta_k(a,b)) \subset \mathcal{G}(I,J). \tag{2.7}$$

Note that $\alpha_k(a,b) \searrow a^2 + b^2$ and $\beta_k(a,b) \searrow a^2 + b^2$ as $k \to \infty$. Moreover, $\alpha_k(a,b) < \beta_{k+1}(a,b)$ for all $k \geqslant n$, since by calculation, we have

$$\lambda^{-k}\left(\beta_{k+1}(a,b) - \alpha_k(a,b)\right) = 2b\lambda - 2(a + \lambda^k)(1 - \lambda - \lambda^2) + (\lambda^2 - 1)^2\lambda^k$$

$$\begin{aligned} &> 2b\lambda - 2(a+\lambda^k)(1-\lambda-\lambda^2) \\ &\geqslant 0, \end{aligned}$$

where the last inequality follows directly from (2.6). Therefore, by (2.7), we conclude that

$$\big(a^2+b^2, \beta_n(a,b)\big) \subset \mathcal{G}(I,J),$$

as desired. ☐

Lemma 2.8. *Let $I, J \in \mathcal{F}_n$ with $I = [a, a+\lambda^n]$, $J = [b, b+\lambda^n]$, and $a \leqslant b$. Suppose that (2.6) in Lemma 2.7 holds. Then, we have*

$$\mathcal{G}(I,J) = \big(a^2+b^2, a^2+b^2+2(a+b)\lambda^n + 2\lambda^{2n}\big) = \mathrm{int}(g(I\times J)).$$

Proof. For $k \geqslant n$, let $\widetilde{I}'_k = [a_k, a_k+\lambda^k]$ and $\widetilde{J}'_k = [b_k, b_k+\lambda^k]$, where $a_k = a+\lambda^n-\lambda^k$ and $b_k = b+\lambda^n-\lambda^k$. Note that $\widetilde{I}'_k \in \mathcal{F}^I_k$ and $\widetilde{J}'_k \in \mathcal{F}^J_k$. Thus, we have

$$\bigcup_{k=n}^{\infty} \mathcal{G}(\widetilde{I}'_k, \widetilde{J}'_k) \subset \mathcal{G}(I,J).$$

By (2.6), we have

$$\frac{1-\lambda-\lambda^2}{\lambda}(a_k+\lambda^k) = \frac{1-\lambda-\lambda^2}{\lambda}(a+\lambda^n) \leqslant b \leqslant b_k,$$

and

$$b_k+\lambda^k = b+\lambda^n \leqslant \frac{\lambda a}{1-2\lambda} \leqslant \frac{\lambda a_k}{1-2\lambda}.$$

Applying Lemma 2.7 for $\widetilde{I}'_k, \widetilde{J}'_k$, we obtain that $\big(a_k^2+b_k^2, \beta_k(a_k,b_k)\big) \subset \mathcal{G}(\widetilde{I}'_k, \widetilde{J}'_k)$. It follows that

$$\bigcup_{k=n}^{\infty} \big(a_k^2+b_k^2, \beta_k(a_k,b_k)\big) \subset \mathcal{G}(I,J). \tag{2.8}$$

Note that

$$\begin{aligned} &\lambda^{-k}\big(\beta_k(a_k,b_k) - a_{k+1}^2 - b_{k+1}^2\big) \\ &= 2b\lambda - 2a(1-2\lambda) + 2(3\lambda-1)\lambda^n + (1-2\lambda-\lambda^2)\lambda^k \\ &= 2b\lambda - 2(a+\lambda^n)(1-2\lambda) + 2(\lambda^{n+1}-\lambda^{k+1}) + (1-\lambda^2)\lambda^k \\ &> 2b\lambda - 2(a+\lambda^n)(1-\lambda-\lambda^2) \\ &\geqslant 0, \end{aligned}$$

where the last inequality follows from (2.6). This implies that $\beta_k(a_k, b_k) > a_{k+1}^2 + b_{k+1}^2$ for each $k \geqslant n$. Note that $a_n = a$, $b_n = b$, and the sequence $\{\beta_k(a_k, b_k)\}_{k=n}^{\infty}$ is increasing to $(a+\lambda^n)^2 + (b+\lambda^n)^2$ as $k \to \infty$. By (2.8), we conclude that

$$\left(a^2 + b^2, a^2 + b^2 + 2(a+b)\lambda^n + 2\lambda^{2n}\right) \subset \mathcal{G}(I, J).$$

By definition, the inverse inclusion is obvious. Thus, we obtain the desired result. □

