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Random beta-transformations on fat Sierpinski gaskets

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Abstract. We consider the iterated function system (IFS)

$$f_{\vec{q}}(\vec{z}) = \frac{\vec{z} + \vec{q}}{\beta},\ \vec{q} \in \{(0,0),(1,0),(0,1)\}.$$

As is well known, for $\beta = 2$ the attractor, S_β, is a fractal called the Sierpiński gasket (or sieve) and for $\beta > 2$ it is also a fractal. Our goal is to study random β-transformations on the attractor for this IFS with $1 < \beta \leq 3/2$. In this case, S_β is a triangle. We show that all β-expansions of a point $\vec{z}$ in S_β can be generated by a random map K_β defined on $\{0,1\}^{\mathbb{N}} \times \{0,1,2\}^{\mathbb{N}} \times S_\beta$ and K_β has a unique invariant measure of maximal entropy. Furthermore, we show the existence of a K_β-invariant probability measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where m_1, m_2 are product measures on $\{0,1\}^{\mathbb{N}}, \{0,1,2\}^{\mathbb{N}}$, respectively, and μ_β is absolutely continuous with respect to the two-dimensional Lebesgue measure λ_2.

1. Introduction

Let $\beta > 1$ and consider the *iterated function system* (IFS):

$$f_{\vec{q}_0}(\vec{z}) = \frac{\vec{z} + \vec{q}_0}{\beta}, \quad f_{\vec{q}_1}(\vec{z}) = \frac{\vec{z} + \vec{q}_1}{\beta}, \quad f_{\vec{q}_2}(\vec{z}) = \frac{\vec{z} + \vec{q}_2}{\beta}, \tag{1.1}$$

where the coordinates of the three points $\vec{q}_0, \vec{q}_1, \vec{q}_2$ are $(0,0),(1,0),(0,1)$, respectively. It is well known that there exists a unique nonempty compact set $S_\beta \subset \mathbb{R}^2$ such that $S_\beta = \cup_{i=0}^2 f_{\vec{q}_i}(S_\beta)$; see [**F**] for further details. The *attractor* for the IFS, S_β, is a Sierpinski gasket. Denote the convex hull of S_β by Δ which is a triangle with vertices at $(0,0), (\frac{1}{\beta-1}, 0)$ and $(0, \frac{1}{\beta-1})$. For every point $\vec{z} \in S_\beta$, there exists a sequence $(a_i)_{i=1}^\infty \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}}$ such that

$$\vec{z} = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(\vec{q}_0) = \sum_{i=1}^\infty \frac{a_i}{\beta^i}.$$

We call $(a_i)_{i=1}^\infty$ a *coding* of $\vec{z}$ and $\sum_{i=1}^\infty a_i \beta^{-i}$ a *representation* of $\vec{z}$ in base β, or simple a β-expansion of $\vec{z}$.

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Let $i \in \{0,1,2\}$. For $\beta > 2$, the images $f_{\vec{q}_i}(\Delta)$ are disjoint. In this case the IFS $\{f_{\vec{q}_i}\}$ satisfies the strong separation condition and each point in S_β has a unique coding. For $\beta = 2$, the sets $f_{\vec{q}_i}(\Delta)$ overlap only at the vertices. Therefore only countably many points in S_β have two codings, and all other points have a unique coding. When $\beta \in (1,2)$, we call S_β a *fat Sierpinski gasket* and we distinguish two cases. For $1 < \beta \leq 3/2$, we have a non-empty double and triple overlaps and $S_\beta = \Delta$, see Figure 1. Furthermore, Lebesgue almost every point in S_β has a continuum of codings (see [**S1**, Theorem 3.5]). For $3/2 < \beta < 2$, there are holes in S_β as well as overlaps, which makes its structure more complex. In [**BMS**], Broomhead et al. described two special types of structures: those in which holes are radially distributed and those that are totally self-similar. Total self-similarity in our case implies $f_{\vec{q}_i}(S_\beta) = f_{\vec{q}_i}(\Delta) \cap S_\beta$. For more results on the Hausdorff dimension of the attractors, see [**KL**, **HP**, **SS**, **JP**, **H**].

In this article, we focus on the case $1 < \beta \leq 3/2$. In order to capture all possible β-expansions and to describe their statistical properties, we take an ergodic view. We start by defining a map K_β whose iterations generate all possible β-expansions of points in S_β. Dynamical properties of this map give information on the asymptotic properties of these expansions. The definition of our map is motivated from an analogous study of random β-expansions for points on an interval with digits in $\{0,1,\cdots,\lfloor\beta\rfloor\}$, see [**DK2**, **DV1**, **DV2**]. Our main aim is to generalise their results, in particular exhibit natural invariant ergodic measures for the random β-transformation K_β.

The rest of the article is organized as follows. In Section 2, we give the definition of the random transformation K_β on $\{0,1\}^{\mathbb{N}} \times \{0,1,2\}^{\mathbb{N}} \times S_\beta$ and prove basic properties. In Section 3, we prove that K_β has a unique invariant measure of maximal entropy. In Section 4, we give a position-dependent random map R on S_β. With two skew product transformations, we establish a connection between R and K_β, and finally prove that K_β has an invariant measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where m_1 is the product measure on $\{0,1\}^{\mathbb{N}}$ with weights $\{p, 1-p\}$, m_2 is the product measure on $\{0,1,2\}^{\mathbb{N}}$ with weights $\{s, t, 1-s-t\}$, and μ_β is R-invariant and absolutely continuous with respect to λ_2, the normalized Lebesgue measure on S_β.

2. Random beta-transformations

Given $1 < \beta \leq 3/2$, recall that the fat Sierpinski gasket S_β is the self-similar set in $\mathbb{R}^2$ generated by the IFS (1.1). For every point $\vec{z} \in S_\beta$, there exists a sequence $(a_i)_{i=1}^\infty \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}}$ such that $\vec{z} = \sum_{i=1}^\infty a_i \beta^{-i}$. Notice that S_β and its convex hull Δ are identical, both being an isosceles right triangle. We denote the Borel σ-algebra on S_β by $\mathcal{S}$. We also consider the following ordering of points in the plane. We write $(x_1, y_1) < (x_2, y_2)$ if $x_1 + y_1 < x_2 + y_2$, or $x_1 + y_1 = x_2 + y_2$ and $y_1 < y_2$. Notice that $\vec{q}_0 < \vec{q}_1 < \vec{q}_2$.

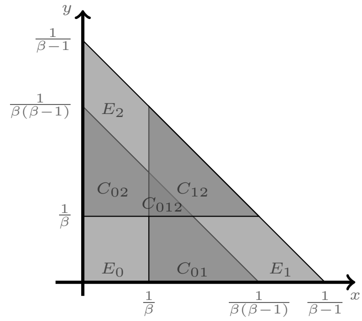

FIGURE 1. S_β for $1 < \beta \leq \frac{3}{2}$

Divide S_β into the following sets according to the overlapping structure of $f_{\vec{q}_i}(S_\beta)$ (see Figure 1):

$$E_0 = [0, \frac{1}{\beta}) \times [0, \frac{1}{\beta}),$$
$$E_1 = \{(x,y) : 0 \leq y < \frac{1}{\beta}, \frac{1}{\beta(\beta-1)} < x+y \leq \frac{1}{\beta-1}\},$$
$$E_2 = \{(x,y) : 0 \leq x < \frac{1}{\beta}, \frac{1}{\beta(\beta-1)} < x+y \leq \frac{1}{\beta-1}\},$$
$$C_{01} = \{(x,y) : x \geq \frac{1}{\beta}, 0 \leq y < \frac{1}{\beta}, x+y \leq \frac{1}{\beta(\beta-1)}\}, \tag{2.1}$$
$$C_{12} = \{(x,y) : x \geq \frac{1}{\beta}, y \geq \frac{1}{\beta}, \frac{1}{\beta(\beta-1)} < x+y \leq \frac{1}{\beta-1}\},$$
$$C_{02} = \{(x,y) : 0 \leq x < \frac{1}{\beta}, y \geq \frac{1}{\beta}, x+y \leq \frac{1}{\beta(\beta-1)}\},$$
$$C_{012} = \{(x,y) : x \geq \frac{1}{\beta}, y \geq \frac{1}{\beta}, x+y \leq \frac{1}{\beta(\beta-1)}\}.$$

Notice that $C_{012} = \{(\frac{2}{3}, \frac{2}{3})\}$ is a single point set if $\beta = 3/2$. These regions specify the digits that our random map assigns to points in S_β. For points in E_i the digit assigned is q_i, while in the double overlapping region C_{ij} we have two choices, q_i or q_j, and in the triple overlapping C_{012} we can choose q_0, q_1 or q_2. The choices will be dictated by either a double-sided coin or a triple-sided coin. To incorporate these choices in our definition of the random map K_β, we introduce two shift spaces representing the required coin tosses.

Let $\Omega = \{0,1\}^{\mathbb{N}}$ with the product σ-algebra $\mathcal{A}$ and $\Upsilon = \{0,1,2\}^{\mathbb{N}}$ with the product σ-algebra $\mathcal{B}$. Define metrics d and ρ on Ω and Υ, respectively, by

$$d((\omega_i),(\omega_i')) = 2^{-\inf\{k:\omega_k \neq \omega_k'\}} \quad \text{and} \quad \rho((\upsilon_i),(\upsilon_i')) = 2^{-\inf\{k:\upsilon_k \neq \upsilon_k'\}}.$$

Let $\sigma : \Omega \to \Omega$ and $\sigma' : \Upsilon \to \Upsilon$ be the left shifts.

Throughout the article, the lexicographical ordering on Ω, Υ and $\{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}}$ are all denoted by $\prec$ and $\preceq$. More precisely, for two sequences $(c_i), (d_i)$ we write $(c_i) \prec (d_i)$ if $c_1 < d_1$, or there exists $k \geq 2$ such that $c_i = d_i$ for all $1 \leq i < k$ and $c_k < d_k$. Similarly, we write $(c_i) \preceq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$.

