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HAUSDORFF DIMENSION OF UNIVOQUE SETS OF SELF-SIMILAR SETS WITH COMPLETE OVERLAPS

MOHAMMAD GAREEB*
Department of Mathematics, East China Normal University
Shanghai 200062, P. R. China
52160601023@stu.ecnu.edu.cn; silver_man321@hotmail.com

WENXIA LI
Department of Mathematics, Shanghai Key Laboratory of PMMP
East China Normal University, Shanghai 200062, P. R. China
wxli@math.ecnu.edu.cn

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Abstract

Let $\lambda \in (0,1)$ and $m \geq 3$ an integer. We consider the collection $\mathcal{A}$ of homogeneous self-similar sets on the line such that every two of copies $f_i(K), f_j(K)$ of the self-similar set K are either separated or overlapped with rank k in $\{2,\ldots,m\}$. For $K \in \mathcal{A}$ generated by n similitudes, we denote by n_j the number of overlaps with rank $j \in \{2,\ldots,m\}$. The set of points in the self-similar set having a unique coding is called the univoque set and denoted by $\mathcal{U}$. In this paper, we investigate a uniform method to calculate the Hausdorff dimension of the set $\mathcal{U}$.

Keywords: Iterated Function System (IFS); Graph-Directed Self-Similar Set; Univoque Set; Configuration of Finite Pattern; Hausdorff Dimension.

*Corresponding author.

1. INTRODUCTION

Let $\{g_j\}_{j=1}^n$ be an iterated function system (IFS) of similitudes defined on $\mathbb{R}$ by

$$g_j(x) = r_j x + a_j,$$

where the similarity ratios r_j satisfy $0 < |r_j| < 1$, and $a_j \in \mathbb{R}$, $1 \le j \le n$. Hutchinson[1] proved that there exists a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^n g_j(K).$$

We call K the self-similar set or the attractor generated by the IFS $\{g_j\}_{j=1}^n$. For any $x \in K$, there exists at least one sequence $(i_k)_{k=1}^\infty \in \{1, \ldots, n\}^{\mathbb{N}}$ such that

$$x = \lim_{k\to\infty} g_{i_1} \circ \cdots \circ g_{i_k}(0) := \Pi((i_k)_{k=1}^\infty).$$

Thus, $\Pi : \{1, \ldots, n\}^{\mathbb{N}} \to K$ is surjective and continuous. We call such sequence a coding of x. A point $x \in K$ is called univoque point if its coding is unique. We denote by $\mathcal{U}$ the set of all univoque points in K. For the univoque set, in the setting of β-expansions, there are many results.[2–4] But there are few results in the setting of general self-similar sets.[5,6]

In this paper, we consider a class of overlapping self-similar sets as follows:

Fix an integer $m \ge 3$ and fix a $\lambda \in (0, 1)$. Let $\mathcal{A}$ be the collection of all self-similar sets K generated by the IFSs $\{f_i(x) = \lambda x + b_i\}_{i=1}^n$, where $n \ge 3$ and $b_i \in \mathbb{R}$ for every $1 \le i \le n$, satisfying the following conditions:

(I) $0 = b_1 < b_2 < \cdots < b_n = 1 - \lambda$;
(II) $f_i([0, 1]) \cap f_j([0, 1]) = \emptyset$ for any $1 \le i < j \le n$ with $j - i \ge 2$;
(III) There exist $i, j \in \{1, \ldots, n-1\}$ such that

$$f_i([0,1]) \cap f_{i+1}([0,1]) = \emptyset \quad \text{and}$$
$$f_j([0,1]) \cap f_{j+1}([0,1]) \ne \emptyset;$$

(IV) If $f_i([0, 1]) \cap f_{i+1}([0, 1]) \ne \emptyset$, then $|f_i([0, 1]) \cap f_{i+1}([0, 1])| = \lambda^j$ with $j \in \{2, 3, \ldots, m\}$, where $|\cdot|$ stands for the length of an interval.

The above conditions (I)–(IV) imply the fact: for a $K \in \mathcal{A}$, if $|f_i([0, 1]) \cap f_{i+1}([0, 1])| = \lambda^j$ with $j \ge 2$, then

$$K \cap (f_i([0,1]) \cap f_{i+1}([0,1])) = f_{in^{j-1}}(K) = f_{(i+1)1^{j-1}}(K).$$

This will be proved in Proposition 2.1. Thus we have $\mathcal{U} \cap (f_i([0, 1]) \cap f_{i+1}([0, 1])) = \emptyset$.

We now introduce some notations:

Let

$$n_j := \|\{1 \le i \le n-1 : |f_i([0,1]) \cap f_{i+1}([0,1])| = \lambda^j\}\|, \quad j = 2, 3, \ldots, m,$$

$$\Sigma := \sum_{j=2}^m n_j,$$

$$s_0 := \min\{1 \le i \le n-1 : f_i([0,1]) \cap f_{i+1}([0,1]) = \emptyset\},$$

$$t_0 := \max\{2 \le i \le n : f_{i-1}([0,1]) \cap f_i([0,1]) = \emptyset\}, \tag{1}$$

where $\|\cdot\|$ denotes the cardinality of a set. Thus, $n - \Sigma$ is just the number of the connected components of $\bigcup_{i=1}^n f_i([0, 1])$.

We classify the digit set $\{1, 2, \ldots, n\}$. For $k, j \in \{2, \ldots, m\}$ let

$$J_{kj} := \{1 \le i \le n : |f_i([0,1]) \cap f_{i-1}([0,1])| = \lambda^k \text{ and } |f_i([0,1]) \cap f_{i+1}([0,1])| = \lambda^j\},$$

$$J_{0j} := \{1 \le i \le n : f_i([0,1]) \cap f_{i-1}([0,1]) = \emptyset \text{ and } |f_i([0,1]) \cap f_{i+1}([0,1])| = \lambda^j\},$$

$$J_{k0} : Z = \{1 \le i \le n : |f_i([0,1]) \cap f_{i-1}([0,1])| = \lambda^k \text{ and } f_i([0,1]) \cap f_{i+1}([0,1]) = \emptyset\},$$

$$J_{00} := \{1 \le i \le n : f_i([0,1]) \cap f_{i-1}([0,1]) = f_i([0,1]) \cap f_{i+1}([0,1]) = \emptyset\}, \tag{2}$$

where we adopt the convention that $f_0([0, 1]) = f_{n+1}([0, 1]) = \emptyset$.

Thus we have

$$\{1, 2, \ldots, n\} = J_{00} \cup \bigcup_{2\le k,j\le m} (J_{kj} \cup J_{k0} \cup J_{0j})$$

with pairwise disjoint union.

It is easy to observe that for each $i \in J_{0j}$ with $j \in \{2, \ldots, m\}$ there exists a unique $i^* \in J_{k0}$ for some $k \in \{2, \ldots, m\}$ such that $i < i^*$ and $\bigcup_{l=i}^{i^*} f_l([0, 1])$ is a closed interval. We call i^* the dual of i.

Notice that

$$\sum_{j=2}^{m} \|J_{0j}\| = \sum_{k=2}^{m} \|J_{k0}\|,$$

$$\|J_{00}\| + \sum_{j=2}^{m} \|J_{0j}\| = n - \Sigma,$$

$$\sum_{j=0,2,3,\ldots,m} \sum_{k=2}^{m} \|J_{kj}\| = \Sigma, \tag{3}$$

and

$$\sum_{j=0,2,3,\ldots,m} \|J_{kj}\| = n_k, \quad \text{for each } k = 2, 3, \ldots, m. \tag{4}$$

In this paper, we give a formula for the Hausdorff dimension of the univoque set $\mathcal{U}$.

Theorem 1.1. *Let $K \in \mathcal{A}$. Then*

$$\dim_H \mathcal{U} = \frac{\log \gamma}{-\log \lambda},$$

and $\mathcal{H}^{\dim_H \mathcal{U}}(\mathcal{U}) > 0$, where γ is the largest positive root of the equation:

(I)

$$x^m - nx^{m-1} + 2n_2x^{m-2} + 2n_3x^{m-3} + 2n_4x^{m-4} + \cdots + 2n_{m-1}x + 2n_m = 0,$$

when $f_1([0,1]) \cap f_2([0,1]) = f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset$;

(II)

$$x^{m+t-1} - nx^{m+t-2} + (n_2x^{m-2} + n_3x^{m-3} + \cdots + n_m)(2x^{t-1} - 1) = 0,$$

when $|f_1([0,1]) \cap f_2([0,1])| = \lambda^t$ and $f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset$, or $|f_{n-1}([0,1]) \cap f_n([0,1])| = \lambda^t$ and $f_1([0,1]) \cap f_2([0,1]) = \emptyset$, for some $t \in \{2, 3, \ldots, m\}$;

(III)

$$x^m(x^{t+q-2} - 1) + nx^{m-1}(1 - x^{t+q-2}) + (n_2x^{m-2} + n_3x^{m-3} + \cdots + n_m)(2x^{t+q-2} - x^{q-1} - x^{t-1}) = 0,$$

when $|f_1([0,1]) \cap f_2([0,1])| = \lambda^t$ and $|f_{n-1}([0,1]) \cap f_n([0,1])| = \lambda^q$, or $|f_1([0,1]) \cap f_2([0,1])| = \lambda^q$ and $|f_{n-1}([0,1]) \cap f_n([0,1])| = \lambda^t$, for some $t, q \in \{2, 3, \ldots, m\}$.

The rest of this paper is arranged as follows. In Sec. 2, *we prove an important property of the collection $\mathcal{A}$ and introduce the concept of configuration. The proof of Theorem* 1.1 *is given in Sec.* 3.