Proposition 2.9. *Let $I, J \in \mathcal{F}_n$ with $I = [a, a+\lambda^n]$, $J = [b, b+\lambda^n]$, and $a < b$. Suppose that (2.6) in Lemma 2.7 holds. Then for any $a^2 + b^2 < r < a^2 + b^2 + 2(a+b)\lambda^n + 2\lambda^{2n}$, the intersection $\left\{(x, y) : x^2 + y^2 = r\right\} \cap (K_\lambda \times K_\lambda)$ is of cardinality continuum.*

Proof. For $k \geqslant n$ and for $I' = [a', a' + \lambda^k] \in \mathcal{F}_k^I$ and $J' = [b', b' + \lambda^k] \in \mathcal{F}_k^J$, noting that $a \leqslant a' \leqslant a + \lambda^n - \lambda^k$ and $b \leqslant b' \leqslant b + \lambda^n - \lambda^k$, by (2.6) we have

$$\frac{1-\lambda-\lambda^2}{\lambda}(a' + \lambda^k) \leqslant b' < b' + \lambda^k \leqslant \frac{\lambda a'}{1-2\lambda}.$$

By Lemma 2.8, we conclude that

$$\mathcal{G}(I', J') = \left(a'^2 + b'^2, a'^2 + b'^2 + 2(a'+b')\lambda^k + 2\lambda^{2k}\right) = \operatorname{int}\left(g(I' \times J')\right). \tag{2.9}$$

Fix $a^2 + b^2 < r < a^2 + b^2 + 2(a+b)\lambda^n + 2\lambda^{2n}$. Next, we will inductively define two nested sequences of basic intervals $\left\{I_{i_1 \dots i_n}\right\}_n$ and $\left\{J_{i_1 \dots i_n}\right\}_n$ with each $i_1 \dots i_n \in \{1, 2\}^n$ for $n \in \mathbb{N}$ such that

- $I_{i_1 \dots i_n}, J_{i_1 \dots i_n}$ are basic intervals of K_λ at the same level;
- $I_{i_1 \dots i_n 1}, I_{i_1 \dots i_n 2} \subset I_{i_1 \dots i_n}$;
- $J_{i_1 \dots i_n 1}, J_{i_1 \dots i_n 2} \subset J_{i_1 \dots i_n}$;
- $r \in \mathcal{G}(I_{i_1 \dots i_n}, J_{i_1 \dots i_n})$.

By Lemma 2.8, we have $r \in \mathcal{G}(I, J)$. Then we can find two distinct pairs $(I_1, J_1), (I_2, J_2) \in \mathcal{F}_k^I \times \mathcal{F}_k^J$ for some $k \in \mathbb{N}$ such that r is an interior point of both $g(I_1 \times J_1)$ and $g(I_2 \times J_2)$. By (2.9), we have $r \in \mathcal{G}(I_1, J_1)$ and $r \in \mathcal{G}(I_2, J_2)$. Assume that $I_{i_1 \dots i_n}, J_{i_1 \dots i_n}$ have been defined. Since $r \in \mathcal{G}(I_{i_1 \dots i_n}, J_{i_1 \dots i_n})$, we can find two distinct pairs

$$(I_{i_1 \dots i_n 1}, J_{i_1 \dots i_n 1}), (I_{i_1 \dots i_n 2}, J_{i_1 \dots i_n 2}) \in \mathcal{F}_{k'}^{I_{i_1 \dots i_n}} \times \mathcal{F}_{k'}^{J_{i_1 \dots i_n}}$$

for some $k' \in \mathbb{N}$ such that r is an interior point of both $g(I_{i_1 \dots i_n 1} \times J_{i_1 \dots i_n 1})$ and $g(I_{i_1 \dots i_n 2} \times J_{i_1 \dots i_n 2})$. By (2.9), we have $r \in \mathcal{G}(I_{i_1 \dots i_n 1}, J_{i_1 \dots i_n 1})$ and $r \in \mathcal{G}(I_{i_1 \dots i_n 2}, J_{i_1 \dots i_n 2})$.