Let $C = C_{01} \cup C_{12} \cup C_{02}$ and $E = \cup_{i=0}^{2} E_i$. Define $K_\beta : \Omega \times \Upsilon \times S_\beta \to \Omega \times \Upsilon \times S_\beta$ by

$$K_\beta(\omega, \upsilon, \vec{z}) = \begin{cases} (\omega, \upsilon, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ (\sigma\omega, \upsilon, \beta\vec{z} - \vec{q}_i), & \text{if } \omega_1 = 0 \text{ and } \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}, \\ (\sigma\omega, \upsilon, \beta\vec{z} - \vec{q}_j), & \text{if } \omega_1 = 1 \text{ and } \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}, \\ (\omega, \sigma'\upsilon, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in C_{012} \text{ and } \upsilon_1 = i \in \{0, 1, 2\}. \end{cases}$$

The digits are given by

$$d_1 = d_1(\omega, \upsilon, \vec{z}) = \begin{cases} \vec{q}_i, & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ & \text{or } (\omega, \upsilon, \vec{z}) \in \Omega \times \{\upsilon_1 = i\} \times C_{012}, \\ & \text{or } (\omega, \upsilon, \vec{z}) \in \{\omega_1 = 0\} \times \Upsilon \times C_{ij},\ ij \in \{01, 12, 02\}, \\ \vec{q}_j, & \text{if } (\omega, \upsilon, \vec{z}) \in \{\omega_1 = 1\} \times \Upsilon \times C_{ij},\ ij \in \{01, 12, 02\}. \end{cases}$$

Then

$$K_\beta(\omega, \upsilon, \vec{z}) = \begin{cases} (\omega, \upsilon, \beta\vec{z} - d_1), & \text{if } \vec{z} \in E, \\ (\sigma\omega, \upsilon, \beta\vec{z} - d_1), & \text{if } \vec{z} \in C, \\ (\omega, \sigma'\upsilon, \beta\vec{z} - d_1), & \text{if } \vec{z} \in C_{012}. \end{cases}$$

Set $d_n = d_n(\omega, \upsilon, \vec{z}) = d_1(K_\beta^{n-1}(\omega, \upsilon, \vec{z}))$, and $\pi_3 : \Omega \times \Upsilon \times S_\beta \to S_\beta$ be the canonical projection onto the third coordinate. Then

$$\pi_3(K_\beta^n(\omega, \upsilon, \vec{z})) = \beta^n \vec{z} - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$\vec{z} = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_3(K_\beta^n(\omega, \upsilon, \vec{z}))}{\beta^n}.$$

Since $\pi_3(K_\beta^n(\omega, \upsilon, \vec{z})) \in S_\beta$ and S_β is a bounded set in $\mathbb{R}^2$, it follows that

$$\|\vec{z} - \sum_{i=1}^{n} \frac{d_i}{\beta^i}\|_1 = \frac{\|\pi_3(K_\beta^n(\omega, \upsilon, \vec{z}))\|_1}{\beta^n} \to 0,$$

where $\|\cdot\|_1$ denotes the L_1 norm, i.e., the sum of the absolute values of the vector elements. This shows that for all $\omega \in \Omega$, $\upsilon \in \Upsilon$ and for all $\vec{z} \in S_\beta$ one has

$$\vec{z} = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, \upsilon, \vec{z})}{\beta^i}.$$

For each point $\vec{z} \in S_\beta$, consider the set

$$D_{\vec{z}} = \{(d_1(\omega, \upsilon, \vec{z}), d_2(\omega, \upsilon, \vec{z}), \ldots) : \omega \in \Omega, \upsilon \in \Upsilon\}.$$

Theorem 2.1 shows how the lexicographical ordering on Ω and Υ affect the ordering of the elements in $D_{\vec{z}}$.

THEOREM 2.1. *Suppose $\omega, \omega' \in \Omega, \upsilon, \upsilon' \in \Upsilon$ are such that $\omega \prec \omega'$ and $\upsilon \prec \upsilon'$. Then for $\vec{z} \in S_\beta$,*

$$(d_1(\omega, \upsilon, \vec{z}), d_2(\omega, \upsilon, \vec{z}), \ldots) \preceq (d_1(\omega', \upsilon', \vec{z}), d_2(\omega', \upsilon', \vec{z}), \ldots).$$

PROOF. Let $m := \inf\{i : \omega_i < \omega'_i\}$, $n := \inf\{i : \upsilon_i < \upsilon'_i\}$. Then we have $\omega_m < \omega'_m$ and $\upsilon_n < \upsilon'_n$. Denote by t_i the time of the ith visit to the region $\Omega \times \Upsilon \times C$ of the orbit of $(\omega, \upsilon, \vec{z})$ under K_β. Denote by s_j the time of the jth visit to the region $\Omega \times \Upsilon \times C_{012}$ of the orbit of $(\omega, \upsilon, \vec{z})$ under K_β. One can see that

$K_\beta^{t_i}(\omega, \upsilon, \vec{z})$ hits $\Omega \times \Upsilon \times C$ for the ith time and $K_\beta^{s_j}(\omega, \upsilon, \vec{z})$ hits $\Omega \times \Upsilon \times C_{012}$ for the jth time.

Let $l = \min\{t_m, s_n\}$. Then $\pi_3(K_\beta^i(\omega, \upsilon, \vec{z})) = \pi_3(K_\beta^i(\omega', \upsilon', \vec{z}))$ for $i = 0, \dots, l$. It follows that $d_i(\omega, \upsilon, \vec{z}) = d_i(\omega', \upsilon', \vec{z})$ for $i = 0, \dots, l$.

If $l = \infty$, then $d_i(\omega, \upsilon, \vec{z}) = d_i(\omega', \upsilon', \vec{z})$ for all i. If $l < +\infty$, then $K_\beta^l(\omega, \upsilon, \vec{z}) = K_\beta^l(\omega', \upsilon', \vec{z})$ hits $\Omega \times \Upsilon \times C$ for the mth time or $\Omega \times \Upsilon \times C_{012}$ for the nth time. Since $\omega_m < \omega'_m$ and $\upsilon_n < \upsilon'_n$, then

$$d_{l+1}(\omega, \upsilon, \vec{z}) = d_1(K_\beta^l(\omega, \upsilon, \vec{z})) < d_1(K_\beta^l(\omega', \upsilon', \vec{z})) = d_{l+1}(\omega', \upsilon', \vec{z}). \qquad \square$$

Now we show that any representation of $\vec{z}$ can be generated from the map K_β by choosing appropriate $\omega \in \Omega$ and $\upsilon \in \Upsilon$. We need the following lemma.

LEMMA 2.2. *Let $\beta \in (1, 3/2]$. Let $(x, y) \in S_\beta$ and $(x, y) = \sum_{i=1}^{\infty} a_i \beta^{-i}$ with $a_i \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}$ be a representation of (x, y) in base β. One has,*

(i) *If $(x, y) \in E_i$ for some $i \in \{0, 1, 2\}$, then $a_1 = \vec{q}_i$;*
(ii) *If $(x, y) \in C_{ij}$ for some $ij \in \{01, 12, 02\}$, then $a_1 \in \{\vec{q}_i, \vec{q}_j\}$;*
(iii) *If $(x, y) \in C_{012}$, then $a_1 \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}$.*

PROOF.

(i) Suppose $a_1 \neq \vec{q}_0$. From $a_1 \in \{\vec{q}_1, \vec{q}_2\}$ and $(x, y) = \sum_{i=1}^{\infty} a_i \beta^{-i}$ we have

$$x \geq \frac{1}{\beta} \text{ or } y \geq \frac{1}{\beta}.$$

Then $(x, y) \notin E_0$.

Suppose $a_1 \neq \vec{q}_1$. If $a_1 = \vec{q}_0$, then

$$x + y = \sum_{i=2}^{\infty} \frac{\| a_i \|_1}{\beta^i} \leq \frac{1}{\beta(\beta - 1)}.$$

If $a_1 = \vec{q}_2$, then $y \geq \frac{1}{\beta}$. In both cases, $(x, y) \notin E_1$.

Suppose $a_1 \neq \vec{q}_2$. By a similar proof we have $(x, y) \notin E_2$.

(ii) If $a_1 = \vec{q}_2$, then $y \geq \frac{1}{\beta}$, which implies $(x, y) \notin C_{01}$. If $a_1 = \vec{q}_0$, then $x + y \leq \frac{1}{\beta(\beta-1)}$, which implies $(x, y) \notin C_{12}$. If $a_1 = \vec{q}_1$, then $x \geq \frac{1}{\beta}$, which implies $(x, y) \notin C_{02}$.

(iii) holds trivially. $\square$

Using the above lemma and a construction similar to the one used in [**DV1**, Theorem 2], we have the following theorem.

THEOREM 2.3. *For $\beta \in (1, 3/2]$, let $\vec{z} \in S_\beta$ and $\vec{z} = \sum_{i=1}^{\infty} a_i \beta^{-i}$ with $a_i \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}$ be a representation of $\vec{z}$ in base β. Then there exists an $\omega \in \Omega$ and an $\upsilon \in \Upsilon$ such that $a_i = d_i(\omega, \upsilon, \vec{z})$.*

3. Unique invariant measure of maximal entropy for random beta-transformations

Equip Υ with the uniform product measure $\mathbb{P}$ and recall that σ' is the left shift on Υ. On the set $\Omega \times \Upsilon \times S_\beta$ we consider the product σ-algebra $\mathcal{A} \times \mathcal{B} \times \mathcal{S}$. Define the function $\rho_1 : \Omega \times \Upsilon \times S_\beta \to \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}}$ by

$$\rho_1(\omega, \upsilon, \vec{z}) = (d_1(\omega, \upsilon, \vec{z}), d_2(\omega, \upsilon, \vec{z}), \dots).$$

Define the function $\rho_2 : \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}} \to \Upsilon$ by

$$\rho_2(\vec{q}_{b_1}, \vec{q}_{b_2}, \vec{q}_{b_3}, \ldots) = (b_1, b_2, b_3, \ldots).$$

Denote by $\varphi = \rho_2 \circ \rho_1$ which is a function from $\Omega \times \Upsilon \times S_\beta$ to Υ. Then $\varphi \circ K_\beta = \sigma' \circ \varphi$, and φ is surjective from Theorem 2.3.