2. PRELIMINARIES

In this section, we first give a property of the collection $\mathcal{A}$, and then introduce the concept of configuration set.[7]

Lemma 2.1 (Ref. 8). *The conditions* (I) *and* (IV) *imply that: If $|f_i([0,1]) \cap f_{i+1}([0,1])| = \lambda^j$ for some $1 \le i \le n-1$ and $j \ge 2$ an integer, then*

$$f_{in^{j-1}}(x) = f_{(i+1)1^{j-1}}(x).$$

Proof. In fact, we have $f_1(0) = 0$ and $f_n(1) = 1$ by (I). Thus

$$|f_i([0,1]) \cap f_{i+1}([0,1])| = |[f_{i+1}(0), f_i(1)]| = |[b_{i+1}, \lambda + b_i]| = \lambda + b_i - b_{i+1} = \lambda^j. \tag{5}$$

Let $f_{in^{j-1}}(x) = \lambda^j x + \alpha$ and let $f_{(i+1)1^{j-1}}(x) = \lambda^j x + \beta$. Then

$$\lambda^j + \alpha = f_{in^{j-1}}(1) = f_i(1) = \lambda + b_i$$

and

$$\beta = f_{(i+1)1^{j-1}}(0) = f_{i+1}(0) = b_{i+1}.$$

Hence, $\alpha = \beta$ by (5). □

Denote $Q_{i,i+1} = f_i([0,1]) \cap f_{i+1}([0,1])$. When $Q_{i,i+1}$ is not empty, we denote by $Q'_{i,i+1}$ the set obtained by deleting the right endpoint of $Q_{i,i+1}$, by $Q''_{i,i+1}$ the set obtained by deleting the left endpoint of $Q_{i,i+1}$. We have $Q'_{i,i+1} = Q''_{i,i+1} = \emptyset$ when $Q_{i,i+1} = \emptyset$.

Lemma 2.2. *Let $K \in \mathcal{A}$. Let $|f_i([0,1]) \cap f_{i+1}([0,1])| = |Q_{i,i+1}| = \lambda^{u+1}$ for some $u \in \mathbb{N}^+$. Then:*

(I) *If $(f_i(K) \cap Q_{i,i+1}) \backslash (f_{i+1}(K) \cap Q_{i,i+1}) \neq \emptyset$, then $(f_{n-1}(K) \cap Q_{n-1,n}) \backslash (f_n(K) \cap Q_{n-1,n}) \neq \emptyset$;*

(II) *If $(f_{i+1}(K) \cap Q_{i,i+1}) \backslash (f_i(K) \cap Q_{i,i+1}) \neq \emptyset$, then $(f_2(K) \cap Q_{1,2}) \backslash (f_1(K) \cap Q_{1,2}) \neq \emptyset$;*

(III) *Suppose that $|f_{n-1}([0,1]) \cap f_n([0,1])| = |Q_{n-1,n}| = \lambda^{l+1}$ with $l \in \mathbb{N}^+$. If $x \in (f_i(K) \backslash f_{i+1}(K)) \cap Q_{i,i+1}$, then x has a unique coding $in^{u-1}((n-1)n^{l-1})^\infty$;*

(IV) *Suppose that $|f_1([0,1]) \cap f_2([0,1])| = |Q_{1,2}| = \lambda^{h+1}$ with $h \in \mathbb{N}^+$. If $x \in (f_{i+1}(K) \backslash f_i(K)) \cap Q_{i,i+1}$, then x has a unique coding $(i+1)1^{u-1}(21^{h-1})^\infty$.*

Proof. (I) Take $x \in (f_i(K) \cap Q_{i,i+1}) \backslash (f_{i+1}(K) \cap Q_{i,i+1})$. Then the coding of x must begin with in^{u-1} and so $x = f_{in^{u-1}}(y)$ with $y \in f_n([0,1]) \cap K$. Since $y \notin f_n(K)$, we have $y \in (f_{n-1}(K) \cap Q_{n-1,n})$. Therefore,

$$(f_{n-1}(K) \cap Q_{n-1,n}) \backslash (f_n(K) \cap Q_{n-1,n}) \neq \emptyset.$$

(II) Take $x \in (f_{i+1}(K) \cap Q_{i,i+1}) \backslash (f_i(K) \cap Q_{i,i+1})$. Then the coding of x must begin with $(i+1)1^{u-1}$ and so $x = f_{(i+1)1^{u-1}}(y)$ with $y \in f_1([0,1]) \cap K$. Since $y \notin f_1(K)$, we have $y \in (f_2(K) \cap Q_{1,2})$. Therefore,

$$(f_2(K) \cap Q_{1,2}) \backslash (f_1(K) \cap Q_{1,2}) \neq \emptyset.$$

(III) Take $x \in (f_i(K) \cap Q_{i,i+1}) \backslash (f_{i+1}(K) \cap Q_{i,i+1})$. Then the coding of x must begin with in^{u-1} and so $x = f_{in^{u-1}}(y)$ with $y \in f_n([0,1]) \cap K$. Since $y \notin f_n(K)$, we have $y \in f_{n-1}(K) \cap Q_{n-1,n}$. Thus, the coding of y must begin with $(n-1)n^{l-1}$. Let $y = f_{(n-1)n^{l-1}}(z)$ with $z \in f_n([0,1]) \cap K$. Note that $z \notin f_n(K)$. We repeat the above process as done on y, we have $z = f_{(n-1)n^{l-1}}(w)$ with $w \in (f_n([0,1]) \cap K) \backslash f_n(K)$. Finally we have x has a unique coding $in^{u-1}((n-1)n^{l-1})^\infty$.

(IV) Take $x \in (f_{i+1}(K) \cap Q_{i,i+1}) \backslash (f_i(K) \cap Q_{i,i+1})$. Then the coding of x must begin with $(i+1)1^{u-1}$ and so $x = f_{(i+1)1^{u-1}}(y)$ with $y \in f_1([0,1]) \cap K$. Since $y \notin f_1(K)$, we have $y \in f_2(K) \cap Q_{1,2}$. Thus, the coding of y must begin with 21^{h-1}. Let $y = f_{21^{h-1}}(z)$ with $z \in f_1([0,1]) \cap K$. Note that $z \notin f_1(K)$. We repeat the above process as done on y, we have $z = f_{21^{h-1}}(w)$ with $w \in (f_1([0,1]) \cap K) \backslash f_1(K)$. Finally we have x has a unique coding $(i+1)1^{u-1}(21^{h-1})^\infty$. □

Corollary 2.1. *Let $K \in \mathcal{A}$. If $|f_i([0,1]) \cap f_{i+1}([0,1])| = \lambda^j$ with $j \geq 2$, then*

$$f_i(K) \cap Q_{i,i+1} = f_{i+1}(K) \cap Q_{i,i+1}.$$

Proof. Suppose that it is not true. Without loss of generality, assume that

$$(f_{i+1}(K) \cap Q_{i,i+1}) \backslash (f_i(K) \cap Q_{i,i+1}) \neq \emptyset.$$

From Lemma 2.2(II) and (IV) it follows that

$$(f_{i+1}(K) \cap Q_{i,i+1}) \backslash (f_i(K) \cap Q_{i,i+1})$$
$$= \{x\} \text{ and } x \text{ has a coding } (i+1)1^{j-2}(21^{h-1})^\infty.$$

Let $x_k = f_{(i+1)1^{j-2}(21^{h-1})^k}(f_1(1))$. Then $x = \lim_{k\to\infty} x_k$. Notice that

$$\begin{aligned} x_k &= f_{(i+1)1^{j-2}(21^{h-1})^k}(f_1(1)) \\ &= f_{(i+1)1^{j-2}(21^{h-1})^{k-1}}(f_{21^h}(1)) \\ &= f_{(i+1)1^{j-2}(21^{h-1})^{k-1}}(f_{1n^h}(1)) \\ &= f_{(i+1)1^{j-2}(21^{h-1})^{k-1}}(f_1(1)) \\ &= x_{k-1} = \cdots = f_{(i+1)1^{j-2}}(f_1(1)) \\ &= f_{(i+1)1^{j-1}}(1), \end{aligned} \tag{6}$$

where $f_{(i+1)1^{j-1}}(1) = f_{in^{j-1}}(1) = f_i(1) \in f_i(K)$, leading to a contradiction. □

Proposition 2.1. *Let $K \in \mathcal{A}$. If $|f_i([0,1]) \cap f_{i+1}([0,1])| = \lambda^j$ with $j \geq 2$, then*

$$\begin{aligned} &K \cap (f_i([0,1]) \cap f_{i+1}([0,1])) \\ &\quad = f_{in^{j-1}}(K) = f_{(i+1)1^{j-1}}(K). \end{aligned}$$

Proof. The second equality is obtained by Lemma 2.1. From the proof of Lemma 2.1 it follows that

$$f_{(i+1)1^{j-1}}(x) = \lambda^j x + b_{i+1} \quad \text{and so}$$
$$f_{(i+1)1^{j-1}}([0,1]) = f_i([0,1]) \cap f_{i+1}([0,1]).$$

Thus

$$\begin{aligned} f_{(i+1)1^{j-1}}(K) &\subseteq K \cap (f_i([0,1]) \cap f_{i+1}([0,1])) \\ &= K \cap f_{(i+1)1^{j-1}}([0,1]). \end{aligned}$$

From Corollary 2.1 it follows that

$$\begin{aligned} &K \cap (f_i([0,1]) \cap f_{i+1}([0,1])) \\ &\quad = (f_i(K) \cap Q_{i,i+1}) \cup (f_{i+1}(K) \cap Q_{i,i+1}) \\ &\quad = f_i(K) \cap Q_{i,i+1} = f_{i+1}(K) \cap Q_{i,i+1}. \end{aligned} \tag{7}$$

Now take $x \in K \cap (f_i([0,1]) \cap f_{i+1}([0,1])) = f_i(K) \cap Q_{i,i+1}$. Then x has a coding begins with in^{j-2}. Let $x = f_{in^{j-2}}(y)$ with $y \in f_n([0,1])$ and $y \in K$. Thus

$$\begin{aligned} y &\in f_n([0,1]) \cap K \\ &= f_n([0,1]) \cap (f_{n-1}(K) \cup f_n(K)) \\ &= (f_{n-1}(K) \cap Q_{n-1,n}) \cup f_n(K) \\ &= (f_n(K) \cap Q_{n-1,n}) \cup f_n(K) \end{aligned} \tag{8}$$

by Corollary 2.1. Thus $x \in f_{in^{j-1}}(K)$. □

The key idea of this paper is the configuration set.[7]