For any sequence $\mathbf{i} = i_1 i_2 \dots \in \{1, 2\}^{\mathbb{N}}$, we define $x_{\mathbf{i}}$ and $y_{\mathbf{i}}$ to be the unique point in

$$\bigcap_{n=1}^{\infty} I_{i_1 \dots i_n} \quad \text{and} \quad \bigcap_{n=1}^{\infty} J_{i_1 \dots i_n},$$

respectively. Then, we have $x_{\mathbf{i}}, y_{\mathbf{i}} \in K_\lambda$. Note that

$$r \in \mathcal{G}(I_{i_1\dots i_n}, J_{i_1\dots i_n}) \subset g(I_{i_1\dots i_n} \times J_{i_1\dots i_n}).$$

By Lemma 2.3, we have

$$\{r\} \subset \bigcap_{n=1}^{\infty} g(I_{i_1\dots i_n} \times J_{i_1\dots i_n}) = g\left(\bigcap_{n=1}^{\infty} I_{i_1\dots i_n} \times \bigcap_{n=1}^{\infty} J_{i_1\dots i_n}\right) = \{x_{\mathbf{i}}^2 + y_{\mathbf{i}}^2\}.$$

That is, $x_{\mathbf{i}}^2 + y_{\mathbf{i}}^2 = r$. It follows that

$$\left\{(x_{\mathbf{i}}, y_{\mathbf{i}}) : \mathbf{i} \in \{1,2\}^{\mathbb{N}}\right\} \subset \left\{(x,y) : x^2 + y^2 = r\right\} \cap (K_\lambda \times K_\lambda).$$

Note that any two points $(x_{\mathbf{i}}, y_{\mathbf{i}}), (x_{\mathbf{i}'}, y_{\mathbf{i}'})$ for $\mathbf{i} \neq \mathbf{i}' \in \{1,2\}^{\mathbb{N}}$ are different. Thus, we conclude that the intersection $\left\{(x,y) : x^2 + y^2 = r\right\} \cap (K_\lambda \times K_\lambda)$ is of cardinality continuum. □

Proposition 2.10. *For* $0.407493 \leqslant \lambda < 1/2$*, the intersection* $\mathbb{S} \cap (K_\lambda \times K_\lambda)$ *is of cardinality continuum.*

Proof.

(i) For $0.415 \leqslant \lambda < 1/2$, let $I = [1-\lambda+\lambda^2-\lambda^3, 1-\lambda+\lambda^2-\lambda^3+\lambda^4]$, and $J = [1-\lambda+\lambda^2-\lambda^4, 1-\lambda+\lambda^2]$. That is, $n = 4$, $a = 1-\lambda+\lambda^2-\lambda^3$, and $b = 1-\lambda+\lambda^2-\lambda^4$. In order to apply Proposition 2.9, we need to verify the following inequalities.

$$\begin{cases} \frac{1-\lambda-\lambda^2}{\lambda}(1-\lambda+\lambda^2-\lambda^3+\lambda^4) \leqslant 1-\lambda+\lambda^2-\lambda^4, \\ 1-\lambda+\lambda^2 \leqslant \frac{\lambda}{1-2\lambda}(1-\lambda+\lambda^2-\lambda^3), \\ (1-\lambda+\lambda^2-\lambda^3)^2 + (1-\lambda+\lambda^2-\lambda^4)^2 < 1, \\ (1-\lambda+\lambda^2-\lambda^3+\lambda^4)^2 + (1-\lambda+\lambda^2)^2 > 1. \end{cases} \tag{2.10}$$

(ii) For $0.407494 \leqslant \lambda < 0.415$, let $I = [1-\lambda, 1-\lambda+\lambda^3]$ and $J = [1-\lambda+\lambda^2-\lambda^3, 1-\lambda+\lambda^2]$. That is, $n = 3$, $a = 1-\lambda$, and $b = 1-\lambda+\lambda^2-\lambda^3$. In order to apply Proposition 2.9, we need to verify the following inequalities.

$$\begin{cases} \frac{1-\lambda-\lambda^2}{\lambda}(1-\lambda+\lambda^3) \leqslant 1-\lambda+\lambda^2-\lambda^3, \\ 1-\lambda+\lambda^2 \leqslant \frac{\lambda}{1-2\lambda}(1-\lambda), \\ (1-\lambda)^2 + (1-\lambda+\lambda^2-\lambda^3)^2 < 1, \\ (1-\lambda+\lambda^3)^2 + (1-\lambda+\lambda^2)^2 > 1. \end{cases} \tag{2.11}$$

The inequalities in (2.10) and (2.11) can be easily checked with the assistance of computers. The desired result follows from Proposition 2.9. □

Proof of Theorem 1.1. It follows directly from Propositions 2.1, 2.5, and 2.10. □

3 | INTERSECTION WITH A CURVE OR A SURFACE

Recall that K_λ is a self-similar set generated by the IFS $\{f_0(x) = \lambda x,\ f_1(x) = \lambda x + 1 - \lambda\}$ for $\lambda \in (0, 1/2)$. Each point $x \in K_\lambda$ corresponds to a unique sequence $(x_n) \in \{0,1\}^{\mathbb{N}}$, called the *coding of* x, such that

$$x = (1-\lambda)\sum_{n=1}^{\infty} x_n \lambda^{n-1}.$$

We will use the following Bernoulli inequality: for $n \in \mathbb{N}_{\geqslant 2}$,

$$(1+x)^n > 1 + nx, \quad \forall x > -1 \text{ with } x \neq 0. \tag{3.1}$$

In this section, we always assume that $0 < \lambda \leqslant 1/3$, and $q \in \mathbb{N}$.