It is easily seen that φ is measurable. In fact, the inverse image of the cylinder set with the first digit fixed is measurable in $\Omega \times \Upsilon \times S_\beta$:

$$\begin{aligned}
&\varphi^{-1}(\{(b_1, b_2, \ldots) \in \Upsilon : b_1 = 0\}) \\
&\quad = (\Omega \times \Upsilon \times E_0) \cup (\{\omega \in \Omega : \omega_1 = 0\} \times \Upsilon \times (C_{01} \cup C_{02})) \\
&\qquad \cup (\Omega \times \{\upsilon \in \Upsilon : \upsilon_1 = 0\} \times C_{012}), \\
&\varphi^{-1}(\{(b_1, b_2, \ldots) \in \Upsilon : b_1 = 1\}) \\
&\quad = (\Omega \times \Upsilon \times E_1) \cup (\{\omega \in \Omega : \omega_1 = 1\} \times \Upsilon \times C_{01}) \\
&\qquad \cup (\{\omega \in \Omega : \omega_1 = 0\} \times \Upsilon \times C_{12}) \cup (\Omega \times \{\upsilon \in \Upsilon : \upsilon_1 = 1\} \times C_{012}), \\
&\varphi^{-1}(\{(b_1, b_2, \ldots) \in \Upsilon : b_1 = 2\}) \\
&\quad = (\Omega \times \Upsilon \times E_2) \cup (\{\omega \in \Omega : \omega_1 = 1\} \times \Upsilon \times (C_{02} \cup C_{12})) \\
&\qquad \cup (\Omega \times \{\upsilon \in \Upsilon : \upsilon_1 = 2\} \times C_{012}).
\end{aligned}$$

To show that φ is an isomorphism, let

$$Z_1 = \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ infinitely often}\},$$

$$Z_2 = \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ infinitely often}\},$$

$$D_1 = \{(b_1, b_2, \ldots) \in \Upsilon : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C \text{ for infinitely many } j\},$$

$$D_2 = \{(b_1, b_2, \ldots) \in \Upsilon : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C_{012} \text{ for infinitely many } j\}.$$

Notice that

$$Z_1 = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} K_\beta^{-m}(\Omega \times \Upsilon \times C)$$

and

$$Z_2 = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} K_\beta^{-m}(\Omega \times \Upsilon \times C_{012}),$$

which imply that Z_1 and Z_2 are Borel sets in $\Omega \times \Upsilon \times S_\beta$. Let $Z = Z_1 \cap Z_2$, $D = D_1 \cap D_2$, then we have $K_\beta^{-1}(Z) = Z$, $(\sigma')^{-1}(D) = D$ and $\varphi(Z) = D$. Let $\varphi' = \varphi|_Z$.

Lemma 3.1. *The map $\varphi' : Z \to D$ is a bimeasurable bijection.*

Proof. For any sequence $(b_1, b_2, \ldots) \in D$, we can obtain a point

$$\vec{z} = \sum_{i=1}^{\infty} \vec{q}_{b_i} \beta^{-i}.$$

To determine ω and υ, we could define

$$r_1 = \min\{j \geq 1 : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C\}, \quad r_k = \min\{j > r_{k-1} : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C\},$$

$$s_1 = \min\{j \geq 1 : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C_{012}\}, \quad s_k = \min\{j > s_{k-1} : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C_{012}\}.$$

- If $\sum_{i=1}^{\infty} \vec{q}_{b_{r_k+i-1}} \beta^{-i} \in C_{ij}, ij \in \{01, 12, 02\}$, then $b_{r_k} \in \{i, j\}$ by Lemma 2.2.
  - If $b_{r_k} = i$, we let $\omega_k = 0$.
  - If $b_{r_k} = j$, we let $\omega_k = 1$.
- If $\sum_{i=1}^{\infty} \vec{q}_{b_{s_k+i-1}} \beta^{-i} \in C_{012}$, then $b_{s_k} \in \{0, 1, 2\}$ by Lemma 2.2. If $b_{s_k} = i$, we let $\upsilon_k = i, i \in \{0, 1, 2\}$.

Notice that for any $N > 0$, there exsit $r, s > N$ such that

$$\sum_{i=1}^{\infty} \vec{q}_{b_{r+i-1}} \beta^{-i} \in C \text{ and } \sum_{i=1}^{\infty} \vec{q}_{b_{s+i-1}} \beta^{-i} \in C_{012},$$

which implies that $K_\beta^n(\omega, \upsilon, \vec{z})$ hits both $\Omega \times \Upsilon \times C$ and $\Omega \times \Upsilon \times C_{012}$ infinitely often. Then the infinite sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots) \in \Omega$ and $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3, \dots) \in \Upsilon$ can be uniquely determined. Therefore, we can define the inverse of φ'. Let $(\varphi')^{-1} : D \to Z$ be

$$(\varphi')^{-1}((b_1, b_2, \dots)) = (\omega, \upsilon, \sum_{i=1}^{\infty} \frac{\vec{q}_{b_i}}{\beta^i}).$$

If $(\omega, \upsilon, \vec{z}) = (\omega', \upsilon', \vec{z'})$ then $\varphi'(\omega, \upsilon, \vec{z}) = \varphi'(\omega', \upsilon', \vec{z'})$. Since $Z = Z_1 \cap Z_2$ is a Borel set in $\Omega \times \Upsilon \times S_\beta$, then we have that $(\varphi')^{-1}$ is measurable (see [**S2**, Theorem 4.5.4]). Hence φ' is a bimeasurable bijection. □

Lemma 3.2. *If $1 < \beta < 3/2$, then $\mathbb{P}(D) = 1$.*

Proof. Let us first prove $\mathbb{P}(D_2) = 1$. Let $n \geq 1$ and denote a cylinder set in Υ by

$$[\upsilon_1, \upsilon_2, \dots, \upsilon_n] = \{(b_1, b_2, \dots) \in \Upsilon : b_i = \upsilon_i, i = 1, \dots, n\}.$$

Let

$$S_{\beta,\upsilon_1,\upsilon_2,\dots,\upsilon_n} = \{\vec{z} = \sum_{i=1}^{\infty} \frac{\vec{q}_{b_i}}{\beta^i} : (b_1, b_2, \dots) \in [\upsilon_1, \upsilon_2, \dots, \upsilon_n]\}.$$

Notice that $S_{\beta,\upsilon_1,\upsilon_2,\dots,\upsilon_n}$ is a right triangle with $\sum_{i=1}^{n} \frac{\vec{q}_{b_i}}{\beta^i}$ as its right-angled vertex and a maximum diameter of $\frac{\sqrt{2}}{\beta^n(\beta-1)}$ when $1 < \beta < \frac{3}{2}$. Since $\lim_{n\to\infty} \frac{\sqrt{2}}{\beta^n(\beta-1)} = 0$ and C_{012} has positive Lebesgue measure, then we can find a cylinder set $[c_1, c_2, \dots, c_N]$ such that $S_{\beta,c_1,c_2,\dots,c_N} \subset C_{012}$.

Let

$$D' = \{(b_1, b_2, \dots) \in \Upsilon : b_j b_{j+1} \dots b_{j+N-1} = c_1 c_2 \dots c_N \text{ for infinitely many } j\}$$

then $D' \subset D_2$.

Now we show that $\mathbb{P}(D') = 1$. Notice that $D' = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} \tilde{D}_m$, where $\tilde{D}_m = \{(b_1, b_2, \dots) \in \Upsilon : b_m b_{m+1} \dots b_{m+N-1} = c_1 c_2 \dots c_N\}$. Let $B_n = \cup_{m=n}^{\infty} \tilde{D}_m$. If $(b_1, b_2, \dots) \in \Upsilon \setminus B_n$, then we have that for any $j \geq n$, $b_j b_{j+1} \dots b_{j+N-1} \neq c_1 c_2 \dots c_N$. Clearly,

$$\Upsilon \setminus B_n \subseteq B' := \{(b_1, b_2, \dots) : b_{n+kN} \dots b_{n+(k+1)N-1} \neq c_1 c_2 \dots c_N, k = 0, 1, 2, \dots\}.$$

Since $\mathbb{P}(\Upsilon \setminus B_n) \leq \mathbb{P}(B') = \lim_{k\to\infty}(1 - 1/3^N)^k = 0$, then $\mathbb{P}(B_n) = 1$. It follows from $D' = \cap_{n=1}^{\infty} B_n$ and $B_1 \supseteq B_2 \supseteq \cdots$ that $\mathbb{P}(D') = \lim_{n\to\infty} B_n = 1$. Then we get $\mathbb{P}(D_2) = 1$.

To prove that $\mathbb{P}(D_1) = 1$, we can use a similar approach. Here we construct a specific cylinder. Define

$$\vec{z}_l = (x_l, y_l) = \frac{\vec{q}_1}{\beta} + \frac{\vec{q}_0}{\beta^2} + \cdots + \frac{\vec{q}_0}{\beta^l} + \frac{\vec{q}_{b_1}}{\beta^{l+1}} + \frac{\vec{q}_{b_2}}{\beta^{l+2}} + \cdots .$$

Then we have $x_l \geq \frac{1}{\beta}$, and

$$0 \leq y_l \leq \sum_{i=1}^{\infty} \frac{1}{\beta^{l+i}} = \frac{1}{\beta^l(\beta-1)},$$

$$\frac{1}{\beta} \leq x_l + y_l \leq \frac{1}{\beta} + \sum_{i=1}^{\infty} \frac{1}{\beta^{l+i}} = \frac{1}{\beta} + \frac{1}{\beta^l(\beta-1)}.$$

Since $\lim_{l\to\infty} \frac{1}{\beta^l(\beta-1)} = 0$, then there exists $L > 0$ such that for any $l \geq L$,

$$0 \leq y_l < \frac{1}{\beta}, x_l + y_l \leq \frac{1}{\beta(\beta-1)}.$$

It follows that $(x_l, y_l) \in C_{01}$ for any $l \geq L$. Let

(3.1)
$$D'' = \{(b_1, b_2, \ldots) \in \Upsilon : b_j b_{j+1} \ldots b_{j+L-1} = 1 \underbrace{00\ldots0}_{L-1 \text{ times}} \text{ for infinitely many } j\},$$

then $D'' \subset D_1$. Since $\mathbb{P}(D'') = 1$, we have $\mathbb{P}(D_1) = 1$. Therefore, $\mathbb{P}(D) = 1$. □

Theorem 3.3 can be obtained from Lemmas 3.1 and 3.2.