Definition 2.1. Suppose (X,d) is a compact metric space. Let $|A|$ be the diameter of $A \subset X$, and $\text{dist}(A,B) = \inf_{x\in A, y\in B} d(x,y)$. We say that $(X, d, \{\mathcal{D}^k\}_k, \{\delta_k\}_k)$ (for simplicity we may replace $(X, d, \{\mathcal{D}^k\}_k, \{\delta_k\}_k)$ by X) is a configuration set if there exists a constant $c \geq 1$ such that $\{\delta_k\}_k$ is a decreasing sequence with $\lim_{k\to\infty} \delta_k = 0$, $\delta_{k+1} \geq c^{-1}\delta_k$ for all k, $\mathcal{D}^i$ consists of finitely many compact subsets of X for any $i \geq 0$ with $\mathcal{D}^0 = \{X\}$, and for any $A \in \mathcal{D}^k$,

$$c^{-1}\delta_k \leq |A| \leq c\delta_k,$$

and there exists some $\mathcal{F}(A) \subset \mathcal{D}^{k+1}$ satisfying

$$A = \bigcup_{B\in\mathcal{F}(A)} B \quad \text{and} \quad \operatorname{dist}(B, B') \geq c^{-1}\delta_k, \quad \forall\, B, \quad B' \in \mathcal{F}(A) \text{ with } B \neq B'.$$

Definition 2.2. Let $(X, d, \{\mathcal{D}^k\}_k, \{\delta_k\}_k)$ be a configuration set. We say that X is a configuration set of finite pattern if the following conditions are satisfied:

(1) $\delta_k = \lambda^k$ for some $\lambda \in (0, 1)$;
(2) there is a surjective label mapping $\ell \colon \bigcup_{k=0}^{\infty} \mathcal{D}^k \to \{1, 2, \ldots, m\}$ and a transition matrix $M = (a_{ij})_{m\times m}$ such that for any $1 \leq i, j \leq m$, any $k \geq 0$ and any $A \in \mathcal{D}^k$ with $\ell(A) = i$,

$$\|\{B \in \mathcal{F}(A) : \ell(B) = j\}\| = a_{ij}.$$

The following result was proved in Ref. 7.

Theorem 2.1. *Suppose that X is a configuration set of finite pattern. Let ρ be the spectral radius of the transition matrix M. Then*

$$\dim_H X = \dim_B X = s = \frac{\log \rho}{-\log \lambda},$$

and $\mathcal{H}^s(X) > 0$. Moreover, if the matrix M is irreducible, then

$$0 < \mathcal{H}^s(X) < \infty,$$

where $\mathcal{H}^s(X)$ is the s-dimensional Hausdorff measure of the set X.

3. PROOF OF THEOREM 1.1

Lemma 3.1. *Suppose that $|f_1([0,1]) \cap f_2([0,1])| = \lambda^t$ and $|f_{n-1}([0,1]) \cap f_n([0,1])| = \lambda^q$, or $|f_1([0,1]) \cap f_2([0,1])| = \lambda^q$ and $|f_{n-1}([0,1]) \cap f_n([0,1])| = \lambda^t$ for some $t, q \in \{2, 3, \ldots, m\}$, then*

$$\dim_H \mathcal{U} = \frac{\log \gamma}{-\log \lambda},$$

where γ is the largest positive root of the equation

$$x^m(x^{t+q-2} - 1) + nx^{m-1}(1 - x^{t+q-2}) + (n_2x^{m-2} + n_3x^{m-3} + \cdots + n_m)(2x^{t+q-2} - x^{q-1} - x^{t-1}) = 0.$$

Proof. In the following we only consider the case $|f_1([0,1]) \cap f_2([0,1])| = \lambda^t$ and $|f_{n-1}([0,1]) \cap f_n([0,1])| = \lambda^q$ for some $t, q \in \{2, 3, \ldots, m\}$. Thus $1 \in J_{0t}$ and $n \in J_{q0}$. Without loss of generality we assume that $t \leq q$.

The proof of this lemma is arranged as follows:

- **Construction of sets $\{H_i\}_{i=1}^{n+2m-4}$:** We construct sets $\{H_i\}_{i=1}^{n+2m-4}$ on the intervals $\{[f_i(0), f_i(1)]\}_{i=1}^{n}$.
- **Graph-directed self-similar set structure:** We show that there are non-empty compact sets $\{E_i\}_{i=1}^{n+2m-4}$ such that $E_i \subseteq H_i$ for every $1 \leq i \leq n + 2m - 4$ and the set $K^* := \bigcup_{i=1}^{n+2m-4} E_i$ is a graph-directed self-similar set. Then $\mathcal{U} = K^*$ except for a countable set, hence $\dim_H \mathcal{U} = \dim_H K^*$.
- **Decomposition of the set K^*:** We decompose the set K^* into some groups and find the relation between this groups.
- **K^* has a configuration structure:** We define a label mapping ℓ and show that K^* has a configuration structure of finite pattern. □

Construction of sets $\{H_i\}_{i=1}^{n+2m-4}$:

For the first interval $[f_1(0), f_1(1)]$, we insert the points $f_{1^k}(1), k = 2, \ldots, m-1$ to get $m-1$ number of sub-intervals as follows:

$$[f_1(0), f_{1^{m-1}}(1)]\backslash f_{1^{m-2}}(Q''_{1,2}),$$
$$[f_{1^{m-k+1}}(1), f_{1^{m-k}}(1)]\backslash f_{1^{m-k-1}}(Q''_{1,2})$$
$$\text{for } k = 2, \ldots, m-1.$$

We label them as $1, \ldots, m-1$ from the left to the right order, i.e.

$$H_1 = [f_1(0), f_{1^{m-1}}(1)]\backslash f_{1^{m-2}}(Q''_{1,2}),$$
$$H_k = [f_{1^{m-k+1}}(1), f_{1^{m-k}}(1)]\backslash f_{1^{m-k-1}}(Q''_{1,2}),$$

for $k = 2, \ldots, m-1$.

For each of the middle $n-2$ intervals $f_k([0,1])$, $k = 2, \ldots, n-1$, we remove the intersections if there exist to get a new interval:

$$f_k([0,1])\backslash(Q'_{k-1,k} \cup Q''_{k,k+1}), \quad k = 2, \ldots, n-1.$$

We label them as $m, m+1, \ldots, m+n-3$ from the left to the right order, i.e.

$$H_{m+k-2} = f_k([0,1])\backslash(Q'_{k-1,k} \cup Q''_{k,k+1}), \quad k = 2, \ldots, n-1. \tag{9}$$

For the last interval $[f_n(0), f_n(1)]$, we insert the points $f_{n^k}(0), k = 2, \ldots, m-1$ to get $m-1$ number of sub-intervals as follows:

$$[f_{n^k}(0), f_{n^{k+1}}(0)]\backslash f_{n^{k-1}}(Q'_{n-1,n}) \quad \text{for} \quad k = 1, \dots, m-2,$$
$$[f_{n^{m-1}}(0), f_n(1)]\backslash f_{n^{m-2}}(Q'_{n-1,n}).$$

We label them as $n+m-2, n+m-1, \dots, n+2m-4$ from the left to the right order, i.e.

$$H_{n+m+k-3} = [f_{n^k}(0), f_{n^{k+1}}(0)]\backslash f_{n^{k-1}}(Q'_{n-1,n}) \quad \text{for} \quad k = 1, \dots, m-2,$$
$$H_{n+2m-4} = [f_{n^{m-1}}(0), f_n(1)]\backslash f_{n^{m-2}}(Q'_{n-1,n}),$$

with the convention that f_{1^0} and f_{n^0} are the identity.

Graph-directed self-similar set structure: Note that in (9), for $i = 2, \dots, n-1$

$$\begin{aligned} H_{m+i-2} &= f_i([0,1])\backslash(Q'_{i-1,i} \cup Q''_{i,i+1}) \\ &= \begin{cases} [f_i(0), f_i(1)] \quad \text{if } i \in J_{00}, \\ [f_i(0), f_{in^{j-1}}(0)] \\ \qquad \text{if } i \in J_{0j}\backslash\{1\};\ 2 \le j \le m, \\ [f_{i1^{k-1}}(1), f_i(1)] \\ \qquad \text{if } i \in J_{k0}\backslash\{n\};\ 2 \le k \le m, \\ [f_{i1^{k-1}}(1), f_{in^{j-1}}(0)] \\ \qquad \text{if } i \in J_{kj};\ 2 \le k,\ j \le m. \end{cases} \end{aligned}$$

Then we have

$$H_{m+i-2} \supseteq \begin{cases} f_i\left(\bigcup_{l=1}^{n+2m-4} H_l\right) \quad \text{if } i \in J_{00}, \\ f_i\left(\bigcup_{l=1}^{n+m+j-5} H_l\right) \\ \qquad \text{if } i \in J_{0j}\backslash\{1\};\ 2 \le j \le m, \\ f_i\left(\bigcup_{l=m-k+2}^{n+2m-4} H_l\right) \\ \qquad \text{if } i \in J_{k0}\backslash\{n\};\ 2 \le k \le m, \\ f_i\left(\bigcup_{l=m-k+2}^{n+m+j-5} H_l\right) \\ \qquad \text{if } i \in J_{kj};\ 2 \le k,\ j \le m, \end{cases} \tag{10}$$

and

$$\begin{cases} H_1 \supseteq f_1(H_1 \cup H_2), \\ H_k \supseteq f_1(H_{k+1}); \\ \qquad k = 2, \dots, m-2, \\ H_{m-1} \supseteq f_1\left(\bigcup_{l=m}^{n+m+t-5} H_l\right), \\ H_{n+m-2} \supseteq f_n\left(\bigcup_{l=m-q+2}^{n+m-3} H_l\right), \\ H_{n+m+k-3} \supseteq f_n(H_{n+m+k-4}); \\ \qquad k = 2, \dots, m-2, \\ H_{n+2m-4} \supseteq f_n(H_{n+2m-5} \cup H_{n+2m-4}). \end{cases} \tag{11}$$

