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Spectral analysis for weighted iterated q-triangulations of graphs

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Much information about the structural properties and dynamical aspects of a network is measured by the eigenvalues of its normalized Laplacian matrix. In this paper, we aim to present a first study on the spectra of the normalized Laplacian of weighed iterated q-triangulations of graphs. We analytically obtain all the eigenvalues, as well as their multiplicities from two successive generations. As examples of application of these results, we then derive closed-form expressions for their Kemeny's constant and multiplicative Kirchhoff index. Simulation example is also provided to demonstrate the effectiveness of the theoretical analysis.

Keywords: Weighted networks; normalized Laplacian spectrum; Kemeny's constant; multiplicative Kirchhoff index.

1. Introduction

In the past decade, complex networks have attracted a increased attention from different scientific fields, such as physics, mathematics, computer science, due to their wide applications in this academic field. Among the extensive empirical researches in different scientific fields, spectral analysis of graphs has been a heated subject due to its wide applications in these academic fields.[1–3] Presently, there has been a particular interest in the study of the eigenvalues and eigenvectors of the normalized Laplacian matrix, since various dynamical processes and structural aspects of a graph are related to it.[3–5] In determining the eigenvalues and eigenvectors of the normalized Laplacian matrix associated to their graph representations, there has been great important progress, including the hitting time, mixing time and Kemeny's constant which can be considered as a measure of the efficiency of navigation on the network. Zhang[4] found that the graph spectrum has important applications in exploring relevant structural properties of unweighted graphs. Julaiti[5] studied that the sum of reciprocals of each nonzero eigenvalues of normalized Laplacian matrix for a

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graph determines the eigentime identity for random walks on the graph, which is a global characteristic of the network, and reflects the architecture of the whole network. Recently, a large number of graph operations and products have been introduced or proposed to construct models of complex networks, including triangulation,[6,7] Kronecker product,[8–10] hierarchical product,[11–13] as well as corona product[14–16] Among these graph operations and products, triangulation graphs have been a heated issue, and a variety of algebrasic and combinational properties of triangulation graphs have been extensively studied.[4,17–20] For more convenient and practical applications, an extended triangulation operation called q-triangulations of graphs was proposed.[7,21,22]

However, real networks behave quite differently not only in the aspect of degree distribution but also in the context of weight distribution. Weight is a relative concept for a certain index. The weight of an index refers to its relative importance in the overall evaluation. In the process of evaluation, weight representation is the quantitative distribution of the importance of different sides of the evaluated object, and it treats the role of each evaluation factor in the overall evaluation differently. In fact, an evaluation without focus is not an objective evaluation. Noted that previous works about spectra of the the normalized Laplacian matrix were mostly limited in unweighted triangulation graphs or q-triangulations of graphs, then, we start to investigate the impact of weight distribution on the spectral properties of the normalized Laplacian matrix for weighted q-triangulations of graphs.

In this paper, we will investigate analytically the spectral of weighted iterated q-triangulations of graphs with weight factor r. In view of the mentioned idea of Ref. 1, and based on the particular construction of the graphs, we propose a method to obtain all the eigenvalues and their corresponding multiplicities of graphs. Using the obtained eigenvalues and their corresponding multiplicities, we deduce an explicit expression for their Kemeny's constant and multiplicative Kirchhoff index.

The rest of this paper is organized as follows: Sec. 2, introduces preliminaries of graph and matrix notation, as well the definition of weighted iterated q-triangulations of graphs. In Sec. 3, we start the spectral analysis for weighted iterated q-triangulations of graphs. In Sec. 4, we present some applications of the normalized Laplacian spectra of these graphs. In Sec. 5, simulation example is provided to demonstrate the effectiveness of the theoretical analysis. Finally, Sec. 6 includes discussions and conclusions.