Lemma 3.1. *Let $x, y \in K_\lambda$, and let $(x_n), (y_n) \in \{0,1\}^{\mathbb{N}}$ be the codings of x and y, respectively. Suppose $y = x^q$ for some $2 \leqslant q \leqslant 1/\lambda - 1$. If $x_1 x_2 \dots x_k x_{k+1} = 0^k 1$ for some $k \in \mathbb{N}$, then $y_1 y_2 \dots y_{qk} y_{qk+1} = 0^{qk} 1$.*

Proof. Since $x_1 x_2 \dots x_k = 0^k$, we have

$$x = (1-\lambda)\sum_{n=k+1}^{\infty} x_n \lambda^{n-1} \leqslant \lambda^k.$$

Then $y = x^q \leqslant \lambda^{qk}$. This implies that $y_1 y_2 \dots y_{qk} = 0^{qk}$.

Note that $x_{k+1} = 1$. Then

$$x = (1-\lambda)\sum_{n=k+1}^{\infty} x_n \lambda^{n-1} \geqslant (1-\lambda)\lambda^k.$$

By the Bernoulli inequality (3.1) and using $2 \leqslant q \leqslant 1/\lambda - 1$, we have

$$y = x^q \geqslant (1-\lambda)^q \lambda^{qk} > (1 - q\lambda)\lambda^{qk} \geqslant \lambda^{qk+1}.$$

This together with $y_1 y_2 \dots y_{qk} = 0^{qk}$ implies $y_{qk+1} = 1$. □

Lemma 3.2. *Let $x, y \in K_\lambda$, and suppose that $y = x^q$ for some $2 \leqslant q \leqslant 1/\lambda - 1$. If $x, y \in [1-\lambda, 1] \cap K_\lambda$, then $x = y = 1$.*

Proof. It suffices to show that $x, y \in [1 - \lambda^\ell, 1] \cap K$ for any $\ell \in \mathbb{N}$. Clearly, the conclusion holds for $\ell = 1$. Next, suppose that $x, y \in [1 - \lambda^k, 1] \cap K_\lambda$ for some $k \in \mathbb{N}$, we show that $x, y \in [1 - \lambda^{k+1}, 1] \cap K_\lambda$. If this were done, we would complete the proof by induction.

Observe that

$$x, y \in \left[1 - \lambda^k, 1\right] \cap K_\lambda \subset \left[1 - \lambda^k, 1 - (1-\lambda)\lambda^k\right] \cup \left[1 - \lambda^{k+1}, 1\right].$$

If $x \leqslant 1-(1-\lambda)\lambda^k$, then

$$\begin{aligned} y &\leqslant \left(1-(1-\lambda)\lambda^k\right)^q \leqslant \left(1-(1-\lambda)\lambda^k\right)^2 \\ &= 1-2(1-\lambda)\lambda^k+(1-\lambda)^2\lambda^{2k} \\ &\leqslant 1-2(1-\lambda)\lambda^k+(1-\lambda)^2\lambda^{k+1} \\ &\leqslant 1-\lambda^k-(1-3\lambda+2\lambda^2-\lambda^3)\lambda^k \\ &< 1-\lambda^k. \end{aligned}$$

where the last inequality follows from $0<\lambda\leqslant 1/3$. This leads to a contradiction. Thus, we obtain $x \in [1-\lambda^{k+1},1]\cap K_\lambda$. Then, by the Bernoulli inequality (3.1) and using $2\leqslant q\leqslant 1/\lambda-1$, we have

$$y = x^q \geqslant (1-\lambda^{k+1})^q > 1-q\lambda^{k+1} \geqslant 1-(1-\lambda)\lambda^k.$$

This implies that $y\in[1-\lambda^{k+1},1]\cap K_\lambda$. Therefore, we conclude that $x,y\in[1-\lambda^{k+1},1]\cap K$. □

Proposition 3.3. *Suppose that* $2\leqslant q\leqslant 1/\lambda-1$. *Then we have*

$$\{(x,y) : y=x^q\}\cap(K_\lambda\times K_\lambda) = \left\{(\lambda^k,\lambda^{qk}) : k\in\mathbb{N}\right\}\cup\{(0,0),(1,1)\}.$$

Proof. Take $x,y\in K_\lambda\setminus\{0\}$ with $y=x^q$. Let $(x_n),(y_n)\in\{0,1\}^{\mathbb{N}}$ be the codings of x and y, respectively.