Hence, from (10) and (11) we conclude that there are non-empty compact sets $E_i \subseteq H_i, 1 \le i \le n+2m-4$, i.e. a graph-directed self-similar set, satisfying

$$E_{m+i-2} = \begin{cases} f_i\left(\bigcup_{l=1}^{n+2m-4} E_l\right) \quad \text{if } i \in J_{00}, \\ f_i\left(\bigcup_{l=1}^{n+m+j-5} E_l\right) \\ \qquad \text{if } i \in J_{0j}\backslash\{1\};\ 2 \le j \le m, \\ f_i\left(\bigcup_{l=m-k+2}^{n+2m-4} E_l\right) \\ \qquad \text{if } i \in J_{k0}\backslash\{n\};\ 2 \le k \le m, \\ f_i\left(\bigcup_{l=m-k+2}^{n+m+j-5} E_l\right) \\ \qquad \text{if } i \in J_{kj};\ 2 \le k,\ j \le m, \end{cases} \tag{12}$$

and

$$\begin{cases} E_1 = f_1(E_1 \cup E_2), \\ E_k = f_1(E_{k+1}); \\ \qquad k = 2, \dots, m-2, \\ E_{m-1} = f_1\left(\bigcup_{l=m}^{n+m+t-5} E_l\right), \\ E_{n+m-2} = f_n\left(\bigcup_{l=m-q+2}^{n+m-3} E_l\right), \\ E_{n+m+k-3} = f_n(E_{n+m+k-4}); \\ \qquad k = 2, \dots, m-2, \\ E_{n+2m-4} = f_n(E_{n+2m-4} \cup E_{n+2m-5}). \end{cases} \tag{13}$$

Let $K^* = \bigcup_{i=1}^{n+2m-4} E_i \subset K$. From Proposition 2.1 it follows that

$$\mathcal{U} \cap \left(\bigcup_{i=1}^{n-1} Q_{i,i+1} \cup \bigcup_{k=1}^{m-2} f_{1^k}(Q_{1,2}) \cup \bigcup_{k=1}^{m-2} f_{n^k}(Q_{n-1,n}) \right) = \emptyset.$$

This means $\mathcal{U} \subseteq K^*$. In fact, we have $K^*\backslash\mathcal{U}$ is countable. One can refer to Ref. 8 for more information.

Decomposition of the set K^*:

As we know, determining the Hausdorff dimension of K^* by a routine way requires calculating the spectral radius of a $(n+2m-4)\times(n+2m-4)$ incidence matrix. To reduce the size of this matrix, we group some parts of K^* to show that it is a configuration set of finite pattern defined in Definitions 2.1 and 2.2 (see also Ref. 7).

Note that

$$\bigcup_{l=1}^{m-1} E_l = f_1\left(\bigcup_{l=1}^{n+m+t-5} E_l\right),$$
$$\bigcup_{l=n+m-2}^{n+2m-4} E_l = f_n\left(\bigcup_{l=m-q+2}^{n+2m-4} E_l\right).$$

Denote

$$L = \bigcup_{l=1}^{m+s_0-2} E_l, \quad M = \bigcup_{l=m+s_0-1}^{m+t_0-3} E_l,$$
$$R = \bigcup_{l=m+t_0-2}^{n+2m-4} E_l, \quad A = \bigcup_{l=m}^{m+s_0-2} E_l, \tag{14}$$
$$B = \bigcup_{l=m+t_0-2}^{n+m-3} E_l,$$

and for $p = 2, 3, \ldots, m+1$,

$$L_p = \bigcup_{l=m-p+2}^{m-1} E_l, \quad R_p = \bigcup_{l=n+m-2}^{n+m+p-5} E_l, \tag{15}$$

with the convention that $L_2 := \emptyset, R_2 := \emptyset$. Hence $L_{m+1} \cup A = L, B \cup R_{m+1} = R$. Then we can rewrite (12) and (13) as

$$E_{m+i-2} = \begin{cases} f_i(L \cup M \cup R) \\ \quad \text{if } i \in J_{00}, \\ f_i(L \cup M \cup B \cup R_j) \\ \quad \text{if } i \in J_{0j}\backslash\{1\};\ \ j \in \{2, \ldots, m\}, \\ f_i(L_k \cup A \cup M \cup R) \\ \quad \text{if } i \in J_{k0}\backslash\{n\};\ \ k \in \{2, \ldots, m\}, \\ f_i(L_k \cup A \cup M \cup B \cup R_j) \\ \quad \text{if } i \in J_{kj};\ \ k,\ j \in \{2, \ldots, m\}. \end{cases} \tag{16}$$

$$\begin{cases} E_1 = f_{1^{m-1}}(L \cup M \cup B \cup R_t), \\ E_l = f_{1^{m-l}}(A \cup M \cup B \cup R_t); \\ \quad l = 2, 3, \ldots, m-1, \\ E_{n+2m-4} = f_{n^{m-1}}(L_q \cup A \cup M \cup R), \\ E_{n+m+l-3} = f_{n^l}(L_q \cup A \cup M \cup B); \\ \quad l = 1, 2, \ldots, m-2. \end{cases} \tag{17}$$

Now we show the relation between the groups of K^*:

Statement 1. We have (I)

$$A = \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{3,\ldots,s_0\}} \{f_{(i-1)}(B \cup R_k) \cup f_i(L_k \cup A \cup M)\} \cup \{f_2(L_t \cup A \cup M) \cup f_{s_0}(R)\};$$

$$B = \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n-1\}} \{f_{(i-1)}(B \cup R_k) \cup f_i(L_k \cup A \cup M)\} \cup \{f_{t_0}(L \cup M) \cup f_{(n-1)}(B \cup R_q)\};$$

(II) $L_p = f_1(A \cup M \cup B \cup R_t) \cup f_1(L_{p-1})$, $p = 3, 4, \ldots, m$, while $L_{m+1} = f_1(L \cup M \cup B \cup R_t)$;
(III) $R_p = f_n(L_q \cup A \cup M \cup B) \cup f_n(R_{p-1})$, $p = 3, 4, \ldots, m$, while $R_{m+1} = f_n(L_q \cup A \cup M \cup R)$;
(IV) $L = f_1(L \cup M \cup B \cup R_t) \cup A$;
(V) $R = B \cup f_n(L_q \cup A \cup M \cup R)$;
(VI)

$$M = \bigcup_{i \in J_{00}} f_i(L \cup M \cup R) \cup \bigcup_{j=2}^{m} \bigcup_{i \in J_{0j} \cap \{s_0+1,\ldots,t_0-2\}} \{f_i(L \cup M) \cup f_i^*(R)\}$$

$$\cup \bigcup_{j=0,2,\ldots,m} \bigcup_{k=2}^{m} \bigcup_{i\in J_{kj}\cap\{s_0+2,\ldots,t_0-1\}} \{f_{(i-1)}(B\cup R_k) \cup f_i(L_k\cup A\cup M)\},$$

where we adopt the convention $\{i,\ldots,j\}=\emptyset$ when $i>j$.

Proof of Statement 1. (I) By (14), (16) we have

$$A = \bigcup_{l=m}^{m+s_0-2} E_l = \bigcup_{l=m}^{m+s_0-3} E_l \cup E_{m+s_0-2} = \bigcup_{k=2}^{m}\bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{2,\ldots,s_0-1\}} f_i(L_k\cup A\cup M\cup B\cup R_j)$$
$$\cup \bigcup_{k=2}^{m} \bigcup_{i\in J_{k0}\cap\{s_0\}} f_i(L_k\cup A\cup M\cup R)$$
$$= \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{3,\ldots,s_0\}} \{f_{(i-1)}(B\cup R_k) \cup f_i(L_k\cup A\cup M)\} \cup \{f_2(L_t\cup A\cup M)\cup f_{s_0}(R)\},$$

$$B = \bigcup_{l=m+t_0-2}^{n+m-3} E_l = E_{m+t_0-2} \cup \bigcup_{l=m+t_0-1}^{n+m-3} E_l$$
$$= \bigcup_{j=2}^{m} \bigcup_{i\in J_{0j}\cap\{t_0\}} f_i(L\cup M\cup B\cup R_j) \cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n-1\}} f_i(L_k\cup A\cup M\cup B\cup R_j)$$
$$= \bigcup_{k=2}^{m}\bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n-1\}} \{f_{(i-1)}(B\cup R_k) \cup f_i(L_k\cup A\cup M)\} \cup \{f_{t_0}(L\cup M)\cup f_{(n-1)}(B\cup R_q)\}.$$

By (13)–(15) and (17) we have

$$L_p = \bigcup_{l=m-p+2}^{m-1} E_l = E_{m-1} \cup \bigcup_{l=m-p+2}^{m-2} E_l$$
$$= f_1(A\cup M\cup B\cup R_t) \cup f_1\left(\bigcup_{l=m-p+3}^{m-1} E_l\right)$$
$$= f_1(A\cup M\cup B\cup R_t) \cup f_1(L_{p-1}), \quad \text{for } p=3,\ldots,m,$$

$$L_{m+1} = \bigcup_{l=1}^{m-1} E_l = f_1\left(\bigcup_{l=1}^{m+s_0-2} E_l \cup \bigcup_{l=m+s_0-1}^{m+t_0-3} E_l \cup \bigcup_{l=m+t_0-2}^{n+m+t-5} E_l\right)$$
$$= f_1(L\cup M\cup B\cup R_t),$$

and

$$R_p = \bigcup_{l=n+m-2}^{n+m+p-5} E_l = E_{n+m-2} \cup \bigcup_{l=n+m-1}^{n+m+p-5} E_l = f_n(L_q\cup A\cup M\cup B)\cup f_n\left(\bigcup_{l=n+m-2}^{n+m+p-6} E_l\right)$$
$$= f_n(L_q\cup A\cup M\cup B) \cup f_n(R_{p-1}), \quad \text{for } p=3,\ldots,m,$$

$$R_{m+1} = \bigcup_{l=n+m-2}^{n+2m-4} E_l = f_n\left(\bigcup_{l=m-q+2}^{m-1} E_l \cup \bigcup_{l=m}^{m+s_0-2} E_l \cup \bigcup_{l=m+s_0-1}^{m+t_0-3} E_l \cup \bigcup_{l=m+t_0-2}^{n+2m-4} E_l\right)$$
$$= f_n(L_q\cup A\cup M\cup R),$$

which proves (II) and (III).

The proof of (IV) and (V) is direct from (II), (III) and the facts $L=L_{m+1}\cup A, R=B\cup R_{m+1}$.