2. Weighted Iterated q-Triangulations of Graphs

Let $G = G(V, E)$ be any simple connected graph of order N, with vertex set $V = \{v_1, v_2, \ldots, v_N\}$ denoting the vertex set and $E \in V \times V$ denoting the edge set. An edge e_{ij} in the graph G is defined as the unordered pair of nodes (v_i, v_j). Denote a path between nodes v_i and v_j in G as a sequence of edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_l}, v_j)$ in the graph with distinct nodes v_{i_k}, $k = 1, 2, \ldots, l$.

Definition 2.1. The weighted iterated q-triangulations of graphs, parameterized by a positive number r, are built in an iterative way.

(i) Let G be any simple connected graph, called the initial graph.
(ii) The q-triangulation graph of G, denoted by $\tau_q(G)$, is the graph obtained by adding q new vertices corresponding to each edge with weight $w = 1$ of G and by joining each new vertex to the end vertices of the edge corresponding to it, and the new edges carries weight $rw = r$, and for each of three old edges of initial graph G, the weight remains the same. For convenience, each edge in G is called the father edge of the new vertex.
(iii) For $n \geq 1$, $\tau_q^n(G)$ is obtained from $\tau_q^{n-1}(G)$ by performing the following operations: For each edge with weight w in $\tau_q^{n-1}(G)$, which is called the father edge of the next new vertex, we add q new vertices and link each of them to either end of the edge, respectively, and each newly generated edge carries weight rw. And for every old edge existed in $\tau_q^{n-1}(G)$, the weight remains the same.

Weight representation is the quantitative distribution of the importance of different sides of the evaluated object, and it treats the role of each evaluation factor differently. Consider every two successive generation graphs of the weighed iterated q-triangulations of graphs, for $n \geq 1$, $\tau_q^n(G)$ is obtained from $\tau_q^{n-1}(G)$ by performing the above q-triangulation operation. Generally speaking, in a pseudofractal scale-free network, for example $\tau_q^n(G)$ in this paper, father edge with weight w plays a more important role than its corresponding newly generated edges with weight rw, that is $w \geq wr$, then $0 < r \leq 1$. In this paper, first of all, we study analytically to obtain the formula relationship between the eigenvalues, as well as their multiplicities of τ_q^{n-1} (G) and that of $\tau_q^n(G)$, which is the basis of the paper, then by iterative operation, we can obtain all the eigenvalues of $\tau_q^{n-1}(G)$ from initial graph $\tau_q^0(G)$. The important formula relationship mentioned above is only decided by way of graph construction or q-triangulation operation, which will be proved in Lemma 3.1. In other words, whatever initial graph is or whether initial graph is weighted, it has no influence in obtaining the formula relationship for the eigenvalues, as well as their multiplicities between $\tau_q^{n-1}(G)$ and $\tau_q^n(G)$ and other main results in this paper, only the simulation example is exactly influenced, which has no negative impact to our main idea.

When $q = 1$, Fig. 1 shows an example of the first three iterations where the initial graph is the triangle graph consisting of three vertices and three edges with unit weight. In this particular case, the resulting graph is known as the scale-free pseudofractal graph that exhibits a scale-free and small-world topology. The study of its structural and dynamic properties has produced abundant literature, since it is a good deterministic model for many real-life networks.

Denoting $\tau_q^n(G) = (V_n, E_n)$ with vertex set V_n and edge set E_n, and $|N_n|$, $|E_n|$ denote, respectively, the total number of vertices and the total number of edges of the graph $\tau_q^n(G)$ in generation n.

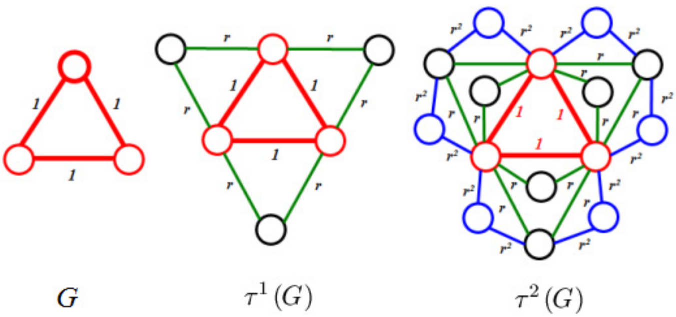


Fig. 1. Iterative construction method for weighted iterated triangulations of graphs from generation $n = 0$ to $n = 2$, where the initial graph is the triangle graph consisting three vertices and three edges with unit weight when $q = 1$.