If $x_1=1$, then $x\geqslant 1-\lambda$. By the Bernoulli inequality (3.1) and using $2\leqslant q\leqslant 1/\lambda-1$, we have

$$y = x^q \geqslant (1-\lambda)^q > 1-q\lambda \geqslant \lambda.$$

This implies that $y_1=1$. Thus, we have $x,y\in[1-\lambda,1]\cap K_\lambda$. Hence, by Lemma 3.2, we conclude that $x=y=1$.

If $x_1=0$, then since $x\neq 0$, there exists $k\in\mathbb{N}$ such that

$$x_1x_2\cdots x_k = 0^k \text{ and } x_{k+1}=1.$$

Then, by Lemma 3.1, we have

$$y_1y_2\cdots y_{qk} = 0^{qk} \text{ and } y_{qk+1}=1.$$

Let $\tilde{x}=\lambda^{-k}x$ and $\tilde{y}=\lambda^{-qk}y$. Then $\tilde{x},\tilde{y}\in[1-\lambda,1]\cap K_\lambda$, and $\tilde{y}=\tilde{x}^q$. Hence, by Lemma 3.2, we conclude that $\tilde{x}=\tilde{y}=1$. That is, $x=\lambda^k$ and $y=\lambda^{qk}$. This completes the proof. □

Proposition 3.4. *For* $0<\lambda\leqslant 0.187915$*, we have*

$$\{(x,y,z) : x^2+y^2=z^2\}\cap(K_\lambda\times K_\lambda\times K_\lambda) = \{(x,0,x),(0,x,x) : x\in K_\lambda\}.$$

Proof. The ideas are similar to the proof of Proposition 3.3. Let $x, y, z \in K_\lambda$ with $x^2 + y^2 = z^2$. By symmetry, we can assume $x \leqslant y$. We will show that $x = 0$. If $z = 0$, then $x = y = z = 0$. In the following, we assume that $z \neq 0$.

We first assume that $z \in [1-\lambda, 1] \cap K_\lambda$. If $x \geqslant 1 - \lambda$, then using $0 < \lambda < 1 - \sqrt{2}/2$, we have

$$x^2 + y^2 \geqslant 2x^2 \geqslant 2(1-\lambda)^2 > 1 \geqslant z^2.$$

Thus, we conclude that $x \in [0, \lambda]$. Next, suppose that $x \in [0, \lambda^k]$ for some $k \in \mathbb{N}$. We shall show $x \in [0, \lambda^{k+1}]$. If this were done, then we could conclude $x \in [0, \lambda^\ell]$ for any $\ell \in \mathbb{N}$, which implies that $x = 0$.

Observe that

$$x \in [0, \lambda^k] \cap K_\lambda \subset [0, \lambda^{k+1}] \cup [(1-\lambda)\lambda^k, \lambda^k].$$

Suppose on the contrary that $x \in [(1-\lambda)\lambda^k, \lambda^k]$. If $y \geqslant z - \lambda^{2k+1}$, then we have

$$\begin{aligned} x^2 + y^2 &\geqslant (1-\lambda)^2\lambda^{2k} + (z - \lambda^{2k+1})^2 \\ &> z^2 + (\lambda^2 - 2\lambda + 1 - 2z\lambda)\lambda^{2k} \\ &\geqslant z^2 + (\lambda^2 - 4\lambda + 1)\lambda^{2k} \\ &> z^2, \end{aligned}$$

where the last inequality follows from $\lambda < 2 - \sqrt{3}$. If $y < z - \lambda^{2k+1}$, then let $(y_n), (z_n) \in \{0,1\}^{\mathbb{N}}$ be the codings of y and z, respectively. That is,

$$y = (1-\lambda)\sum_{n=1}^{\infty} x_n\lambda^{n-1} \quad \text{and} \quad z = (1-\lambda)\sum_{n=1}^{\infty} z_n\lambda^{n-1}.$$

Since $y < z$, there exists $\ell \in \mathbb{N}$ such that

$$y_n = z_n \text{ for } 1 \leqslant n \leqslant \ell - 1, \quad \text{and} \quad y_\ell < z_\ell.$$

Then, we have $y_\ell = 0$ and $z_\ell = 1$. Observe that

$$\lambda^{2k+1} < z - y \leqslant (1-\lambda)\sum_{n=\ell}^{\infty} z_n\lambda^{n-1} \leqslant \lambda^{\ell-1}.$$