(VI) By (14), (16) we have

$$M = \bigcup_{l=m+s_0-1}^{m+t_0-3}$$

$$
\begin{aligned}
E_l = & \bigcup_{i\in J_{00}} f_i(L\cup M\cup R)\cup \bigcup_{j=2}^{m} \bigcup_{i\in J_{0j}\cap\{s_0+1,\ldots,t_0-2\}} f_i(L\cup M\cup B\cup R_j) \\
& \cup \bigcup_{k=2}^{m} \bigcup_{i\in J_{k0}\cap\{s_0+2,\ldots,t_0-1\}} f_i(L_k\cup A\cup M\cup R) \\
& \cup \bigcup_{k=2}^{m}\bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{s_0+2,\ldots,t_0-2\}} f_i(L_k\cup A\cup M\cup B\cup R_j) \\
= & \bigcup_{i\in J_{00}} f_i(L\cup M\cup R)\cup \bigcup_{j=2}^{m} \bigcup_{i\in J_{0j}\cap\{s_0+1,\ldots,t_0-2\}} \{f_i(L\cup M)\cup f_{i^*}(R)\} \\
& \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{s_0+2,\ldots,t_0-1\}} \{f_i(L_k\cup A\cup M) \\
& \cup f_{(i-1)}(B\cup R_k)\}.
\end{aligned}
$$

K^* has a configuration structure:

Now we are ready to construct the collections $\{\mathcal{D}^k\}_{k\geq 0}$ and to establish a label mapping $\ell : \bigcup_{k\geq 0} \mathcal{D}^k \to \{1,2,\ldots,m+t+q-2\}$. We first define the label mapping ℓ on certain subsets of K^*, and then construct the collections $\{\mathcal{D}^k\}_{k\geq 0}$ according to ℓ. A compact subset $A\subseteq K^*$ is said to be of pattern k if $\ell(A)=k$.

Define the mapping ℓ as follows: for any $I, J\in \bigcup_{s=0}^{\infty}\{1,2,\ldots,n\}^s$ with same length

$$
\begin{cases}
\ell(f_I(L\cup M)\cup f_J(R)) = 1, \\
\ell(f_I(B\cup R_p)\cup f_J(L_p\cup A\cup M)) = p; \\
\qquad p = 2,\ldots,m, \\
\ell(f_I(L_{t-p+1}\cup A\cup M)\cup f_J(R)) = m+p; \\
\qquad p = 1,\ldots,t-1, \\
\ell(f_I(L\cup M)\cup f_J(B\cup R_{q-p+1})) = m+t+p-1; \\
\qquad p = 1,\ldots,q-1,
\end{cases} \tag{18}
$$

with the convention that f_I and f_J are the identity when $s=0$.

Let $\mathcal{D}^0 = \{K^*\}$ and so $\ell(K^*)=1$ by (14) and (18). We call K^* the 0-level pattern 1 set.

The set K^* can be decomposed into a union of disjoint subsets, i.e.

$$
\begin{aligned}
K^* = & \bigcup_{i\in J_{00}} f_i\left(\bigcup_{l=1}^{n+2m-4} E_l\right)\cup \bigcup_{j=2}^{m}\bigcup_{i\in J_{0j}} \left\{ f_i\left(\bigcup_{l=1}^{m+t_0-3} E_l\right) \right. \\
& \left. \cup f_{i^*}\left(\bigcup_{l=m+t_0-2}^{n+2m-4} E_l\right)\right\} \cup \bigcup_{k=2}^{m}\bigcup_{j=0,2,3,\ldots,m} \\
& \bigcup_{i\in J_{kj}} \left\{ f_{(i-1)}\left(\bigcup_{l=m+t_0-2}^{n+m+k-5} E_l\right)\cup f_i\left(\bigcup_{l=m-k+2}^{m+t_0-3} E_l\right)\right\} \\
= & \bigcup_{i\in J_{00}} \{f_i(L\cup M\cup R)\}\cup \bigcup_{j=2}^{m} \\
& \bigcup_{i\in J_{0j}} \{f_i(L\cup M)\cup f_{i^*}(R)\} \\
& \cup \bigcup_{k=2}^{m}\bigcup_{j=0,2,3,\ldots,m}\bigcup_{i\in J_{kj}} \{f_{(i-1)}(B\cup R_k) \\
& \cup f_i(L_k\cup A\cup M)\}.
\end{aligned}
$$

We take $\mathcal{D}^1$ to be the collection of sets in the braces. Hence, by (3), (4) and (18), $\mathcal{D}^1$ consists of $(n-\Sigma)$ number of 1-level pattern 1 sets and n_p number of 1-level pattern p sets for each $p\in\{2,3,\ldots,m\}$.

In the following statement we show that for each $1\leq k\leq m+t+q-2$, an s-level pattern k set can be decomposed into a disjoint union of certain $(s+1)$-level sets of patterns in $\{1,2,\ldots,m+t+q-2\}$.

Statement 2. (I) Each s-level pattern 1 set can be represented as a disjoint union of $(n-\Sigma)$ number of $(s+1)$-level pattern 1 sets and n_j number of $(s+1)$-level pattern j sets for each $j\in\{2,3,\ldots,m\}$;

(II) Each s-level pattern p with $p\in\{3,\ldots,m\}$ can be represented as a disjoint union of $(n-\Sigma-1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j\in\{2,3,\ldots,p-2,p,\ldots,m\}$ and $(n_{p-1}+1)$ number of $(s+1)$-level pattern $(p-1)$ sets;

(III) Each s-level pattern 2 can be represented as a disjoint union of $(n-\Sigma-2)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j\in\{2,3,\ldots,t-1,t+1,\ldots,q-1,q+1,\ldots,m\}$,$(n_t-1)$ number of $(s+1)$-level pattern t sets, (n_q-1) number of $(s+1)$-level pattern q sets, one $(s+1)$-level pattern $(m+1)$ set and one $(s+1)$-level pattern $(m+t)$ set;

(IV) Each s-level pattern $(m+p)$ set with $p \in \{1,2,\ldots,t-2\}$ can be represented as a disjoint union of $(n-\Sigma-1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2,3,\ldots,m\}$ and one $(s+1)$-level of pattern $(m+p+1)$ set;

(V) Each s-level pattern $(m+t-1)$ set can be represented as a disjoint union of $(n-\Sigma-1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2,\ldots,t-1,t+1,\ldots,m\}$, (n_t-1) number of $(s+1)$-level pattern t sets and one $(s+1)$-level pattern $(m+1)$ set;

(VI) Each s-level pattern $(m+p+t-1)$ set with $p \in \{1,2,\ldots,q-2\}$ can be represented as a disjoint union of $(n-\Sigma-1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2,3,\ldots,m\}$ and one $(s+1)$-level of pattern $(m+p+t)$ set;

(VII) Each s-level pattern $(m+t+q-2)$ set can be represented as a disjoint union of $(n-\Sigma-1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2,\ldots,q-1,q+1,\ldots,m\}$, (n_q-1) number of $(s+1)$-level pattern q sets and one $(s+1)$-level pattern $(m+t)$ set.

Proof of Statement 2. In the following proof we adopt the convention that $\{i,\ldots,j\}=\emptyset$ when $i<j$.

Let $I, J \in \bigcup_{s=0}^{\infty}\{1,2,\ldots,n\}^s$ with same length, then by Statement 1(I) and (IV) we have

$$\begin{aligned}
f_J(A) = {} & \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{3,\ldots,s_0\}} \{f_{J(i-1)}(B\cup R_k) \\
& \cup f_{Ji}(L_k \cup A \cup M)\} \\
& \cup \{f_{J2}(L_t \cup A \cup M) \cup f_{Js_0}(R)\},
\end{aligned}$$

$$\begin{aligned}
f_I(B) = {} & \bigcup_{k=2}^{m}\bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n-1\}} \{f_{I(i-1)}(B\cup R_k) \\
& \cup f_{Ii}(L_k\cup A\cup M)\} \\
& \cup \{f_{It_0}(L\cup M) \cup f_{I(n-1)}(B\cup R_t)\},
\end{aligned}$$

$$\begin{aligned}
f_J(M) = {} & \bigcup_{i\in J_{00}} f_{Ji}(L\cup M\cup R) \cup \bigcup_{j=2}^{m} \bigcup_{i\in J_{0j}\cap\{s_0+1,\ldots,t_0-2\}} \{f_{Ji}(L\cup M)\cup f_{Ji^*}(R)\} \\
& \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{s_0+2,\ldots,t_0-1\}} \{f_{J(i-1)}(B\cup R_k) \\
& \cup f_{Ji}(L_k\cup A\cup M)\}.
\end{aligned}$$

Hence, by (3), (4) and (18), $f_I(B)\cup f_J(A\cup M)$ can be represented as a disjoint union of $(n-\Sigma-2)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j\in\{2,3,\ldots,t-1,t+1,\ldots,q-1,q+1,\ldots,m\}$, (n_t-1) number of $(s+1)$-level pattern t sets, (n_q-1) number of $(s+1)$-level pattern q sets, one $(s+1)$-level pattern $(m+1)$ set and one $(s+1)$-level pattern $(m+t)$ set, which proves (III).