It is easy to check that

$$|E_n| = (2q+1)^n |E_0|, \quad |N_n| = |N_0| + \frac{(2q+1)^n - 1}{2}|E_0|. \tag{1}$$

Let $s_n(i)$ be the strength of vertex i in $\tau_q^n(G)$, which is defined by the sum of its linked edges' weight, then S_n denote the diagonal strength matrix of $\tau_q^n(G)$ with its ith diagonal entry being the strength $s_n(i)$ of vertex i. Let W_n be the generalized adjacency matrix (weight matrix) of $\tau_q^n(G)$, the entries W_{ij} of W_n are defined as follows: $W_{ij} = w_{ij}$ if vertices i and j are adjacent in $\tau_q^n(G)$, or $W_{ij} = 0$ otherwise, where w_{ij} is the weight of edge linking vertices i and j.

Then, the transition matrix for biased random walks in $\tau_q^n(G)$, denoted by T_n, is defined as $T_n = S_n^{-1} W_n$, where S_n is the diagonal strength matrix of $\tau_q^n(G)$ with its ith diagonal entry being the strength s_i of vertex i. T_n can be normalized to obtain a real and symmetric matrix P_n defined as

$$P_n = S_n^{-\frac{1}{2}} W_n S_n^{-\frac{1}{2}} = S_n^{\frac{1}{2}} T_n S_n^{-\frac{1}{2}}.$$

By definition, the (i,j)th entry of P_n is $P_n(i,j) = \frac{w_n(i,j)}{\sqrt{s_n(i)s_n(j)}}$. Since P_n is similar to T_n, they have the same set of eigenvalues.

Definition 2.2. The normalized Laplacian matrix of $\tau_q^n(G)$ is

$$L_n = I_n - S_n^{\frac{1}{2}} T_n S_n^{-\frac{1}{2}} = I_n - P_n, \tag{2}$$

where I_n is the identify matrix with the same order as P_n.

3. Spectral Analysis for Weighted Iterated *q*-Triangulations of Graphs

In this section, we address the eigenvalue spectrum problem of L_n. Let λ signify the eigenvalue of L_n. We denote the spectrum of L_n by $\sigma_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{|N_n|}^{(n)}\}$ and the multiplicity of $\lambda_i^{(n)}$ by $m_{L_n}(\lambda_i^{(n)})$. Inspired by the methods of Ref. 1, we will derive closed-form expressions for the spectrum σ_n of the normalized Laplacian matrix L_n.

It's revealed that this spectrum can be obtained iteratively from the spectrum of any simple connected graphs.

Lemma 3.1. *Let λ be any eigenvalue of the normalized Laplacian matrix $L_n, n > 0$, such that $\lambda \neq 1$ and $\lambda \neq 1 + \frac{r}{2}$. Then here is the result that $\lambda(1 + r + \frac{r-r^2}{2-2\lambda+r})$ is an eigenvalue of L_{n-1}, and its multiplicity, denoted by $m_{L_{n-1}}[\lambda(1 + r + \frac{r-r^2}{2-2\lambda+r})]$, is the same with the multiplicity $m_{L_n}(\lambda)$ of the eigenvalue λ of L_n.*

Proof. Let the vertices of $\tau_q^n(G)$ fall into two groups V_{old}^n and V_{new}^n, where V_{new}^n denotes the set of all the newly added vertices in $\tau_q^n(G)$ and V_{old}^n contains all the vertices inherited from $\tau^{n-1}(G)$. Meanwhile, we also divided the vertices of $\tau_q^n(G)$ into groups as

$$V_n = V_0 \cup V_1 \cup V_r \cup \cdots \cup V_{r^{n-1}},$$

where V_0 and $V_{r^k}(0 \leq k \leq n-1)$ denotes, respectively, the vertex set of initial graph G and the vertex set of vertices whose father edge with weight r^k. In $\tau_q^n(G)$, we obviously have

$$V_0 \subseteq V_{\text{old}}^n, \quad V_{r^{n-1}} \subseteq V_{\text{new}}^n.$$