THEOREM 3.3. *Let $\beta \in (1, 3/2]$ and set $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$. The dynamical systems $(\Omega \times \Upsilon \times S_\beta, \mathcal{A} \times \mathcal{B} \times \mathcal{S}, \nu_\beta, K_\beta)$ and $(\Upsilon, \mathcal{B}, \mathbb{P}, \sigma')$ are isomorphic.*

REMARK 3.4.

(i) Notice that Lemma 3.2 and Theorem 3.3 remain true if one replaces $\mathbb{P}$ by any other non-uniform product measure on Υ giving a positive weight to each symbol.

(ii) Since $\mathbb{P}$ is the unique measure of maximal entropy on Υ, the above theorem implies that any other K_β-invariant measure with support Z has entropy strictly less than $\log 3$. We now investigate the entropy of K_β-invariant measure μ for which $\mu(Z^c) > 0$.

Divide Z^c into three Borel sets as follows:

$$\begin{aligned} Z^c &= (Z_1 \cap Z_2)^c \\ &= (Z_1^c \setminus Z_2^c) \cup (Z_2^c \setminus Z_1^c) \cup (Z_1^c \cap Z_2^c) \\ &= (Z_2 \setminus Z_1) \cup (Z_1 \setminus Z_2) \cup (Z_1^c \cap Z_2^c) \\ &:= Z_3 \cup Z_4 \cup Z_5, \end{aligned}$$

where

$$\begin{aligned}
Z_3 =& Z_2 \setminus Z_1 \\
=& \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ for finitely many } n \\
& \text{and } K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ infinitely often}\}, \\
Z_4 =& Z_1 \setminus Z_2 \\
=& \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ for finitely many } n, \\
& \text{and } K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ infinitely often}\}, \\
Z_5 =& Z_1^c \cap Z_2^c \\
=& \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ for finitely many } n, \\
& \text{and } K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ for finitely many } n\}.
\end{aligned}$$

We first prove Lemma 3.5.

LEMMA 3.5. *Let $\beta \in (1, 3/2)$. Let μ_3 be a K_β-invariant measure for which $\mu_3(Z_3) = 1$. Then $h_{\mu_3}(K_\beta) < \log 3$. Similarly, let μ_4 and μ_5 be K_β-invariant measures for which $\mu_4(Z_4) = \mu_5(Z_5) = 1$. Then $h_{\mu_4}(K_\beta), h_{\mu_5}(K_\beta) < \log 3$ also holds.*

PROOF. Let

$$H_3 = \{\vec{z} = \sum_{i=1}^{\infty} \frac{\vec{q}_{b_i}}{\beta^i} \in S_\beta : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \text{ belongs to } C_{012} \text{ for infinitely many } j, \\ \text{and never belongs to } C\}.$$

Then $\Omega \times \Upsilon \times H_3 \subseteq K_\beta^{-1}(\Omega \times \Upsilon \times H_3)$ and $\cup_{i=0}^{\infty} K_\beta^{-i}(\Omega \times \Upsilon \times H_3) = Z_3$. It follows that $\mu_3(Z_3) = \lim_{i\to\infty} \mu_3(K_\beta^{-i}(\Omega \times \Upsilon \times H_3)) = 1$. Since μ_3 is K_β-invariant, then

$$\mu_3(\Omega \times \Upsilon \times H_3) = \mu_3(K_\beta^{-1}(\Omega \times \Upsilon \times H_3)) = \mu_3(K_\beta^{-2}(\Omega \times \Upsilon \times H_3)) = \cdots = 1.$$

Thus it is enough to study the entropy with respect to μ_3 of the map K_β restricted to $\Omega \times \Upsilon \times H_3$. Let π_1, π_2, π_3 be the canonical projection onto the three coordinates respectively. Notice that the action of the transformation K_β on the first coordinate is an identity, which implies that K_β is essentially a product transformation $I_\Omega \times K'_\beta$, where $K'_\beta = (\pi_2 \circ K_\beta) \times (\pi_3 \circ K_\beta)$ on $\Upsilon \times H_3$ and I_Ω is the identity on Ω. Since $(\upsilon, \vec{z}) \in \Upsilon \times S_\beta$ and $\omega \in \Omega$ are independent, and $h_\mu(I_\Omega) = 0$ for any measure μ on $(\Omega, \mathcal{A})$, we see that $h_{\mu_3}(K_\beta) = h_{\mu'_3}(K'_\beta)$, where $\mu'_3(B \times H) = \mu_3(\Omega \times B \times H)$ for $B \in \mathcal{B}, H \in (H_3 \cap \mathcal{S})$.

Let

$$D_3 = \{(b_1, b_2, \dots) \in \Upsilon : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \text{ belongs to } C_{012} \text{ for infinitely many } j, \\ \text{and never belongs to } C\}.$$

Define a map ϕ from $(\Upsilon \times H_3, \mathcal{B} \times (H_3 \cap \mathcal{S}), \mu'_3, K'_\beta)$ to $(D_3, D_3 \cap \mathcal{B}, \mu'_3 \circ \phi^{-1}, \sigma')$ as

$$\phi(\upsilon, \vec{z}) = \rho_2(\rho_1(0^\infty, \upsilon, \vec{z})).$$

Since $\vec{z} \in H_3$, then ϕ is well defined and bijective. ϕ is measurable and the inverse is also measurable (see [**S2**, Theorem 4.5.4]). Finally, ϕ preserves the measure and

$\phi \circ K'_\beta = \sigma' \circ \phi$. Then ϕ is an isomorphism and it follows that

$$h_{\mu_3}(K_\beta) = h_{\mu'_3}(K'_\beta) = h_{\mu'_3 \circ \phi^{-1}}(\sigma') \leq h_{\mathbb{P}}(\sigma') = \log 3.$$

Since $\mathbb{P}$ is the unique measure of maximal entropy on D_3, to show $h_{\mu_3}(K_\beta) < \log 3$, it is enough to prove that $\mu'_3 \circ \phi^{-1} \neq \mathbb{P}$. This is done by contradiction. If $\mu'_3 \circ \phi^{-1} = \mathbb{P}$, then

$$\mathbb{P}(D_3) = \mu'_3(\Upsilon \times H_3) = \mu_3(\Omega \times \Upsilon \times H_3) = 1.$$

Since $D_3 \subset (D'')^c$, where D'' is defined as in (3.1), then $\mathbb{P}((D'')^c) = 1$, which is a contradiction to $\mathbb{P}(D'') = 1$. Therefore, $h_{\mu_3}(K_\beta) < \log 3$. Let

$$\begin{aligned} H_4 = \{\vec{z} = \sum_{i=1}^{\infty} \frac{\vec{q}_{b_i}}{\beta^i} \in S_\beta : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} &\text{ belongs to } C \text{ for infinitely many } j, \\ &\text{and never belongs to } C_{012}\}, \\ H_5 = \{\vec{z} = \sum_{i=1}^{\infty} \frac{\vec{q}_{b_i}}{\beta^i} \in S_\beta : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} &\text{ never belongs to } C \\ &\text{and never belongs to } C_{012}\}, \\ = \{\vec{z} \in S_\beta : \vec{z} \text{ has a unique } \beta\text{-expansion}\}. \end{aligned}$$

Then it follows that

$$\mu_4(\Omega \times \Upsilon \times H_4) = \mu_5(\Omega \times \Upsilon \times H_5) = 1.$$

We can also obtain that $h_{\mu_4}(K_\beta), h_{\mu_5}(K_\beta) < \log 3$ using the similar method. □

From Lemma 3.5 we can obtain the upper bound of the entropy of K_β-invariant measure for which Z^c has positive measure.

Lemma 3.6. *Let $\beta \in (1, 3/2]$. Let μ be a K_β-invariant measure for which $\mu(Z^c) > 0$. Then $h_\mu(K_\beta) < \log 3$.*

Proof. Notice that Z, Z_3, Z_4 and Z_5 are pairwise disjoint and the union is $\Omega \times \Upsilon \times S_\beta$. Since Z, Z_3, Z_4 and Z_5 are K_β-invariant, then there exist K_β-invariant probability measures μ_{12}, μ_3, μ_4 and μ_5 concentrated on Z, Z_3, Z_4 and Z_5, respectively, such that

$$\mu = (1 - \alpha_3 - \alpha_4 - \alpha_5)\mu_{12} + \alpha_3\mu_3 + \alpha_4\mu_4 + \alpha_5\mu_5,$$

where $0 \leq \alpha_3, \alpha_4, \alpha_5 \leq 1$ and $0 < \alpha_3 + \alpha_4 + \alpha_5 \leq 1$. Then

$$h_\mu(K_\beta) = (1 - \alpha_3 - \alpha_4 - \alpha_5)h_{\mu_{12}}(K_\beta) + \alpha_3 h_{\mu_3}(K_\beta) + \alpha_4 h_{\mu_4}(K_\beta) + \alpha_5 h_{\mu_5}(K_\beta).$$

Since $h_{\mu_{12}}(K_\beta) \leq \log 3$ by Remark 3.4 and $h_{\mu_3}(K_\beta), h_{\mu_4}(K_\beta), h_{\mu_5}(K_\beta) < \log 3$ by Lemma 3.5, then the result follows. □

Now we obtain the main result in this section.