By Statement 1(IV) and (V), we have

$$\begin{aligned}
& f_I(L\cup M)\cup f_J(R) \\
&\quad = f_{I1}(L\cup M)\cup f_{I1}(B\cup R_t)\cup f_I(A)\cup f_I(M) \\
&\qquad \cup f_J(B)\cup f_{Jn}(L_q\cup A\cup M)\cup f_{Jn}(R) \\
&\quad = \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{3,\ldots,s_0\}} \{f_{I(i-1)}(B\cup R_k) \\
&\qquad \cup f_{Ii}(L_k\cup A\cup M)\} \\
&\qquad \cup \{f_{I2}(L_t\cup A\cup M)\cup f_{Is_0}(R)\} \\
&\qquad \cup f_{I1}(L\cup M)\cup f_{I1}(B\cup R_t)\cup f_I(M) \\
&\qquad \cup \{f_{Jt_0}(L\cup M)\cup f_{J(n-1)}(B\cup R_q)\} \\
&\qquad \cup \bigcup_{k=2}^{m}\bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n-1\}} \{f_{J(i-1)}(B\cup R_k) \\
&\qquad \cup f_{Ji}(L_k\cup A\cup M)\} \\
&\qquad \cup f_{Jn}(L_q\cup A\cup M)\cup f_{Jn}(R) \\
&\quad = \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{2,\ldots,s_0\}} \{f_{I(i-1)}(B\cup R_k) \\
&\qquad \cup f_{Ii}(L_k\cup A\cup M)\} \\
&\qquad \cup \{f_{I1}(L\cup M)\cup f_{Is_0}(R)\}
\end{aligned}$$

$$\cup f_I(M) \cup \{f_{Jt_0}(L \cup M) \cup f_{Jn}(R)\} \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n\}} \{f_{J(i-1)}(B \cup R_k) \cup f_{Ji}(L_k \cup A \cup M)\},$$

which, by the decomposition of $f_I(M)$ and Eqs. (3), (4) and (18), proves (I)

By Statement 1(II) and (III), we get

$$\begin{aligned}
&f_I(B \cup R_p) \cup f_J(L_p \cup A \cup M)\\
&= f_I(B) \cup f_{In}(L_q \cup A \cup M \cup B \cup R_{p-1})\\
&\quad \cup f_J(A) \cup f_J(M)\\
&\quad \cup f_{J1}(L_{p-1} \cup A \cup M \cup B \cup R_t)\\
&= \{f_{It_0}(L \cup M) \cup f_{I(n-1)}(B \cup R_q)\}\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n-1\}} \{f_{I(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ii}(L_k \cup A \cup M)\} \cup f_{In}(L_q \cup A \cup M)\\
&\quad \cup f_{In}(B \cup R_{p-1})\\
&\quad \cup \{f_{J2}(L_t \cup A \cup M) \cup f_{Js_0}(R)\}\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{3,\ldots,s_0\}} \{f_{J(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ji}(L_k \cup A \cup M)\}\\
&\quad \cup f_J(M) \cup f_{J1}(L_{p-1} \cup A \cup M) \cup f_{J1}(B \cup R_t)\\
&= \{f_{It_0}(L \cup M) \cup f_{Js_0}(R)\}\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n\}} \{f_{I(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ii}(L_k \cup A \cup M)\}\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{2,\ldots,s_0\}} \{f_{J(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ji}(L_k \cup A \cup M)\} \cup f_J(M) \cup \{f_{In}(B \cup R_{p-1})\\
&\quad \cup f_{J1}(L_{p-1} \cup A \cup M)\};\quad p = 3, 4, \ldots, m,
\end{aligned}$$

which, by the decomposition of $f_J(M)$ and Eqs. (3), (4) and (18), proves (II).

By Statement 1(II),

$$\begin{aligned}
&f_I(L_{t-p+1} \cup A \cup M) \cup f_J(R)\\
&= f_{I1}(L_{t-p} \cup A \cup M \cup B \cup R_t) \cup f_I(A)\\
&\quad \cup f_I(M) \cup f_J(B) \cup f_{Jn}(L_q \cup A \cup M \cup R)\\
&= f_{I1}(L_{t-p} \cup A \cup M) \cup f_{I1}(B \cup R_t)\\
&\quad \cup \{f_{I2}(L_t \cup A \cup M) \cup f_{Is_0}(R)\}\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{3,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ii}(L_k \cup A \cup M)\} \cup \{f_{Jt_0}(L \cup M)\\
&\quad \cup f_{J(n-1)}(B \cup R_q)\}\\
&\quad \cup f_{Jn}(L_q \cup A \cup M) \cup f_{Jn}(R)\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n-1\}} \{f_{J(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ji}(L_k \cup A \cup M)\}\\
&= \{f_{I1}(L_{t-p} \cup A \cup M) \cup f_{Is_0}(R)\} \cup f_I(M)\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{2,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ii}(L_k \cup A \cup M)\} \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m}\\
&\quad \bigcup_{i\in J_{kj}\cap\{t_0+1,\ldots,n\}} \{f_{J(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ji}(L_k \cup A \cup M)\} \cup \{f_{Jt_0}(L \cup M)\\
&\quad \cup f_{Jn}(R)\};\quad p = 1, 2, \ldots, t-2,
\end{aligned}$$

which, by the decomposition of $f_I(M)$ and Eqs. (3), (4) and (18), proves (IV).

By Statement 1(V),

$$\begin{aligned}
&f_I(A \cup M) \cup f_J(R)\\
&= f_I(A) \cup f_I(M) \cup f_J(B)\\
&\quad \cup f_{Jn}(L_q \cup A \cup M \cup R)\\
&= \{f_{I2}(L_t \cup A \cup M) \cup f_{Is_0}(R)\}\\
&\quad \cup f_I(M) \cup \{f_{Jt_0}(L \cup M) \cup f_{J(n-1)}(B \cup R_q)\}\\
&\quad \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i\in J_{kj}\cap\{3,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)\\
&\quad \cup f_{Ii}(L_k \cup A \cup M)\}
\end{aligned}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n-1\}} \{f_{J(i-1)}(B \cup R_k)$$

$$\cup f_{Ji}(L_k \cup A \cup M)\}$$

$$\cup f_{Jn}(L_q \cup A \cup M) \cup f_{Jn}(R)$$

$$= \{f_{I2}(L_t \cup A \cup M) \cup f_{Is_0}(R)\} \cup f_I(M)$$

$$\cup \{f_{Jt_0}(L \cup M) \cup f_{Jn}(R)\}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{3,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)$$

$$\cup f_{Ii}(L_k \cup A \cup M)\}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n\}} \{f_{J(i-1)}(B \cup R_k)$$

$$\cup f_{Ji}(L_k \cup A \cup M)\},$$

which, by the decomposition of $f_I(M)$ and Eqs. (3), (4) and (18), proves (V).

By Statement 1(III) and (IV), we get

$$f_I(L \cup M) \cup f_J(B \cup R_{q-p+1})$$

$$= f_{I1}(L \cup M \cup B \cup R_t) \cup f_I(A) \cup f_I(M)$$

$$\cup f_J(B) \cup f_{Jn}(L_q \cup A \cup M \cup B \cup R_{q-p})$$

$$= f_{I1}(L \cup M) \cup f_{I1}(B \cup R_t) \cup f_I(M)$$

$$\cup \{f_{I2}(L_t \cup A \cup M) \cup f_{Is_0}(R)\}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{3,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)$$

$$\cup f_{Ii}(L_k \cup A \cup M)\}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n-1\}} \{f_{J(i-1)}(B \cup R_k)$$

$$\cup f_{Ji}(L_k \cup A \cup M)\} \cup \{f_{Jt_0}(L \cup M)$$

$$\cup f_{J(n-1)}(B \cup R_q)\} \cup f_{Jn}(L_q \cup A \cup M)$$

$$\cup f_{Jn}(B \cup R_{q-p})$$

$$= \{f_{I1}(L \cup M) \cup f_{Is_0}(R)\} \cup f_I(M)$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{2,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)$$

$$\cup f_{Ii}(L_k \cup A \cup M)\} \cup$$

$$\bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n\}} \{f_{J(i-1)}(B \cup R_k)$$

$$\cup f_{Ji}(L_k \cup A \cup M)\} \cup \{f_{Jt_0}(L \cup M)$$

$$\cup f_{Jn}(B \cup R_{q-p})\}; \quad p = 1, 2, \ldots, t-2,$$

which, by the decomposition of $f_I(M)$ and Eqs. (3), (4) and (18), proves (VI).

By Statement 1(IV), we have

$$f_I(L \cup M) \cup f_J(B)$$

$$= f_{I1}(L \cup M \cup B \cup R_t) \cup f_I(A) \cup f_I(M) \cup f_J(B)$$

$$= f_{I1}(L \cup M) \cup f_{I1}(B \cup R_t)$$

$$\cup \{f_{I2}(L_t \cup A \cup M) \cup f_{Is_0}(R)\} \cup f_I(M)$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{3,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)$$

$$\cup f_{Ii}(L_k \cup A \cup M)\}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n-1\}} \{f_{J(i-1)}(B \cup R_k)$$

$$\cup f_{Ji}(L_k \cup A \cup M)\}$$

$$\cup \{f_{Jt_0}(L \cup M) \cup f_{J(n-1)}(B \cup R_q)\}$$

$$= \{f_{I1}(L \cup M) \cup f_{Is_0}(R)\} \cup \{f_{Jt_0}(L \cup M)$$

$$\cup f_{J(n-1)}(B \cup R_q)\} \cup f_I(M)$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,\ldots,m} \bigcup_{i \in J_{kj} \cap \{2,\ldots,s_0\}} \{f_{I(i-1)}(B \cup R_k)$$

$$\cup f_{Ii}(L_k \cup A \cup M)\}$$

$$\cup \bigcup_{k=2}^{m} \bigcup_{j=2}^{m} \bigcup_{i \in J_{kj} \cap \{t_0+1,\ldots,n-1\}} \{f_{J(i-1)}(B \cup R_k)$$

$$\cup f_{Ji}(L_k \cup A \cup M)\},$$

which, by the decomposition of $f_I(M)$ and Eqs. (3), (4) and (18), proves (VII). Thus according to above Statement 2 we can define $\mathcal{D}^k$ inductively.

Now we take $\delta_k = \lambda^k, \forall k \geq 1$. For $A \in \mathcal{D}^s; s \geq 0$ we have $c_1\lambda^s \leq |A| \leq n\lambda^s$, where $c_1 = \min\{\lambda, 1-\lambda\}$. On the other hand, for every $A \in \mathcal{D}^s; s \geq 0$ and $B, B' \in \mathcal{F}(A)$ with $B \neq B'$ we have

$$\mathrm{dist}(B, B') \geq c_2\lambda^s,$$

where $c_2 = \lambda \min\{\mathrm{dist}(f_i(1), f_{(i+1)}(0)); 1 \leq i \leq n-1, f_i([0,1]) \cap f_{(i+1)}([0,1]) = \emptyset\}$.

Therefore, K^* satisfies the conditions in Definitions (2.1) and (2.2) for $c = \max\{n, c_1^{-1}, c_2^{-1}\}$ and $\delta_k = \lambda^k; k \geq 1$.