We here introduce, respectively, the old vertices and the new vertices of V_{r^k} as V'_{r^k} and V''_{r^k}, $(0 \leq k \leq n-2)$, then one has

$$V_{r^k} = V'_{r^k} \cup V''_{r^k}, \quad \text{where } V'_{r^k} = V_{r^k} \cap V_{\text{old}}^n, \quad V''_{r^k} = V_{r^k} \cap V_{\text{new}}^n \quad (0 \leq k \leq n-2).$$

Suppose that λ is an eigenvalue of $\tau_q^n(G)$, and $\psi = (\psi_1, \psi_2, \dots, \psi_{|N_n|})$ is its corresponding eigenvector, where φ_i is the component corresponding to vertex i in $\tau_q^n(G)$. For $L_n = I_n - P_n$, ψ is also an eigenvector of P_n associated with eigenvalue $1 - \lambda$. By definition, we have

$$P_n \psi = (1 - \lambda)\psi. \tag{3}$$

For any old vertex, $i \in V_0 \cup \{V'_{r^1}, V'_{r^2}, \dots, V'_{r^{n-2}}\}$. We will consider them, respectively, by classifying them into two cases: Case I. $i \in V_0$; Case II. $i \in \{V'_{r^1}, V'_{r^2}, \dots, V'_{r^{n-2}}\}$.

Case I. For any old vertex $i \in V_0$. According to Eq. (3), it has

$$(1 - \lambda)\psi_i = \sum_{j=1}^{|N_n|} P_n(i, j)\psi_j \tag{4}$$

then

$$\begin{aligned}(1 - \lambda)\psi_i = & \sum_{j'_0 \in V_0} \frac{1}{\sqrt{s_n(i)s_n(j'_0)}}\psi_{j'_0} + \sum_{J_1 \in V_1} \frac{r}{\sqrt{s_n(i)s_n(J_1)}}\psi_{J_1} \\ & + \sum_{J_r \in V_r} \frac{r^2}{\sqrt{s_n(i)s_n(J_r)}}\psi_{J_r} + \cdots + \sum_{J_{r^{n-2}} \in V_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_n(i)s_n(J_{r^{n-2}})}}\psi_{J_{r^{n-2}}} \\ & + \sum_{j_{r^{n-1}} \in V_{r^{n-1}}} \frac{r^n}{\sqrt{s_n(i)s_n(j_{r^{n-1}})}}\psi_{j_{r^{n-1}}}.\end{aligned} \tag{5}$$

From the construction of $\tau_q^n(G)$, for $0 \leq k \leq n-1$, we can see the strength difference of the vertices in $\tau_q^n(G)$,

$$\begin{cases} s_n(i) = (1+qr)s_{n-1}(i), & \text{if } i \in V_{\text{old}}^n, \\ s_n(j_{r^k}) = 2r^k, & \text{if } j_{r^k} \in V_{\text{new}}^n. \end{cases} \tag{6}$$

So Eq. (5) leads to

$$\begin{aligned} (1-\lambda)\psi_i = &\sum_{j'_0 \in V_0} \frac{1}{\sqrt{s_n(i)s_n(j'_0)}}\psi_{j'_0} \\ &+ \left(\sum_{j'_1 \in V'_1} \frac{r}{\sqrt{s_n(i)s_n(j'_1)}}\psi_{j'_1} + \sum_{j_1 \in V''_1} \frac{r}{\sqrt{s_n(i)s_n(j_1)}}\psi_{j_1}\right) \\ &+ \left(\sum_{j'_r \in V'_r} \frac{r^2}{\sqrt{s_n(i)s_n(j'_r)}}\psi_{j'_r} + \sum_{j_r \in V''_r} \frac{r^2}{\sqrt{s_n(i)s_n(j_r)}}\psi_{j_r}\right) + \cdots \\ &+ \left(\sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_n(i)s_n(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}} + \sum_{j_{r^{n-2}} \in V''_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_n(i)s_n(j_{r^{n-2}})}}\psi_{j_{r^{n-2}}}\right) \\ &+ \sum_{j_{r^{n-1}} \in V_{r^{n-1}}} \frac{r^n}{\sqrt{s_n(i)s_n(j_{r^{n-1}})}}\psi_{j_{r^{n-1}}}. \end{aligned}$$