Theorem 3.7. *Let $\beta \in (1, 3/2)$. The measure $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$ is the unique K_β-invariant measure of maximal entropy.*

Remark 3.8. The measure ν_β is not self-similar, but the projection in the third coordinate is a self-similar measure defined on S_β. To be more precise, define

$h : \Upsilon \to S_\beta$ by $h(b_1, b_2, \ldots) = \sum_{i=1}^{\infty} \frac{\vec{q}_{b_i}}{\beta^i}$ and consider the commuting diagram

$$\begin{array}{ccc} \Omega \times \Upsilon \times S_\beta & \xrightarrow{\pi_2} & S_\beta \\ & \searrow \varphi & \uparrow h \\ & & \Upsilon, \end{array}$$

Then, $\nu_\beta \circ \pi_2^{-1}$ satisfies $\nu_\beta \circ \pi_2^{-1} = \frac{1}{3} \sum_{i=0}^{2} \nu_\beta \circ f_i^{-1}$.

4. An absolutely continuous invariant measure for random beta-transformations

We start by recalling that for $\beta = 3/2$, the region $C_{012} = \{(\frac{2}{3}, \frac{2}{3})\}$ is a point. As a result the analysis for this case is slightly different from the one conducted for a general $\beta \in (1, 3/2)$. In this section, we concentrate on the case $\beta \in (1, 3/2)$ and in Remark 4.9, we give a brief description of the case $\beta = 3/2$.

Endow $\Omega = \{0, 1\}^{\mathbb{N}}$ with the product measure m_1 giving the symbol 0 probability p and the symbol 1 probability $1 - p$, and $\Upsilon = \{0, 1, 2\}^{\mathbb{N}}$ with the product measure m_2 giving the symbol 0 probability s, the symbol 1 probability t and the symbol 2 probability $1 - s - t$. Consider the measure space $(S_\beta, \mathcal{S}, \lambda_2)$, where λ_2 is the normalized Lebesgue measure. In this section we will prove that K_β has an invariant measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where μ_β is absolutely continuous with respect to λ_2. We will show the result by several steps.

Step 1. A position-dependent random transformation R.

Bahsoun and Góra [**BG**] gave a sufficient condition for the existence of an absolutely continuous invariant measure for a random map with position-dependent probabilities on a bounded domain of $\mathbb{R}^N$. We take some of their results a little further.

For $k = 1, \ldots, K$, let $\tau_k : S_\beta \to S_\beta$ be piecewise one-to-one and C^2, non-singular transformations on a common partition $\mathcal{P}$ of $S_\beta : \mathcal{P} = \{S_1, \ldots, S_q\}$ and $\tau_{k,i} = \tau_k|_{S_i}, i = 1, \ldots, q$. Let $p_k : S_\beta \to [0, 1]$ be piecewise C^1 functions such that $\sum_{k=1}^{K} p_k = 1$. Denote by $R = \{\tau_1, \ldots, \tau_K; p_1(\vec{z}), \ldots, p_K(\vec{z})\}$ the position-dependent random map, i.e., $R(\vec{z}) = \tau_k(\vec{z})$ with probability $p_k(\vec{z})$. Define the transition function for R as follows:

$$\mathbf{P}(\vec{z}, A) = \sum_{k=1}^{K} p_k(\vec{z}) \mathbb{1}_A(\tau_k(\vec{z})),$$

where A is any measurable set and $\mathbb{1}_A$ denotes the indicator function of the set A.

The iteration of R is denoted by $R^n := \{\tau_{k_1 k_2 \cdots k_n}; p_{k_1 k_2 \cdots k_n}\}, k_1 k_2 \cdots k_n \in \{1, 2, \ldots, K\}^n$, where $\tau_{k_1 k_2 \cdots k_n}(\vec{z}) = \tau_{k_n} \circ \tau_{k_{n-1}} \circ \cdots \circ \tau_{k_1}(\vec{z})$ and

$$p_{k_1 k_2 \cdots k_n}(\vec{z}) = p_{k_n}(\tau_{k_{n-1}} \circ \cdots \circ \tau_{k_1}(\vec{z})) \cdot p_{k_{n-1}}(\tau_{k_{n-2}} \circ \cdots \circ \tau_{k_1}(\vec{z})) \cdots p_{k_1}(\vec{z}).$$

The transition function $\mathbf{P}$ induces an operator $\mathbf{P}_*$ on the set of probability measures on $(S_\beta, \mathcal{S})$ defined by

$$\mathbf{P}_* \mu(A) = \int \mathbf{P}(\vec{z}, A) d\mu(\vec{z}) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(\vec{z}) d\mu(\vec{z}) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(\vec{z}) d\mu(\vec{z}).$$

We say that the measure μ is R-invariant iff $\mathbf{P}_* \mu = \mu$.

If μ has density f with respect to λ_2, then $\mathbf{P}_*\mu$ also has a density which we denote by $P_R f$, i.e.,

$$\int_A P_R f(\vec{z}) d\lambda_2(\vec{z}) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(\vec{z}) f(\vec{z}) d\lambda_2(\vec{z}).$$

We call P_R the Perron–Frobenius operator of the random map R and it has very useful properties[**BG**]:

(i) P_R is linear;
(ii) P_R is nonnegative;
(iii) $P_R f = f \iff \mu = f \cdot \lambda_2$ is R-invariant;
(iv) $\|P_R f\|_1 \leq \|f\|_1$, where $\|\cdot\|_1$ denotes the L^1 norm;
(v) $P_{R \circ T} = P_R \circ P_T$. In particular, $P_R^N = P_{R^N}$.

Let each S_i be a bounded closed domain having a piecewise C^2 boundary of finite 1-dimensional measure. Assume that the faces of ∂S_i meet at angles bounded uniformly away from 0 and the probabilities $p_k(\vec{z})$ are piecewise C^1 functions on the partition $\mathcal{P}$. We assume:

Condition (A).

$$\max_{1 \leq i \leq q} \sum_{k=1}^{K} p_k(\vec{z}) \|D\tau_{k,i}^{-1}(\tau_{k,i}(\vec{z}))\| < c < 1,$$

where $D\tau_{k,i}^{-1}(\vec{z})$ is the derivative matrix of $\tau_{k,i}^{-1}$ at $\vec{z}$.

Using the multidimensional notion of variation [**G**]:

$$V(f) = \int_{\mathbb{R}^N} \|Df\| d\lambda_N = \sup\{\int_{\mathbb{R}^N} f \operatorname{div}(g) d\lambda_N : g = (g_1, \dots, g_N) \in C_0^1(\mathbb{R}^N, \mathbb{R}^N) \text{ and } |g(x)| \leq 1 \text{ for } x \in \mathbb{R}^N\}$$

where $f \in L_1(\mathbb{R}^N)$ has bounded support, Df denotes the gradient of f in the distributional sense, $\operatorname{div}(g) = \nabla \cdot g = \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \cdots + \frac{\partial g_N}{\partial x_N}$ is the divergence operator, and $C_0^1(\mathbb{R}^N, \mathbb{R}^N)$ is the space of continuously differentiable functions from $\mathbb{R}^N$ into $\mathbb{R}^N$ having compact support. Consider the Banach space [**G**, Remark 1.12],

$$BV(S_\beta) = \{f \in L_1(S_\beta) : V(f) < +\infty\},$$

with the norm $\|f\|_{BV} = \|f\|_{L_1} + V(f)$.

Fix $1 \leq i \leq q$. Let F denote the set of singular points of ∂S_i. At any $x \in F$ we construct the largest cone having a vertex at x and which lies completely in S_i. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\gamma(S_i) = \min_{x \in F} \theta(x).$$

Since the faces of ∂S_i meet at angles bounded uniformly away from 0, $\gamma(S_i) > 0$. Let $\alpha(S_i) = \pi/2 + \gamma(S_i)$ and $a(S_i) = |\cos(\alpha(S_i))|$.

Now we start at points $y \in F$, where the minimal angle $\gamma(S_i)$ is attained, defining L_y to be central rays of the largest regular cones contained in S_i. Then we extend this field of segments to a C^1 field of segments $L_y, y \in \partial S_i$, every L_y being a central ray of a regular cone contained in S_i, with angle subtended at the vertex

y greater than or equal to $\beta(S_i)$. We make L_y short enough to avoid overlapping. Let $\delta(y)$ be the length of $L_y, y \in \partial S_i$. By the compactness of ∂S_i we have

$$\delta(S_i) := \inf_{y \in \partial(S_i)} \delta(y) > 0.$$

Let $\vec{z}$ be a point in ∂S_i and $J_{k,i}$ the Jacobian of $\tau_{k|S_i}$ at $\vec{z}$.

We recall the following two theorems.

THEOREM 4.1 ([**BG**, Theorem 6.3]). *If R is a random map which satisfies Condition (A), then*

$$V(P_R f) \leq c(1 + 1/a)V(f) + (M + \frac{c}{a\delta})\|f\|_1 \quad \text{for all } f \in BV(S_\beta),$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0, \delta = \min\{\delta(S_i) : i = 1 \ldots, q\} > 0, M_{k,i} = \sup_{\vec{z} \in S_i}(Dp_k(\vec{z}) - \frac{DJ_{k,i}}{J_{k,i}} p_k(\vec{z}))$ *and* $M = \sum_{k=1}^{K} \max_{1 \leq i \leq q} M_{k,i}$.