From the above analysis we find that K^* has a configuration structure of $(m+t+q-2)$ patterns and the corresponding $(m+t+q-2)\times(m+t+q-2)$ matrix is

$$\begin{pmatrix}
n-\Sigma & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
n-\Sigma-2 & n_2 & \cdots & n_t-1 & n_{t+1} & \cdots & n_q-1 & n_{q+1} & \cdots & n_m & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
n-\Sigma-1 & n_2 & \cdots & n_t+1 & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q+1 & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & \cdots & n_t-1 & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1\\
n-\Sigma-1 & n_2 & \cdots & n_t & n_{t+1} & \cdots & n_q-1 & n_{q+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0
\end{pmatrix}. \tag{19}$$

The spectral radius of the above matrix is just the largest positive root of the equation:

$$\begin{aligned}
x^m(x^{t+q-2}-1) &+ nx^{m-1}(1-x^{t+q-2})\\
&+ (n_2x^{m-2}+n_3x^{m-3}+\cdots+n_m)\\
&\times (2x^{t+q-2}-x^{q-1}-x^{t-1}) = 0.
\end{aligned}$$

This finishes the proof of the lemma. □

Lemma 3.2. *Suppose that* $|f_1([0,1])\cap f_2([0,1])| = \lambda^t$ *and* $f_{n-1}([0,1])\cap f_n([0,1]) = \emptyset$, *or* $f_1([0,1])\cap f_2([0,1]) = \emptyset$ *and* $|f_{n-1}([0,1])\cap f_n([0,1])| = \lambda^t$ *for some* $t\in\{2,3,\ldots,m\}$, *then*

$$\dim_H \mathcal{U} = \frac{\log\gamma}{-\log\lambda},$$

where γ *is the largest positive root of the equation*

$$\begin{aligned}
x^{m+t-1} - nx^{m+t-2} &+ (n_2x^{m-2}\\
&+ n_3x^{m-3}+\cdots+n_m)(2x^{t-1}-1) = 0.
\end{aligned}$$

Proof. In the following we only consider the case $f_{n-1}([0,1])\cap f_n([0,1]) = \emptyset$ and $|f_1([0,1])\cap f_2([0,1])| = \lambda^t$ for some $t\in\{2,3,\ldots,m\}$. Thus $1\in J_{0t}$ and $n\in J_{00}$.

Since $f_{n-1}([0,1])\cap f_n([0,1]) = \emptyset$ then we have $t_0 = n,\ B = \emptyset,\ R_{m+1} = R$ and an s-level set of pattern 2 does not generate any $(s+1)$-level set of pattern $m+t$ for any $s\geq 0$, then we do not get any of the patterns $m+t, m+t+1,\ldots,m+t+q-2$.

Thus, the proof of this lemma is just a special case of the proof of Lemma 3.1. In this case we get $K^* = \bigcup_{i=1}^{n+2m-4} E_i \subset K$, where

$$E_{m+i-2} = \begin{cases}
f_i\left(\bigcup_{l=1}^{n+2m-4} E_l\right) & \text{if } i\in J_{00}\backslash\{n\},\\
f_i\left(\bigcup_{l=1}^{n+m+j-5} E_l\right) & \\
\quad \text{if } i\in J_{0j}\backslash\{1\};\quad 2\leq j\leq m, & \\
f_i\left(\bigcup_{l=m-k+2}^{n+2m-4} E_l\right) & \\
\quad \text{if } i\in J_{k0}\backslash;\quad 2\leq k\leq m, & \\
f_i\left(\bigcup_{l=m-k+2}^{n+m+j-5} E_l\right) & \\
\quad \text{if } i\in J_{kj};\quad 2\leq k,\ j\leq m, &
\end{cases} \tag{20}$$

and

$$\begin{cases}
E_1 = f_1(E_1\cup E_2),\\
E_k = f_1(E_{k+1});\quad k=2,\ldots,\ m-2,\\
E_{m-1} = f_1\left(\bigcup_{l=m}^{n+2m+t-5} E_l\right),\\
E_{n+m-2} = f_n\left(\bigcup_{l=1}^{n+m-3} E_l\right),\\
E_{n+m+k-3} = f_n(E_{n+m+k-4});\quad k=2,\ldots,\ m-2,\\
E_{n+2m-4} = f_n(E_{n+2m-4}\cup E_{n+2m-5}).
\end{cases} \tag{21}$$

We also have

$$\begin{aligned}
L &= \bigcup_{l=1}^{m+s_0-2} E_l, \quad M = \bigcup_{l=m+s_0-1}^{m+n-3} E_l,\\
R &= \bigcup_{l=m+n-2}^{n+2m-4} E_l, \quad A = \bigcup_{l=m}^{m+s_0-2} E_l,
\end{aligned} \tag{22}$$

and for $p=2,3,\ldots,m+1$,

$$L_p = \bigcup_{l=m-p+2}^{m-1} E_l, \quad R_p = \bigcup_{l=n+m-2}^{n+m+p-5} E_l, \tag{23}$$

with the convention that $L_2 := \emptyset, R_2 := \emptyset$. Hence $L_{m+1}\cup A = L, R_{m+1} = R$. Then we can rewrite

(20) and (21) as

$$E_{m+i-2} = \begin{cases} f_i(L \cup M \cup R) \quad \text{if } i \in J_{00}\backslash\{n\}, \\ f_i(L \cup M \cup R_j) \\ \qquad \text{if } i \in J_{0j}\backslash\{1\};\ \ j \in \{2, \dots, m\}, \\ f_i(L_k \cup A \cup M \cup R) \\ \qquad \text{if } i \in J_{k0};\ \ k \in \{2, \dots, m\}, \\ f_i(L_k \cup A \cup M \cup R_j) \\ \qquad \text{if } i \in J_{kj}; k, j \in \{2, \dots, m\}, \end{cases} \tag{24}$$

$$\begin{cases} E_1 = f_{1^{m-1}}(L \cup M \cup R_t), \\ E_l = f_{1^{m-l}}(A \cup M \cup R_t); \\ \qquad l = 2, 3, \dots, m-1, \\ E_{n+2m-4} = f_{n^{m-1}}(L \cup M \cup R), \\ E_{n+m+l-3} = f_{n^l}(L \cup M); \\ \qquad l = 1, 2, \dots, m-2. \end{cases} \tag{25}$$

Then we can define the label mapping as follows: for any $I, J \in \bigcup_{s=0}^{\infty}\{1, 2, \dots, n\}^s$ with same length

$$\begin{cases} \ell(f_I(L \cup M) \cup f_J(R)) = 1, \\ \ell(f_I(R_p) \cup f_J(L_p \cup A \cup M)) = p; \\ \qquad p = 2, \dots, m, \\ \ell(f_I(L_{t-p+1} \cup A \cup M) \cup f_J(R)) = m + p; \\ \qquad p = 1, \dots, t-1, \end{cases} \tag{26}$$

with the convention that f_I and f_J are the identity when $s = 0$.

Let $\mathcal{D}^0 = \{K^*\}$ and so $\ell(K^*) = 1$ by (22) and (26). We call K^* the 0-level pattern 1 set.

The set K^* can be decomposed into a union of disjoint subsets, i.e.

$$\begin{aligned} K^* = &\bigcup_{i \in J_{00}} f_i\left(\bigcup_{l=1}^{n+2m-4} E_l\right) \cup \bigcup_{j=2}^{m} \\ &\bigcup_{i \in J_{0j}} \left\{ f_i\left(\bigcup_{l=1}^{m+n-3} E_l\right) \cup f_{i^*}\left(\bigcup_{l=m+n-2}^{n+2m-4} E_l\right)\right\} \\ &\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,3,\dots,m} \bigcup_{i \in J_{kj}} \left\{ f_{(i-1)}\left(\bigcup_{l=m+n-2}^{n+m+k-5} E_l\right)\right. \\ &\left.\cup f_i\left(\bigcup_{l=m-k+2}^{m+n-3} E_l\right)\right\} \\ = &\bigcup_{i \in J_{00}} f_i\{(L \cup M \cup R)\} \\ &\cup \bigcup_{j=2}^{m} \bigcup_{i \in J_{0j}} \{f_i(L \cup M) \cup f_{i^*}(R)\} \\ &\cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,3,\dots,m} \bigcup_{i \in J_{kj}} \{f_{(i-1)}(R_k) \\ &\cup f_i(L_k \cup A \cup M)\}. \end{aligned}$$

We take $\mathcal{D}^1$ to be the collection of sets in the braces. Hence, by (3), (4) and (26), $\mathcal{D}^1$ consists of $(n - \Sigma)$ number of 1-level pattern 1 sets and n_p number of 1-level pattern p sets for each $p \in \{2, 3, \dots, m\}$. This way we can construct $\mathcal{D}^k, k \geq 0$ inductively.

Statement 2 can be reformulated as follows.

Statement 2. (I) Each s-level pattern 1 set can be represented as a disjoint union of $(n - \Sigma)$ number of $(s+1)$-level pattern 1 sets and n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \dots, m\}$;

(II) Each s-level pattern p with $p \in \{3, \dots, m\}$ can be represented as a disjoint union of $(n - \Sigma - 1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \dots, p-2, p, \dots, m\}$ and $(n_{p-1} + 1)$ number of $(s+1)$-level pattern $(p-1)$ sets;

(III) Each s-level pattern 2 can be represented as a disjoint union of $(n - \Sigma - 2)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \dots, t-1, t+1, \dots, m\}$, $(n_t - 1)$ number of $(s+1)$-level pattern t sets and one $(s+1)$-level pattern $(m+1)$ set;

(IV) Each s-level pattern $(m+p)$ set with $p \in \{1, 2, \dots, t-2\}$ can be represented as a disjoint union of $(n - \Sigma - 1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \dots, m\}$ and one $(s+1)$-level of pattern $(m+p+1)$ set;

(V) Each s-level pattern $(m+t-1)$ set can be represented as a disjoint union of $(n - \Sigma - 1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, \dots, t-1, t+1, \dots, m\}$, $(n_t - 1)$ number of $(s+1)$-level pattern t sets and one $(s+1)$-level pattern $(m+1)$ set.