From the detachment of old vertices and new vertices, one has

$$\begin{aligned} (1-\lambda)\psi_i = &\left(\sum_{j'_0 \in V_0} \frac{1}{\sqrt{s_n(i)s_n(j'_0)}}\psi_{j'_0} + \sum_{j'_1 \in V'_1} \frac{r}{\sqrt{s_n(i)s_n(j'_1)}}\psi_{j'_1}\right. \\ &\left. + \cdots + \sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_n(i)s_n(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right) \\ &+ \left(\sum_{j_1 \in V''_1} \frac{r}{\sqrt{s_n(i)s_n(j_1)}}\psi_{j_1} + \sum_{j_r \in V''_r} \frac{r^2}{\sqrt{s_n(i)s_n(j_r)}}\psi_{j_r}\right. \\ &\left. + \cdots + \sum_{j_{r^{n-1}} \in V_{r^{n-1}}} \frac{r^n}{\sqrt{s_n(i)s_n(j_{r^{n-1}})}}\psi_{j_{r^{n-1}}}\right). \end{aligned} \tag{7}$$

Referring to Eqs. (6) and (7), we have

$$\begin{aligned} (1-\lambda)\psi_i = \frac{1}{1+qr}&\left(\sum_{j'_0 \in V_0} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0} + \sum_{j'_1 \in V'_1} \frac{r}{\sqrt{s_{n-1}(i)s_{n-1}(j'_1)}}\psi_{j'_1}\right. \\ &\left. + \cdots + \sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right) + \frac{1}{\sqrt{2(1+qr)}} \end{aligned}$$

$$\times\left(\sum_{j_1\in V_1''}\frac{\sqrt{r}}{\sqrt{s_{n-1}(i)s_{n-1}(j_1)}}\psi_{j_1}+\sum_{j_r\in V_r''}\frac{\sqrt{r^2}}{\sqrt{s_{n-1}(i)s_{n-1}(j_r)}}\psi_{j_r}\right.$$
$$\left.+\cdots+\sum_{j_{r^{n-1}}\in V_{r^{n-1}}}\frac{\sqrt{r^n}}{\sqrt{s_{n-1}(i)s_{n-1}(j_{r^{n-1}})}}\psi_{j_{r^{n-1}}}\right). \tag{8}$$

For any vertex in V_{new}^n have and only have two old adjacent vertices, so from Eq. (4), one has

$$\begin{aligned}(1-\lambda)\psi_{j_1} &= \frac{\sqrt{r}}{\sqrt{2(1+qr)}}\left(\frac{1}{\sqrt{s_{n-1}(i)}}\psi_i+\frac{1}{\sqrt{s_{n-1}(j_0')}}\psi_{j_0'}\right),\\ (1-\lambda)\psi_{j_r} &= \frac{\sqrt{r^2}}{\sqrt{2(1+qr)}}\left(\frac{1}{\sqrt{s_{n-1}(i)}}\psi_i+\frac{1}{\sqrt{s_{n-1}(j_1')}}\psi_{j_1'}\right),\\ &\vdots\\ (1-\lambda)\psi_{j_{r^{n-1}}} &= \frac{\sqrt{r^n}}{\sqrt{2(1+qr)}}\left(\frac{1}{\sqrt{s_{n-1}(i)}}\psi_i+\frac{1}{\sqrt{s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right).\end{aligned}$$

Let $n_{j_0'}$ be the total number of the neighbors in V_0 of i, and $n_{j'_{r^k}}$ be the total number of the neighbors in V_{r^k} $(0\le k\le n-2)$ of i.