THEOREM 4.2 ([**BG**, Theorem 6.4]). *If R is a random map which satisfies Condition (A) and $c(1 + 1/a) < 1$, then the random map R preserves a measure which is absolutely continuous with respect to Lebesgue measure. Furthermore, the associated random Perron Frobenius operator P_R is quasi compact.*

Now, let R be a random map which is given by $\{\tau_1, \ldots, \tau_6; p_1(\vec{z}), \ldots, p_6(\vec{z})\}$ where

$$\tau_1(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\} \\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\} \\ \beta\vec{z}, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_2(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\} \\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\} \\ \beta\vec{z} - \vec{q}_1, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_3(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\} \\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\} \\ \beta\vec{z} - \vec{q}_2, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_4(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\} \\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\} \\ \beta\vec{z}, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_5(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\} \\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\} \\ \beta\vec{z} - \vec{q}_1, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_6(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\} \\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\} \\ \beta\vec{z} - \vec{q}_2, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

The probabilities are defined as follows.

$$\begin{aligned} p_1(\vec{z}) &= p \cdot s, & p_4(\vec{z}) &= (1 - p) \cdot s, \\ p_2(\vec{z}) &= p \cdot t, & p_5(\vec{z}) &= (1 - p) \cdot t, \\ p_3(\vec{z}) &= p \cdot (1 - s - t), & p_6(\vec{z}) &= (1 - p) \cdot (1 - s - t). \end{aligned} \tag{4.1}$$

We have Lemma 4.3.

LEMMA 4.3. *For any* $\vec{z} \in S_\beta$ *and* $n \in \mathbb{N}$, $\sum_{k_1k_2\cdots k_n \in \{1,\ldots,6\}^n} p_{k_1k_2\ldots k_n}(\vec{z}) = 1$.

PROOF. We prove this lemma by induction. For $n = 1$, $p_1(\vec{z}) + \cdots + p_6(\vec{z}) = 1$. Assume it is true for $n = m$, i.e. for any $\vec{z} \in S_\beta$,

$$\sum_{k_1k_2\cdots k_m \in \{1,\ldots,6\}^n} p_{k_1k_2\ldots k_m}(\vec{z}) = 1.$$

For $n = m + 1$,

$$\begin{aligned}
&\sum_{k_1k_2\cdots k_{m+1} \in \{1,\ldots,6\}^{m+1}} p_{k_1k_2\ldots k_{m+1}}(\vec{z}) \\
=&\sum_{k_1k_2\cdots k_{m+1} \in \{1,\ldots,6\}^{m+1}} p_{k_{m+1}}(\tau_{k_m} \circ \cdots \circ \tau_{k_1}(\vec{z})) \\
&\cdot p_{k_m}(\tau_{k_{m-1}} \circ \cdots \circ \tau_{k_1}(\vec{z})) \cdots p_{k_1}(\vec{z}) \\
=&p_1(\vec{z}) \sum_{k_2\cdots k_{m+1} \in \{1,\ldots,6\}^m} p_{k_2\ldots k_{m+1}}(\tau_1(\vec{z})) \\
&+ p_2(\vec{z}) \sum_{k_2\cdots k_{m+1} \in \{1,\ldots,6\}^m} p_{k_2\ldots k_{m+1}}(\tau_2(\vec{z})) \\
&+ \cdots + p_6(\vec{z}) \sum_{k_2\cdots k_{m+1} \in \{1,\ldots,6\}^m} p_{k_2\ldots k_{m+1}}(\tau_6(\vec{z})) \\
=&p_1(\vec{z}) + \cdots + p_6(\vec{z}) \\
=&1.
\end{aligned}$$

□

To prove the existence of an absolutely continuous invariant measure(acim), we would like to use Theorem 4.2. This cannot be done directly since R does not satisfy the hypothesis of the theorem, however a higher iterate of R does. For the convenience of the reader we supply a complete proof.

THEOREM 4.4. *Let* $R = \{\tau_1, \ldots, \tau_6; p_1(\vec{z}), \ldots, p_6(\vec{z})\}$*, then* R *admits an acim.*

PROOF. Denote the partition (2.1) by $\mathcal{P}$ with

$$S_1 = \overline{E}_0, S_2 = \overline{E}_1, S_3 = \overline{E}_2, S_4 = \overline{C}_{01}, S_5 = \overline{C}_{12}, S_6 = \overline{C}_{02}, S_7 = \overline{C}_{012}.$$

Consider the iteration of the random map, R^n, the corresponding partition is $\vee_{i=0}^{n-1} R^{-i}\mathcal{P}$, where

$$R^{-i}\mathcal{P} = \vee_{k_1k_2\cdots k_i \in \{1,\ldots,6\}^i} \tau^{-1}_{k_1k_2\ldots k_i}\mathcal{P}.$$

For a set $P_i \in \vee_{i=0}^{n-1} R^{-i}\mathcal{P}$ and a sequence $k_1 \ldots k_n \in \{1,\ldots,6\}^n$, let $\tau_{k_1\ldots k_n,i} = \tau_{k_1\ldots k_n}|_{P_i}$ and $M_{k_1\ldots k_n,i} = \sup_{\vec{z}\in P_i}(Dp_{k_1\ldots k_n}(\vec{z}) - \frac{DJ_{k_1\ldots k_n,i}}{J_{k_1\ldots k_n,i}} p_{k_1\ldots k_n,i}(\vec{z}))$, where $J_{k_1\ldots k_n,i}$ is the Jacobian of $\tau_{k_1\ldots k_n,i}$. Let

$$M_n = \sum_{k_1\ldots k_n \in \{0,1,2\}^n} \max_{P_i \in \vee_{i=0}^{n-1} R^{-i}\mathcal{P}} M_{k_1\ldots k_n,i} \quad \text{and} \quad \delta_n = \min_{P_i \in \vee_{i=0}^{n-1} R^{-i}\mathcal{P}} \delta(P_i).$$

For any set $P_i \in \vee_{i=0}^{n-1} R^{-i}\mathcal{P}$, the derivative matrix of $\tau^{-1}_{k_1k_2\ldots k_n}$ is equal to

$$\begin{bmatrix} \frac{1}{\beta^n} & 0 \\ 0 & \frac{1}{\beta^n} \end{bmatrix}.$$

Using Lemma 4.3 we have

$$\max_{P_i \in \vee_{i=0}^{n-1} R^{-i}\mathcal{P}} \sum_{k_1 k_2 \cdots k_n \in \{1,\ldots,6\}^n} p_{k_1 k_2 \ldots k_n}(\vec{z}) \| D(\tau_{k_1 k_2 \ldots k_n}|_{P_i})^{-1} \| = \frac{\sqrt{2}}{\beta^n} < \frac{2\sqrt{2}}{\beta^n} := c_n.$$

For the partition $\vee_{i=0}^{n-1} R^{-i}\mathcal{P}$, we have $a_n = \sqrt{2}/2$ (Here a_n refers to a in Theorem 4.1). Let

$$r_n = c_n(1 + \frac{1}{a_n}) = \frac{2\sqrt{2}+4}{\beta^n}, \quad R_n = M_n + \frac{c_n}{a_n \delta_n}.$$

We can find $l > \log(2\sqrt{2}+4)/\log\beta$ such that $r_l < 1$. Fix this l and let $C_1 = \max\{r_1, r_2, \ldots, r_{l-1}\}, C_2 = \max\{R_1, R_2, \ldots, R_{l-1}\}$. For any integer n, we have $n = jl + i$, where $0 \le i \le l-1$. Notice that $P_{R^n} = (P_{R^l})^j P_{R^i}$. Apply Theorem 4.1 on R^l, then we get

$$\begin{aligned}
V(P_{R^n} f) &= V P_{R^l}^j (P_{R^i} f) \\
&\le r_l \cdot V P_{R^l}^{j-1}(P_{R^i} f) + R_l \|f\|_1 \\
&\le r_l \cdot (r_l \cdot V P_{R^l}^{j-2}(P_{R^i} f) + R_l\|f\|_1) + R_l\|f\|_1 \\
&\cdots \\
&\le r_l^j V(P_{R^i} f) + (r_l^{j-1} + r_l^{j-2} + \cdots + r_l + 1) R_l \|f\|_1 \\
&\le r_l^j (C_1 V(f) + C_2\|f\|_1) + (r_l^{j-1} + r_l^{j-2} + \cdots + r_l + 1) R_l \|f\|_1 \\
&= C_1 r_l^j V(f) + (C_2 r_l^j + r_l^{j-1} + r_l^{j-2} + \cdots + r_l + 1) R_l \|f\|_1 \\
&\le C_1 r_l^j V(f) + (C_2 + \frac{1}{1-r_l}) R_l \|f\|_1.
\end{aligned}$$

By definition of the norm $\|\cdot\|_{BV}$,

$$\begin{aligned}
\|P_{R^n} f\|_{BV} &= \|P_{R^n} f\|_1 + V(P_{R^n} f) \\
&\le \|f\|_1 + C_1 r_l^j V(f) + (C_2 + \frac{1}{1-r_l}) R_l \|f\|_1.
\end{aligned}$$

Then the result follows by the technique in [**GB**, Theorem 1]. We write some details for completeness. From the above inequality it follows that the set $\{P_R^n \mathbf{1}\}_{n \ge l}$ is uniformly bounded, where $\mathbf{1}$ is the constant function equal to 1 on S_β. Hence P_R has a nontrivial fixed point $\mathbf{1}^*$ which is the density of an acim by the Kakutani–Yoshida Theorem (see [**K**, **Y**]). □

Step 2. For the skew product transformation R' on $S_\beta \times [0,1)$.

Let $(I, \mathcal{B}(I), \lambda_1)$ be the unit interval $I = [0,1)$, with $\mathcal{B}(I)$ the Borel σ-algebra on I and λ_1 being Lebesgue measure on $(I, \mathcal{B}(I))$. Let $Y = S_\beta \times I$ and the set J_k be given by $J_k = \{(\vec{z}, w) \colon \sum_{i<k} p_i(\vec{z}) \le w < \sum_{i \le k} p_i(\vec{z})\}$. Define maps $\varphi_k \colon J_k \to I$ by

$$\varphi_k(\vec{z}, w) = \frac{1}{p_k(\vec{z})} w - \frac{\sum_{r=1}^{k-1} p_r(\vec{z})}{p_k(\vec{z})}.$$

Define the skew product transformation $R' : S_\beta \times I \to S_\beta \times I$ by

$$R'(\vec{z}, w) = (\tau_k(\vec{z}), \varphi_k(\vec{z}, w))$$

for $(\vec{z}, w) \in J_k$.