Therefore, K^* has a configuration structure of $(m + t - 1)$ patterns and the corresponding

$(m+t-1)\times(m+t-1)$ matrix is

$$\begin{pmatrix}
n-\Sigma & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 0\\
n-\Sigma-2 & n_2 & n_3 & \cdots & n_t-1 & n_{t+1} & \cdots & n_m & 1 & 0 & \cdots & 0 & 0 & 0\\
n-\Sigma-1 & n_2+1 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 0\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t+1 & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 0\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 1 & \cdots & 0 & 0 & 0\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 1 & 0\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t & n_{t+1} & \cdots & n_m & 0 & 0 & \cdots & 0 & 0 & 1\\
n-\Sigma-1 & n_2 & n_3 & \cdots & n_t-1 & n_{t+1} & \cdots & n_m & 1 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}, \tag{27}$$

where the spectral radius is the largest positive root of the equation:

$$\begin{aligned} &x^{m+t-1} - nx^{m+t-2} \\ &\quad + (n_2x^{m-2} + n_3x^{m-3} + \cdots + n_m) \\ &\quad \times (2x^{t-1} - 1) = 0. \end{aligned}$$

This finishes the proof. □

Lemma 3.3. *Suppose that* $f_1([0,1]) \cap f_2([0,1]) = f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset$. *Then*

$$\dim_H \mathcal{U} = \frac{\log\gamma}{-\log\lambda},$$

where γ *is the largest positive root of the equation*

$$\begin{aligned} &x^m - nx^{m-1} + 2(n_2x^{m-2} + n_3x^{m-3} \\ &\quad + n_4x^{m-4} + \cdots + n_{m-1}x + n_m) = 0. \end{aligned}$$

Proof. Here we have $1, n \in J_{00}$. The proof is also a special case of the proof of Lemma 3.1. Since $f_1([0,1]) \cap f_2([0,1]) = f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset$ then we have $s_0 = 1, t_0 = n, A = B = \emptyset, L_{m+1} = L, R_{m+1} = R$ and an s-level set of pattern 2 does not generate any $(s+1)$-level set of pattern $m+1$ or any $(s+1)$-level set of pattern $m+t$ for any $s \geq 0$, then we do not get any of the patterns $m+1, m+2, \ldots, m+t-1, m+t, m+t+1, \ldots, m+t+q-2$.

In this case we get $K^* = \bigcup_{i=1}^{n+2m-4} E_i \subset K$, where

$$E_{m+i-2} = \begin{cases} f_i\left(\bigcup\limits_{l=1}^{n+2m-4} E_l\right) & \text{if } i \in J_{00}\backslash\{1,n\}, \\ f_i\left(\bigcup\limits_{l=1}^{n+m+j-5} E_l\right) & \text{if } i \in J_{0j};\ \ 2 \leq j \leq m, \\ f_i\left(\bigcup\limits_{l=m-k+2}^{n+2m-4} E_l\right) & \text{if } i \in J_{k0};\ \ 2 \leq k \leq m, \\ f_i\left(\bigcup\limits_{l=m-k+2}^{n+m+j-5} E_l\right) & \text{if } i \in J_{kj}; 2 \leq k, j \leq m, \end{cases} \tag{28}$$

and

$$\begin{cases} E_1 = f_1(E_1 \cup E_2), \\ E_k = f_1(E_{k+1});\quad k = 2, \ldots, m-2, \\ E_{m-1} = f_1\left(\bigcup\limits_{l=m}^{n+2m-4} E_l\right), \\ E_{n+m-2} = f_n\left(\bigcup\limits_{l=1}^{n+m-3} E_l\right), \\ E_{n+m+k-3} = f_n(E_{n+m+k-4}); \\ \qquad k = 2, \ldots, m-2, \\ E_{n+2m-4} = f_n(E_{n+2m-4} \cup E_{n+2m-5}). \end{cases} \tag{29}$$

We also have

$$\begin{aligned} &L = \bigcup_{l=1}^{m-1} E_l, \quad M = \bigcup_{l=m}^{m+n-3} E_l, \\ &R = \bigcup_{l=m+n-2}^{n+2m-4} E_l, \end{aligned} \tag{30}$$

and for $p = 2, 3, \ldots, m$

$$L_p = \bigcup_{l=m-p+2}^{m-1} E_l, \quad R_p = \bigcup_{l=n+m-2}^{n+m+p-5} E_l, \tag{31}$$

with the convention that $L_2 := \emptyset, R_2 := \emptyset$. Then we can rewrite (28) and (29) as

$$E_{m+i-2} = \begin{cases} f_i(L \cup M \cup R) \quad \text{if } i \in J_{00}\backslash\{1,n\}, \\ f_i(L \cup M \cup R_j) \\ \qquad \text{if } i \in J_{0j};\ \ j \in \{2, \ldots, m\}, \\ f_i(L_k \cup M \cup R) \quad \text{if } i \in J_{k0}; \\ \qquad k \in \{2, \ldots, m\}, \\ f_i(L_k \cup M \cup R_j) \quad \text{if } i \in J_{kj}; \\ \qquad k, j \in \{2, \ldots, m\}, \end{cases} \tag{32}$$

$$\begin{cases} E_1 = f_{1^{m-1}}(L \cup M \cup R), \\ E_l = f_{1^{m-l}}(M \cup R); \quad l = 2, 3, \ldots, m-1, \\ E_{n+2m-4} = f_{n^{m-1}}(L \cup M \cup R), \\ E_{n+m+l-3} = f_{n^l}(L \cup M); \\ \qquad l = 1, 2, \ldots, m-2. \end{cases} \tag{33}$$

Then we can define the label mapping as follows: for any $I, J \in \bigcup_{s=0}^{\infty}\{1, 2, \ldots, n\}^s$ with same length

$$\begin{cases} \ell(f_I(L \cup M) \cup f_J(R)) = 1, \\ \ell(f_I(R_p) \cup f_J(L_p \cup A \cup M)) = p; \;\; p = 2, \ldots, m, \end{cases} \tag{34}$$

with the convention that f_I and f_J are the identity when $s = 0$.

Let $\mathcal{D}^0 = \{K^*\}$ and so $\ell(K^*) = 1$ by (30) and (34). We call K^* the 0-level pattern 1 set.

The set K^* can be decomposed into a union of disjoint subsets, i.e.

$$\begin{aligned} K^* = & \bigcup_{i \in J_{00}} f_i \left(\bigcup_{l=1}^{n+2m-4} E_l \right) \cup \bigcup_{j=2}^{m} \\ & \bigcup_{i \in J_{0j}} \left\{ f_i \left(\bigcup_{l=1}^{m+n-3} E_l \right) \cup f_{i^*} \left(\bigcup_{l=m+n-2}^{n+2m-4} E_l \right) \right\} \\ & \cup \bigcup_{k=2}^{m} \bigcup_{j=0,2,3,\ldots,m} \bigcup_{i \in J_{kj}} \left\{ f_{(i-1)} \left(\bigcup_{l=m+n-2}^{n+m+k-5} E_l \right) \right. \\ & \left. \cup f_i \left(\bigcup_{l=m-k+2}^{m+n-3} E_l \right) \right\} \\ = & \bigcup_{i \in J_{00}} f_i\{(L \cup M \cup R)\} \cup \bigcup_{j=2}^{m} \\ & \bigcup_{i \in J_{0j}} \{f_i(L \cup M) \cup f_{i^*}(R)\} \cup \bigcup_{k=2}^{m} \\ & \bigcup_{j=0,2,3,\ldots,m} \bigcup_{i \in J_{kj}} \{f_{(i-1)}(R_k) \cup f_i(L_k \cup M)\}. \end{aligned}$$

We take $\mathcal{D}^1$ to be the collection of sets in the braces. Hence, by (3), (4) and (34), $\mathcal{D}^1$ consists of $(n - \Sigma)$ number of 1-level pattern 1 sets and n_p number of 1-level pattern p sets for each $p \in \{2, 3, \ldots, m\}$. This way we can construct $\mathcal{D}^k, k \geq 0$ inductively.

Statement 2 can be reformulated as follows.

Statement 2. (I) Each s-level pattern 1 set can be represented as a disjoint union of $(n - \Sigma)$ number of $(s+1)$-level pattern 1 sets and n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \ldots, m\}$;

(II) Each s-level pattern p with $p \in \{3, \ldots, m\}$ can be represented as a disjoint union of $(n - \Sigma - 1)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \ldots, p-2, p, \ldots, m\}$, $(n_{p-1} + 1)$ number of $(s+1)$-level pattern $(p-1)$ sets;

(III) Each s-level pattern 2 can be represented as a disjoint union of $(n - \Sigma - 2)$ number of $(s+1)$-level pattern 1 sets, n_j number of $(s+1)$-level pattern j sets for each $j \in \{2, 3, \ldots, m\}$.

Therefore, K^* has a configuration structure of (m) patterns and the corresponding $(m \times m)$ matrix is

$$\begin{pmatrix} n - \Sigma & n_2 & n_3 \cdots n_m \\ n - \Sigma - 2 & n_2 & n_3 \cdots n_m \\ n - \Sigma - 1 & n_2 + 1 & n_3 \cdots n_m \\ \vdots & \vdots & \vdots \;\ldots\; \vdots \\ n - \Sigma - 1 & n_2 & n_3 \cdots n_m \end{pmatrix}, \tag{35}$$

where the spectral radius is the largest positive root of the equation:

$$x^m - nx^{m-1} + 2(n_2 x^{m-2} + n_3 x^{m-3} + \cdots + n_m) = 0.$$

This finishes the proof. □

Proof of Theorem 1.1. It is just based on Lemmas 3.3, 3.2 and 3.1. □

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REFERENCES

1. J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30**(5) (1981) 713–747.
2. M. de Vries and V. Komornik, Unique expansions of real numbers, *Adv. Math.* **221**(2) (2009) 390–427.
3. P. Erdős, M. Horváth and I. Joó, On the uniqueness of the expansions $1 = \sum q^{-n_i}$, *Acta Math. Hungar.* **58**(3–4) (1991) 333–342.
4. P. Glendinning and N. Sidorov, Unique representations of real numbers in non-integer bases, *Math. Res. Lett.* **8**(4) (2001) 535–543.

5. K. Dajani, K. Jiang, D. Kong and W. Li, Multiple expansions of real numbers with digits set $\{0, 1, q\}$, preprint (2015), arXiv:1508.06138.
6. X. Chen, K. Jiang and W. X. Li, Estimating the Hausdorff dimensions of univoque sets for self-similar sets, *Indag. Math.* **30**(5) (2019) 862–873.
7. K. Jiang, S. Wang and L. Xi, On self-similar sets with exact overlaps, submitted (2017).
8. K. Dajani, K. Jiang, D. Kong, W. Li and L. Xi, Multiple codings for self-similar sets with overlaps, preprint (2016), arXiv:1603.09304.