Take the above relationships into Eq. (8),

$$\begin{aligned}(1-\lambda)\psi_i = {}& \frac{1}{1+qr}\left(\sum_{j_0'\in V_0}\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j_0')}}\psi_{j_0'}+\sum_{j_1'\in V_1'}\frac{r}{\sqrt{s_{n-1}(i)s_{n-1}(j_1')}}\psi_{j_1'}\right.\\ &\left.+\cdots+\sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\frac{r^{n-1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right)\\ &+\frac{r}{2(1-\lambda)(1+qr)}\sum_{j_0'\in V_0}\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j_0')}}\psi_{j_0'}\right)\\ &+\frac{r^2}{2(1-\lambda)(1+qr)}\sum_{j_1'\in V_1'}\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j_1')}}\psi_{j_1'}\right)+\cdots\\ &+\frac{r^n}{2(1-\lambda)(1+qr)}\sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right)\\ = {}& \frac{rn_{j_0'}+r^2n_{j_1'}+\cdots+r^nn_{j'_{r^{n-2}}}}{[2(1-\lambda)(1+qr)]s_{n-1}(i)}+\left(\frac{1}{1+qr}+\frac{r}{2(1-\lambda)(1+qr)}\right)\\ &\times\sum_{j_1'\in V_0}\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j_0')}}\psi_{j_0'}+\left(\frac{r}{1+qr}+\frac{r^2}{2(1-\lambda)(1+qr)}\right)\end{aligned}$$

$$\times \sum_{j'_1 \in V'_1} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_1)}}\psi_{j'_1} + \cdots + \left(\frac{r^{n-1}}{1+qr} + \frac{r^n}{2(1-\lambda)(1+qr)}\right) \times \sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}. \quad (9)$$

Notice that the strength of $i \in V_0$ is $s_{n-1}(i) = n_{j'_0} + rn_{j'_1} + \cdots + r^{n-1}n_{j'_{r^{n-2}}}$. Then,

$$2(1-\lambda)^2(1+qr)\psi_i = r\psi_i + (2(1-\lambda)+r)\sum_{j'_0 \in V_0} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0} + (2(1-\lambda)r + r^2)\sum_{j'_1 \in V'_1} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_1)}}\psi_{j'_1} + \cdots + (2(1-\lambda)r^{n-1} + r^n)\sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}.$$

Therefore, for any $i \in V_0$,

$$\frac{2(1-\lambda)^2(1+qr) - r}{2(1-\lambda)+r}\psi_i = \sum_{j'_0 \in V_0} \frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0} + \sum_{j'_1 \in V'_1} \frac{r}{\sqrt{s_{n-1}(i)s_{n-1}(j'_1)}}\psi_{j'_1} + \cdots + \sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}, \quad (10)$$

limited by $\lambda \neq 1 + \frac{r}{2}$.

Case II. For any old vertex $i \in V'_{r^k} \subseteq \{V'_{r^1}, V'_{r^2}, \ldots, V'_{r^{n-2}}\}$. Referring to Eq. (4), one has