This implies that $\ell \leqslant 2k + 1$. It follows that

$$\begin{aligned} z - y &\geqslant (1-\lambda)\lambda^{\ell-1} - (1-\lambda)\sum_{n=\ell+1}^{\infty} y_n\lambda^{n-1} \\ &\geqslant (1-\lambda)\lambda^{\ell-1} - \lambda^\ell \\ &\geqslant (1-2\lambda)\lambda^{2k}, \end{aligned}$$

that is, $y \leqslant z - (1-2\lambda)\lambda^{2k}$. Thus, using $z \geqslant 1-\lambda$, we have

$$\begin{aligned} x^2+y^2 &\leqslant \lambda^{2k} + \left(z-(1-2\lambda)\lambda^{2k}\right)^2 \\ &= z^2 - (2(1-2\lambda)z-1)\lambda^{2k} + (1-2\lambda)^2\lambda^{4k} \\ &\leqslant z^2 - \left(2(1-2\lambda)(1-\lambda) - 1 - (1-2\lambda)^2\lambda^2\right)\lambda^{2k} \\ &= z^2 - (1-6\lambda+3\lambda^2+4\lambda^3-4\lambda^4)\lambda^{2k} \\ &< z^2, \end{aligned}$$

where the last inequality follows from $0<\lambda\leqslant 0.187915$. This leads to a contradiction with $x^2+y^2=z^2$. Thus, we conclude that $x \in [0,\lambda^{k+1}]$.

Next, if $z \in [0,\lambda]\cap K_\lambda$, then since $z \neq 0$, there exists $k \in \mathbb{N}$ such that $z \in [(1-\lambda)\lambda^k, \lambda^k]$. Let $\tilde{x} = \lambda^{-k}x$, $\tilde{y} = \lambda^{-k}y$, and $\tilde{z} = \lambda^{-k}z$. Then we have $\tilde{x},\tilde{y},\tilde{z} \in K_\lambda$ and $\tilde{x}^2+\tilde{y}^2-\tilde{z}^2=0$. Note that $\tilde{z} \in [1-\lambda,1]\cap K_\lambda$. By the above arguments, we have $\tilde{x}=0$. This implies that $x=0$.

Therefore, we conclude that either $x=0, y=z\in K_\lambda$ or $y=0, x=z\in K_\lambda$. □

Proof of Theorem 1.3. It follows directly from Propositions 3.3 and 3.4. □

4 | INTERSECTION WITH A SEQUENCE

We first introduce the quadratic residues and the Legendre symbol. Let $p \in \mathbb{N}$ be an odd prime and let $a \in \mathbb{Z}$ with $p \nmid a$. We say that a is a *quadratic residue* of p if the congruence $x^2 \equiv a \pmod p$ has a solution in $\{1,2,\dots,p-1\}$; otherwise, a is called a *quadratic nonresidue* of p [5]. The Legendre symbol is defined by

$$\left(\frac{a}{p}\right)_L = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Proof of Theorem 1.5. Since $(m-1)^2 \equiv 1 \pmod m$, we have

$$\left(\frac{-(m-1)}{m}\right)_L = \left(\frac{1}{m}\right)_L = 1.$$

This means that $m-1 \notin D$. It follows that $1 \notin K_{m,D}$.

We first show that for $n \in \mathbb{N}_{\geqslant 2}$, if $m \nmid n$, then $1/n^2 \notin K_{m,D}$. Fix $n \in \mathbb{N}_{\geqslant 2}$ with $m \nmid n$. Since m is prime, we have $\gcd(n^2,m)=1$. By [2, Proposition 2.1.2], the expansion of $1/n^2$ in base m is purely periodic, denoted by $(x_1x_2\dots x_q)^\infty$, where $x_1,x_2,\dots,x_q \in \{0,1,\dots,m-1\}$. Then we have

$$\frac{1}{n^2} = \left(\sum_{i=1}^{q}\frac{x_i}{m^i}\right)\Big/\left(1-\frac{1}{m^q}\right).$$

That is,

$$(x_q + x_{q-1}m + \cdots + x_1 m^{q-1})n^2 = m^q - 1.$$

Taking modulo m on the both sides, we obtain

$$x_q n^2 \equiv -1 \pmod{m}. \tag{4.1}$$

Thus, $x_q \neq 0$ and it follows that

$$(x_q n)^2 \equiv -x_q \pmod{m},$$

which yields that

$$\left(\frac{-x_q}{m}\right)_L = 1.$$

Thus, we conclude that $x_q \notin D$. Observe that $(x_1 x_2 \cdots x_q)^\infty$ is the unique expansion of $1/n^2$ in base m. Therefore, we obtain $1/n^2 \notin K_{m,D}$.