Since $p_k(\vec{z})$ is defined as in (4.1), then we have

$$\varphi_1(\vec{z},w)=\frac{w}{ps},\qquad \varphi_4(\vec{z},w)=\frac{w-p}{(1-p)s},$$
$$\varphi_2(\vec{z},w)=\frac{w-ps}{pt},\qquad \varphi_5(\vec{z},w)=\frac{w-p-(1-p)s}{(1-p)t},$$
$$\varphi_3(\vec{z},w)=\frac{w-ps-pt}{p(1-s-t)},\quad \varphi_6(\vec{z},w)=\frac{w-p-(1-p)s-(1-p)t}{(1-p)(1-s-t)}.$$

We denote $p_k(\vec{z})$ and $\varphi_k(\vec{z},w)$ by p_k and $\varphi_k(w)$, respectively, since each $p_k(\vec{z})$ is a constant. Therefore,

$$R'(\vec{z},w)=\begin{cases}(\tau_1(\vec{z}),\varphi_1(w)), & \text{if } w\in[0,ps),\\ (\tau_2(\vec{z}),\varphi_2(w)), & \text{if } w\in[ps,ps+pt),\\ (\tau_3(\vec{z}),\varphi_3(w)), & \text{if } w\in[ps+pt,p),\\ (\tau_4(\vec{z}),\varphi_4(w)), & \text{if } w\in[p,p+(1-p)s),\\ (\tau_5(\vec{z}),\varphi_5(w)), & \text{if } w\in[p+(1-p)s,p+(1-p)s+(1-p)t),\\ (\tau_6(\vec{z}),\varphi_6(w)), & \text{if } w\in[p+(1-p)s+(1-p)t,1).\end{cases}$$

Denote by μ_β an acim for the position-dependent random transformation $R=\{\tau_1,\dots,\tau_6;p_1,\dots,p_6\}$, which means μ_β is R-invariant and absolutely continuous with respect to Lebesgue measure λ_2 in $\mathbb{R}^2$. We start by recalling [**BBQ**, Lemma 3.2].

LEMMA 4.5. *μ_β is invariant for the random map R if and only if $\mu_\beta\otimes\lambda_1$ is invariant for the skew product R'.*

STEP 3. For the skew product transformation R_β on $\Omega\times\Upsilon\times S_\beta$.

Define the *skew product transformation* R_β on $\Omega\times\Upsilon\times S_\beta$ as follows:

$$R_\beta(\omega,\upsilon,\vec{z})=\begin{cases}(\sigma\omega,\sigma'\upsilon,\beta\vec{z}-\vec{q}_i), & \text{if } \vec{z}\in E_i, i\in\{0,1,2\}\\ (\sigma\omega,\sigma'\upsilon,\beta\vec{z}-\vec{q}_i), & \text{if } \vec{z}\in C_{ij}, ij\in\{01,02,12\} \text{ and } \omega_1=0\\ (\sigma\omega,\sigma'\upsilon,\beta\vec{z}-\vec{q}_j), & \text{if } \vec{z}\in C_{ij}, ij\in\{01,02,12\} \text{ and } \omega_1=1\\ (\sigma\omega,\sigma'\upsilon,\beta\vec{z}-\vec{q}_i), & \text{if } \vec{z}\in C_{012}, \upsilon_1=i, i\in\{0,1,2\}.\end{cases}$$

LEMMA 4.6. *$(S_\beta\times I,\mathcal{S}\times\mathcal{B}(I),\mu_\beta\otimes\lambda_1,R')$ and $(\Omega\times\Upsilon\times S_\beta,\mathcal{A}\times\mathcal{B}\times\mathcal{S},m_1\otimes m_2\otimes\mu_\beta,R_\beta)$ are isomorphic.*

PROOF. Let $\pi_2: S_\beta\times I\to I$ be the canonical projection onto the second coordinate. Consider the map $\varphi=\pi_2\circ R'$ on $(I,\mathcal{B}(I),\lambda_1)$. One can see that $\varphi(w)=\varphi_k(w)$ for $w\in I_k$, where $I_1=[0,p_1)$ and $I_k=\left[\sum_{i=1}^{k-1}p_i,\sum_{i=1}^{k}p_i\right)$ for $2\le k\le 6$. Define

$$l(w)=\frac{1}{p_k}\quad\text{and}\quad h(w)=\frac{\sum_{i=1}^{k-1}p_i}{p_k}$$

for $w\in I_k$. It follows that $\varphi(w)=l(w)\cdot w-h(w)$. Let

$$l_n=l_n(w):=l(\varphi^{n-1}(w))\quad\text{and}\quad h_n=h_n(w):=h(\varphi^{n-1}(w)).$$

For $w\in[0,1)$, we can write the generalized Lüroth series (GLS) of w, which is

$$w=\frac{h_1}{l_1}+\frac{h_2}{l_1l_2}+\cdots+\frac{h_n}{l_1\cdots l_n}+\cdots.$$

Consider the system $\{\{0,1,2,3,4,5\}^{\mathbb{N}}, \mathcal{C}, m, \sigma''\}$, where $\mathcal{C}$ is the product σ-algebra, σ'' is the left shift and m is the product measure with weights $\{p_1, \dots, p_6\}$ as in (4.1). Let $\phi_1 : I \to \{0,1,2,3,4,5\}^{\mathbb{N}}$ be given by

$$\phi_1 : w = \sum_{n=1}^{\infty} \frac{h_i}{l_1 l_2 \cdots l_i} \mapsto (\gamma_1, \gamma_2, \dots,),$$

where $\gamma_n = \gamma_n(w), n \geq 1$ is defined as follows:

$$\gamma_n := \gamma_n(w) = k - 1 \iff \varphi^{n-1}(w) \in I_k,$$

for $k \in \{1,2,3,4,5,6\}$. It is known that φ preserves the Lebesgue measure λ_1 and ϕ_1 is an isomorphism between the two dynamical systems $(I, \mathcal{B}(I), \lambda_1, \varphi\}$ and $\{\{0,1,2,3,4,5\}^{\mathbb{N}}, \mathcal{C}, m, \sigma''\}$. See [**BBDK**] for more details.

Next we give a map ϕ_2 from $\{\{0,1,2,3,4,5\}^{\mathbb{N}}, \mathcal{C}, m, \sigma''\}$ to $\{\Omega \times \Upsilon, \mathcal{A} \times \mathcal{B}, m_1 \otimes m_2, \sigma \times \sigma'\}$. Let $h_1 : \{0,1,2,3,4,5\} \to \{0,1\}$ and $h_2 : \{0,1,2,3,4,5\} \to \{0,1,2\}$ be given by

$$h_1(x) = \begin{cases} 0, & \text{if } x = 0,1,2, \\ 1, & \text{if } x = 3,4,5, \end{cases} \quad h_2(x) = \begin{cases} 0, & \text{if } x = 0,3, \\ 1, & \text{if } x = 1,4, \\ 2, & \text{if } x = 2,5. \end{cases}$$

Define $\phi_2 : \{0,1,2,3,4,5\}^{\mathbb{N}} \to \Omega \times \Upsilon$ by $\phi_2(\gamma) = (\omega, \upsilon)$, where

$$\omega = (h_1(\gamma_1), h_1(\gamma_2), h_1(\gamma_3), \dots) := \tilde{h}_1(\gamma),$$
$$\upsilon = (h_2(\gamma_1), h_2(\gamma_2), h_2(\gamma_3), \dots) := \tilde{h}_2(\gamma).$$

One can see that ϕ_2 maps a cylinder of rank n in $\{0,1,2,3,4,5\}^{\mathbb{N}}$ to the product of two cylinders of the same rank n in $\Omega \times \Upsilon$. It follows that ϕ_2 is a bimeasurable bijection. From the definition of the product measure, we can get the measure preservingness on cylinders. Finally, it is easy to see that $\phi_2 \circ \sigma'' = (\sigma \times \sigma') \circ \phi_2$. Therefore, ϕ_2 is an isomorphism.

Now let $\phi : S_\beta \times I \to \Omega \times \Upsilon \times S_\beta$ be given by

$$\phi(\vec{z}, w) = (\tilde{h}_1(\phi_1(w)), \tilde{h}_2(\phi_1(w)), \vec{z}).$$

In fact, $\phi = \iota \circ (I_{S_\beta} \times (\phi_2 \circ \phi_1))$, where I_{S_β} is the identity map on S_β and $\iota(\vec{z}, \omega, \upsilon) = (\omega, \upsilon, \vec{z})$ is a transformation that only changes the order of coordinates. Since $\phi_2 \circ \phi_1$ preserves the dynamics of $\pi_2 \circ R$ and $\sigma \times \sigma'$, i.e.,

$$(\phi_2 \circ \phi_1) \circ (\pi_2 \circ R) = (\sigma \times \sigma') \circ (\phi_2 \circ \phi_1),$$

we have that $\phi \circ R' = R_\beta \circ \phi$. Therefore, the result follows. □

STEP 4. For the random transformation K_β on $\Omega \times \Upsilon \times S_\beta$.

Define a skew product transformation R_β as follows:

$$R_\beta(\omega, \upsilon, \vec{z}) = \begin{cases} (\sigma\omega, \sigma'\upsilon, \tau_1(\vec{z})), & \text{if } \omega_1 = 0, \upsilon_1 = 0, \\ (\sigma\omega, \sigma'\upsilon, \tau_2(\vec{z})), & \text{if } \omega_1 = 0, \upsilon_1 = 1, \\ (\sigma\omega, \sigma'\upsilon, \tau_3(\vec{z})), & \text{if } \omega_1 = 0, \upsilon_1 = 2, \\ (\sigma\omega, \sigma'\upsilon, \tau_4(\vec{z})), & \text{if } \omega_1 = 1, \upsilon_1 = 0, \\ (\sigma\omega, \sigma'\upsilon, \tau_5(\vec{z})), & \text{if } \omega_1 = 1, \upsilon_1 = 1, \\ (\sigma\omega, \sigma'\upsilon, \tau_6(\vec{z})), & \text{if } \omega_1 = 1, \upsilon_1 = 2, \end{cases}$$

Let μ be an arbitrary probability measure on S_β. We will show that any product measure of the form $m_1 \otimes m_2 \otimes \mu$ is K_β-invariant if and only if it is R_β-invariant.