$$(1-\lambda)\psi_i = \left(\sum_{j'_0 \in V_0} \frac{r^{k+1}}{\sqrt{s_n(i)s_n(j'_0)}}\psi_{j'_0} + \cdots + \sum_{j'_{r^{k-1}} \in V'_{r^{k-1}}} \frac{r^{k+1}}{\sqrt{s_n(i)s_n(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}} + \sum_{j'_{r^{k+1}} \in V'_{r^{k+1}}} \frac{r^{k+2}}{\sqrt{s_n(i)s_n(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}} + \cdots + \sum_{j'_{r^{n-2}} \in V'_{r^{n-2}}} \frac{r^{n-1}}{\sqrt{s_n(i)s_n(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right) + \left(\sum_{j_1 \in V''_1} \frac{0}{\sqrt{s_n(i)s_n(j_1)}}\psi_{j_1} + \cdots + \sum_{j_{r^k} \in V''_{r^k}} \frac{0}{\sqrt{s_n(i)s_n(j_{r^k})}}\psi_{j_{r^k}} + \sum_{j_{r^{k+1}} \in V''_{r^{k+1}}} \frac{r^{k+2}}{\sqrt{s_n(i)s_n(j_{r^{k+1}})}}\psi_{j_{r^{k+1}}} + \cdots + \sum_{j_{r^{n-1}} \in V_{r^{n-1}}} \frac{r^n}{\sqrt{s_n(i)s_n(j_{r^{n-1}})}}\psi_{j_{r^{n-1}}}\right). \quad (11)$$

From Eqs. (6) and (11), we have

$$
\begin{aligned}
(1-\lambda)\psi_i = {} & \frac{1}{1+qr}\Bigg(\sum_{j'_0\in V_0}\frac{r^{k+1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0} \\
& + \cdots + \sum_{j'_{r^{k-1}}\in V'_{r^{k-1}}}\frac{r^{k+1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}} \\
& + \sum_{j'_{r^{k+1}}\in V'_{r^{k+1}}}\frac{r^{k+2}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}} \\
& + \cdots + \sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\frac{r^{n-1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\Bigg) \\
& + \frac{1}{\sqrt{2(1+qr)}}\Bigg(\sum_{j_{r^{k+1}}\in V''_{r^{k+1}}}\frac{r^{k+2}}{\sqrt{s_{n-1}(i)s_{n-1}(j_{r^{k+1}})}}\psi_{j_{r^{k+1}}} \\
& + \cdots + \sum_{j_{r^{n-1}}\in V_{r^{n-1}}}\frac{r^{n}}{\sqrt{s_{n-1}(i)s_{n-1}(j_{r^{n-1}})}}\psi_{j_{r^{n-1}}}\Bigg).
\end{aligned}
\tag{12}
$$

Let e_{ij} be the edge linking vertices i and j. By Eqs. (4) and (6), it can be seen from the structure of $\tau_q^n(G)$ that

$$
(1-\lambda)\psi_{j_{r^{k+1}}} = \begin{cases}
\dfrac{\sqrt{r^{k+2}}}{\sqrt{2(1+qr)}}\left(\dfrac{1}{\sqrt{s_{n-1}(i)}}\psi_i + \dfrac{1}{\sqrt{s_{n-1}(j'_0)}}\psi_{j'_0}\right), \\
\qquad \text{if } e_{ij'_0} \text{ is the father edge of } j_{r^{k+1}}; \\
\dfrac{\sqrt{r^{k+2}}}{\sqrt{2(1+qr)}}\left(\dfrac{1}{\sqrt{s_{n-1}(i)}}\psi_i + \dfrac{1}{\sqrt{s_{n-1}(j'_1)}}\psi_{j'_1}\right), \\
\qquad \text{if } e_{ij'_1} \text{ is the father edge of } j_{r^{k+1}}; \\
\qquad \vdots \\
\dfrac{\sqrt{r^{k+2}}}{\sqrt{2(1+qr)}}\left(\dfrac{1}{\sqrt{s_{n-1}(i)}}\psi_i + \dfrac{1}{\sqrt{s_{n-1}(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}}\right), \\
\qquad \text{if } e_{ij'_{r^{k-1}}} \text{ is the father edge of } j_{r^{k+1}};
\end{cases}
$$

$$
\begin{aligned}
(1-\lambda)\psi_{j_{r^{k+2}}} &= \frac{\sqrt{r^{k+3}}}{\sqrt{2(1+qr)}}\left(\frac{1}{\sqrt{s_{n-1}(i)}}\psi_i + \frac{1}{\sqrt{s_{n-1}(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}}\right); \\
&\ \vdots \\
(1-\lambda)\psi_{j_{r^{n-1}}} &= \frac{\sqrt{r^{n}}}{\sqrt{2(1+qr)}}\left(\frac{1}{\sqrt{s_{n-1}(i)}}\psi_i + \frac{1}{\sqrt{s_{n-1}(j'_{r^{n-1}})}}\psi_{j'_{r^{n-1}}}\right).
\end{aligned}
\tag{13}
$$