Next, we show that

$$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap K_{m,D} \subset \left\{\frac{1}{m^{2\ell}} : \ell \in \mathbb{N}\right\}. \tag{4.2}$$

Take $n \in \mathbb{N}$ with $1/n^2 \in K_{m,D}$. By the above arguments, we have $n \geqslant 2$ and $m \mid n$. Write $n = \widetilde{n} m^\ell$ with $\ell \in \mathbb{N}$ and $m \nmid \widetilde{n}$. If $\widetilde{n} \geqslant 2$, then we have $1/(\widetilde{n})^2 \notin K_{m,D}$. However,

$$\frac{1}{(\widetilde{n})^2} = m^{2\ell} \cdot \frac{1}{n^2} \in K_{m,D},$$

leading to a contradiction. Thus, we obtain $\widetilde{n} = 1$, that is, $n = m^\ell$ with $\ell \in \mathbb{N}$.

(i) If $1 \notin D$, then noting that $m - 1 \notin D$, we have

$$\left\{\frac{1}{m^{2\ell}} : \ell \in \mathbb{N}\right\} \cap K_{m,D} = \emptyset.$$

Thus, by (4.2), we conclude that

$$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap K_{m,D} = \emptyset.$$

(ii) If $1 \in D$, then we clearly have

$$\left\{\frac{1}{m^{2\ell}} : \ell \in \mathbb{N}\right\} \subset K_{m,D}.$$

Therefore, by (4.2), we conclude that

$$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap K_{m,D} = \left\{\frac{1}{m^{2\ell}} : \ell \in \mathbb{N}\right\},$$

as desired. □

Next, we give some basic properties about quadratic residues, see [5, Theorem 82, 83, 93].

Proposition 4.1. *Let p be an odd prime. Then*

(i) $\left(\frac{-1}{p}\right)_L = (-1)^{\frac{p-1}{2}}$;

(ii) $\left(\frac{2}{p}\right)_L = (-1)^{\frac{p^2-1}{8}}$;

(iii) *if $a, b \in \mathbb{Z}$ with $p \nmid a$ and $p \nmid b$, then* $\left(\frac{ab}{p}\right)_L = \left(\frac{a}{p}\right)_L \left(\frac{b}{p}\right)_L$.

The most famous theorem in quadratic residues is Gauss's law of reciprocity, see [5, Theorem 98].

Theorem 4.2 (Quadratic reciprocity law). *Let p, q be two different odd primes. Then,*

$$\left(\frac{p}{q}\right)_L \left(\frac{q}{p}\right)_L = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

We utilize Theorem 1.5 and properties of quadratic residue to obtain the following corollary.

Corollary 4.3. *For any $k \in \mathbb{N}_{\geqslant 2}$, there exist infinitely many primes $m > k$ such that the Cantor set $K_{m,D}$ with digit set $D = \{0, 1, \dots, k\}$ satisfying*

$$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap K_{m,D} = \left\{\frac{1}{m^{2\ell}} : \ell \in \mathbb{N}\right\}.$$

Proof. Let $p_1 = 2, p_2, \dots, p_t$ be all primes less than or equal to k. By Dirichlet's theorem, there are infinitely many primes $m > k$ of the form $4np_1p_2 \dots p_t - 1$ with $n \in \mathbb{N}$. Fix a such prime $m = 4np_1p_2 \dots p_t - 1$.

By Proposition 4.1 (i), we have

$$\left(\frac{-1}{m}\right)_L = -1. \tag{4.3}$$

For $2 \leqslant j \leqslant t$, by the quadratic reciprocity law, we have

$$\begin{aligned}\left(\frac{p_j}{m}\right)_L &= \left(\frac{m}{p_j}\right)_L \cdot (-1)^{\frac{(p_j-1)(m-1)}{4}} \\ &= \left(\frac{-1}{p_j}\right)_L \cdot (-1)^{\frac{(p_j-1)(m-1)}{4}}\end{aligned}$$

$$= (-1)^{\frac{p_j-1}{2}} \cdot (-1)^{\frac{(p_j-1)(m-1)}{4}}$$
$$= (-1)^{\frac{(p_j-1)(m+1)}{4}} = 1.$$

Note by Proposition 4.1 (ii) that

$$\left(\frac{2}{m}\right)_L = 1.$$

So, we have showed that

$$\left(\frac{p_j}{m}\right)_L = 1, \quad \forall 1 \leqslant j \leqslant t. \tag{4.4}$$