LEMMA 4.7. *$m_1 \otimes m_2 \otimes \mu \circ K_\beta^{-1} = m_1 \otimes m_2 \otimes \mu \circ R_\beta^{-1} = m_1 \otimes m_2 \otimes \nu$, where*

$$\begin{aligned}\nu =& ps \cdot \mu \circ \tau_1^{-1} + pt \cdot \mu \circ \tau_2^{-1} + p(1-s-t) \cdot \mu \circ \tau_3^{-1} \\ &+ (1-p) \cdot s \cdot \mu \circ \tau_4^{-1} + (1-p) \cdot t \cdot \mu \circ \tau_5^{-1} + (1-p) \cdot (1-s-t) \cdot \mu \circ \tau_6^{-1}.\end{aligned}$$

PROOF. Denote by C_1 and C_2 arbitrary cylinders in Ω and Υ, respectively. Let S be a closed set in S_β. It suffices to verify that the measures coincide on sets of the form $C_1 \times C_2 \times S$, because the collection of these sets forms a generating π-system. Let $[i, C_1] = \{\omega_1 = i\} \cap \sigma^{-1}(C_1)$ for $i = 0, 1$ and $[i, C_2] = \{\upsilon_1 = i\} \cap (\sigma')^{-1}(C_2)$ for $i = 0, 1, 2$. Notice that

$$\begin{aligned}\tau_1^{-1}(S) \cap E = \tau_2^{-1}(S) \cap E = \cdots = \tau_6^{-1}(S) \cap E,\\ \tau_1^{-1}(S) \cap C = \tau_2^{-1}(S) \cap C = \tau_3^{-1}(S) \cap C,\\ \tau_4^{-1}(S) \cap C = \tau_5^{-1}(S) \cap C = \tau_6^{-1}(S) \cap C,\\ \tau_1^{-1}(S) \cap C_{012} = \tau_4^{-1}(S) \cap C_{012},\\ \tau_2^{-1}(S) \cap C_{012} = \tau_5^{-1}(S) \cap C_{012},\\ \tau_3^{-1}(S) \cap C_{012} = \tau_6^{-1}(S) \cap C_{012}.\end{aligned}$$

We can divide $K_\beta^{-1}(C_1 \times C_2 \times S)$ into the union of some disjoint sets as follows:

$$\begin{aligned}&K_\beta^{-1}(C_1 \times C_2 \times S)\\ &\quad= C_1 \times C_2 \times (\tau_1^{-1}(S) \cap E) \cup [0, C_1] \times C_2 \times (\tau_1^{-1}(S) \cap C)\\ &\quad\cup [1, C_1] \times C_2 \times (\tau_4^{-1}(S) \cap C) \cup C_1 \times [0, C_2] \times (\tau_1^{-1}(S) \cap C_{012})\\ &\quad\cup C_1 \times [1, C_2] \times (\tau_2^{-1}(S) \cap C_{012}) \cup C_1 \times [2, C_2] \times (\tau_3^{-1}(S) \cap C_{012})\end{aligned}$$

Hence,

$$\begin{aligned}&m_1 \otimes m_2 \otimes \mu \circ K_\beta^{-1}(C_1 \times C_2 \times S)\\ &= m_1(C_1)m_2(C_2)\mu(\tau_1^{-1}(S) \cap E)\\ &\quad + p \cdot m_1(C_1)m_2(C_2)\mu(\tau_1^{-1}(S) \cap C)\\ &\quad + (1-p) \cdot m_1(C_1)m_2(C_2)\mu(\tau_4^{-1}(S) \cap C)\\ &\quad + s \cdot m_1(C_1)m_2(C_2)\mu(\tau_1^{-1}(S) \cap C_{012})\\ &\quad + t \cdot m_1(C_1)m_2(C_2)\mu(\tau_2^{-1}(S) \cap C_{012})\\ &\quad + (1-s-t) \cdot m_1(C_1)m_2(C_2)\mu(\tau_3^{-1}(S) \cap C_{012})\\ &= ps \cdot m_1(C_1)m_2(C_2)\mu(\tau_1^{-1}(S))\\ &\quad + pt \cdot m_1(C_1)m_2(C_2)\mu(\tau_2^{-1}(S))\\ &\quad + p(1-s-t) \cdot m_1(C_1)m_2(C_2)\mu(\tau_3^{-1}(S))\\ &\quad + (1-p)s \cdot m_1(C_1)m_2(C_2)\mu(\tau_4^{-1}(S))\\ &\quad + (1-p)t \cdot m_1(C_1)m_2(C_2)\mu(\tau_5^{-1}(S))\\ &\quad + (1-p)(1-s-t) \cdot m_1(C_1)m_2(C_2)\mu(\tau_6^{-1}(S))\\ &= m_1 \otimes m_2 \otimes \nu(C_1 \times C_2 \times S).\end{aligned}$$

On the other hand,

$$\begin{aligned}&R_\beta^{-1}(C_1 \times C_2 \times S)\\&\quad= [0, C_1] \times [0, C_2] \times \tau_1^{-1}(S) \cup [0, C_1] \times [1, C_2] \times \tau_2^{-1}(S)\\&\qquad \cup [0, C_1] \times [2, C_2] \times \tau_3^{-1}(S) \cup [1, C_1] \times [0, C_2] \times \tau_4^{-1}(S)\\&\qquad \cup [1, C_1] \times [1, C_2] \times \tau_5^{-1}(S) \cup [1, C_1] \times [2, C_2] \times \tau_6^{-1}(S).\end{aligned}$$

Therefore, we complete the proof. □

Now we give the main result in this section.

THEOREM 4.8. *Let $\beta \in (1, 3/2)$. Then K_β has an invariant measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where μ_β is absolutely continuous with respect to λ_2.*

PROOF. By Theorem 4.4, Lemma 4.5, Lemma 4.6, and Lemma 4.7, we complete the proof. □

REMARK 4.9. When $\beta = 3/2$, $C_{012} = \{(\frac{2}{3}, \frac{2}{3})\}$ is a point. We modify the definition of R, R', R_β, and give relevant conclusions.

(i) Let $R = \{\tau_1, \tau_2; p_1(\vec{z}), p_2(\vec{z})\}$ be a position-dependent random transformation on S_β, where

$$\tau_1(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\},\\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\},\\ \beta\vec{z}, & \text{if } \vec{z} = (\frac{2}{3}, \frac{2}{3}),\end{cases}$$

$$\tau_2(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\},\\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\},\\ \beta\vec{z}, & \text{if } \vec{z} = (\frac{2}{3}, \frac{2}{3}),\end{cases}$$

and $p_1(\vec{z}) = p_2(\vec{z}) = 1/2$ for $\vec{z} \in S_\beta$. Similar to Theorem 4.4, it is not difficult to prove that R has an acim μ_β.

(ii) By [**BBQ**, Lemma 3.2], $\mu_\beta \otimes \lambda_1$ is invariant for the skew product R', where

$$R'(\vec{z}, w) = \begin{cases} (\tau_1(\vec{z}), \frac{w}{p}), & \text{if } w \in [0, p),\\ (\tau_2(\vec{z}), \frac{w-p}{1-p}), & \text{if } w \in [p, 1).\end{cases}$$

(iii) Define the *skew product transformation* R_β on $\Omega \times S_\beta$ as follows:

$$R_\beta(\omega, \vec{z}) = \begin{cases} (\sigma\omega, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in E_i, i \in \{0, 1, 2\},\\ (\sigma\omega, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\}, \omega_1 = 0,\\ (\sigma\omega, \beta\vec{z} - \vec{q}_j), & \text{if } \vec{z} \in C_{ij}, ij \in \{01, 02, 12\}, \omega_1 = 1,\\ (\sigma\omega, \beta\vec{z}), & \text{if } \vec{z} = (\frac{2}{3}, \frac{2}{3}).\end{cases}$$

Then we have that the dynamical systems $(S_\beta \times I, \mathcal{S} \times \mathcal{B}(I), \mu_\beta \otimes \lambda_1, R')$ and $(\Omega \times S_\beta, \mathcal{A} \times \mathcal{S}, m_1 \otimes \mu_\beta, R_\beta)$ are isomorphic. The proof is similar and easier than that of Lemma 4.6.

(iv) Let μ be an arbitrary probability measure on S_β and let $\tilde{K}_\beta : \Omega \times S_\beta \to \Omega \times S_\beta$ be given by

$$\tilde{K}_\beta(\omega, \vec{z}) = \begin{cases} (\omega, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in E_i, i \in \{0,1,2\} \\ (\sigma\omega, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in C_{ij}, ij \in \{01,02,12\}, \omega_1 = 0 \\ (\sigma\omega, \beta\vec{z} - \vec{q}_j), & \text{if } \vec{z} \in C_{ij}, ij \in \{01,02,12\}, \omega_1 = 1 \\ (\omega, \beta\vec{z}), & \text{if } \vec{z} = (\frac{2}{3}, \frac{2}{3}). \end{cases}$$

It is easy to check that

$$m_1 \otimes \mu \circ \tilde{K}_\beta^{-1} = m_1 \otimes \mu \circ R_\beta^{-1} = m_1 \otimes \nu,$$

where

$$\nu = p \cdot \mu \circ \tau_1^{-1} + (1-p) \cdot \mu \circ \tau_2^{-1}$$

by using the same method of calculation in Lemma 4.7. Therefore, it follows from (i)–(iv) that $\tilde{K}_\beta$ has an invariant measure of the form $m_1 \otimes \mu_\beta$.

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