Each $2 \leqslant i \leqslant k$ can be factored as $p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, where $n_1, n_2, \ldots, n_t \geqslant 0$. Thus, by (4.4) and Proposition 4.1 (iii), we obtain that

$$\left(\frac{i}{m}\right)_L = 1, \quad \forall 2 \leqslant i \leqslant k.$$

Again by Proposition 4.1 (iii) and by (4.3), we have

$$\left(\frac{-i}{m}\right)_L = -1, \quad \forall 1 \leqslant i \leqslant k.$$

By Theorem 1.5, we conclude the desired result. □

5 | SOME QUESTIONS

At the end of this paper, we list some questions we are interested in. Let C be the middle-third Cantor set.

Question 5.1. Is the intersection

$$\{(x, y) : x^2 + y^2 = 1\} \cap (C \times C)$$

infinite? Moreover, to determine the Hausdorff dimension of the intersection. We expect the value to be $2 \log 2/ \log 3 - 1$.

Question 5.2. What is the intersection

$$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap C?$$

It is easy to check that

$$\left\{\frac{1}{9^k}, \frac{1}{4\times 9^k}, \frac{1}{121\times 9^k} : k \in \mathbb{N}\cup\{0\}\right\} \subset \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cap C.$$

Are there any other elements in the intersection?

Question 5.3. Determine the intersection

$$\left\{\frac{1}{n!} : n \in \mathbb{N}\right\} \cap C.$$

Note that 1 and $\frac{1}{5!}$ are in this intersection.

ACKNOWLEDGMENTS

The authors thank the anonymous referee for many useful comments. K. Jiang was supported by Zhejiang Provincial Natural Science Foundation of China under Grant No. LMS25A010009. D. Kong was supported by Chongqing NSF No. CQYC20220511052 and Scientific Research Innovation Capacity Support Project for Young Faculty No. ZYGXQNISKYCXNLZCXM-P2P. W. Li was supported by NSFC No. 12471085 and Science and Technology Commission of Shanghai Municipality (STCSM) No. 22DZ2229014. Z. Wang was supported by NSFC No. 12501110 and the China Postdoctoral Science Foundation No. 2024M763857.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. V. Ya Bloshchitsyn, *Rational points in m-adic Cantor sets*, J. Math. Sci. (N.Y.) **211** (2015), no. 6, 747–751.
2. K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Mathematical Association of America, Washington, DC, 2002.
3. C. Du, K. Jiang, and Y. Yao, *Rational and non-trivial solutions to some algebraic equations in the Cantor sets*, Monatsh. Math. **208** (2025), no. 1, 19–63.
4. K. Falconer, *Fractal geometry*, Mathematical Foundations and Applications, Wiley, Chichester, 1990.
5. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 6th ed, Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles, Oxford University Press, Oxford, 2008, pp. xxii+621.
6. J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747.
7. K. Jiang, D. Kong, W. Li, and Z. Wang, *Rational points in translations of the Cantor set*, Indag. Math. (N.S.) **35** (2024), no. 3, 516–522.
8. E. Lehmer, *Rational reciprocity laws*, Amer. Math. Monthly **85** (1978), no. 6, 467–472.
9. B. Li, R. Li, and Y. Wu, *Rational numbers in ×b-invariant sets*, Proc. Amer. Math. Soc. **151** (2023), no. 5, 1877–1887.
10. P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Fractals and rectifiability, 1st paperback ed., Cambridge University Press, Cambridge, 1999.
11. J. Nagy, *Rational points in Cantor sets*, Fibonacci Quart. **39** (2001), no. 3, 238–241.
12. J. Schleischitz, *On intrinsic and extrinsic rational approximation to Cantor sets*, Ergodic Theory Dynam. Systems **41** (2021), no. 5, 1560–1589.

13. P. Shmerkin, *On Furstenberg's intersection conjecture, self-similar measures, and the L^q norms of convolutions*, Ann. of Math. (2) **189** (2019), no. 2, 319–391.
14. I. Shparlinski, *On the arithmetic structure of rational numbers in the Cantor set*, Bull. Aust. Math. Soc. **103** (2021), no. 1, 22–27.
15. C. Wall, *Terminating decimals in the Cantor ternary set*, Fibonacci Quart. **28** (1990), no. 2, 98–101.
16. M. Wu, *A proof of Furstenberg's conjecture on the intersections of $\times p$- and $\times q$-invariant sets*, Ann. of Math. (2) **189** (2019), no. 3, 707–751.
17. H. Yu, *Missing digits points near manifolds*, arXiv:2309.00130, 2023.