Combining Eqs. (12) and (13),

$$
\begin{aligned}
&(1-\lambda)\psi_i \\
&= \frac{1}{1+qr}\left(\sum_{j'_0\in V_0}\frac{r^{k+1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0}+\cdots\right.\\
&\quad+\sum_{j'_{r^{k-1}}\in V'_{r^{k-1}}}\frac{r^{k+1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}}+\sum_{j'_{r^{k+1}}\in V'_{r^{k+1}}}\frac{r^{k+2}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}}\\
&\quad\left.+\cdots+\sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\frac{r^{n-1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right)\\
&\quad+\frac{r^{k+2}}{2(1-\lambda)(1+qr)}\sum_{j'_0\in V_0}\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0}\right)+\cdots\\
&\quad+\frac{r^{k+2}}{2(1-\lambda)(1+qr)}\sum_{j'_{r^{k-1}}\in V'_{r^{k-1}}}\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}}\right)\\
&\quad+\frac{r^{k+3}}{2(1-\lambda)(1+qr)}\sum_{j'_{r^{k+1}}\in V'_{r^{k+1}}}\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}}\right)\\
&\quad+\cdots+\frac{r^{n}}{2(1-\lambda)(1+qr)}\sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\\
&\quad\times\left(\frac{1}{s_{n-1}(i)}\psi_i+\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}\right)\\
&= \frac{r^{k+2}n_{j'_0}+\cdots+r^{k+2}n_{j'_{r^{k-1}}}+r^{k+3}n_{j'_{r^{k+1}}}+\cdots+r^{n}n_{j'_{r^{n-2}}}}{[2(1-\lambda)(1+qr)]s_{n-1}(i)}\psi_i\\
&\quad+\left(\frac{r^{k+1}}{1+qr}+\frac{r^{k+2}}{2(1-\lambda)(1+qr)}\right)\sum_{j'_0\in V_0}\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0}+\cdots\\
&\quad+\left(\frac{r^{k+1}}{1+qr}+\frac{r^{k+2}}{2(1-\lambda)(1+qr)}\right)\sum_{j'_{r^{k-1}}\in V'_{r^{k-1}}}\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}}\\
&\quad+\left(\frac{r^{k+2}}{1+qr}+\frac{r^{k+3}}{2(1-\lambda)(1+qr)}\right)\sum_{j'_{r^{k+1}}\in V'_{r^{k+1}}}\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}}+\cdots\\
&\quad+\left(\frac{r^{n-1}}{1+qr}+\frac{r^{n}}{2(1-\lambda)(1+qr)}\right)\sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\frac{1}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}}.
\end{aligned}
\tag{14}
$$

Notice that the strength of old vertex $i\in V_{r^k}(0\le k\le n-2)$ is

$$s_{n-1}(i)=r^{k+1}n_{j'_0}+\cdots+r^{k+1}n_{j'_{r^{k-1}}}+r^{k+2}n_{j'_{r^{k+1}}}+\cdots+r^{n-1}n_{j'_{r^{n-2}}}.$$

Then we can obtain the following relationship by taking the above equation into Eq. (14):

$$\begin{aligned}\frac{2(1-\lambda)^2(1+qr)-r}{2(1-\lambda)+r}\psi_i = &\sum_{j'_0\in V_0}\frac{r^{k+1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_0)}}\psi_{j'_0}\\ &+\cdots+\sum_{j'_{r^{k-1}}\in V'_{r^{k-1}}}\frac{r^{k+1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k-1}})}}\psi_{j'_{r^{k-1}}}\\ &+\sum_{j'_{r^{k+1}}\in V'_{r^{k+1}}}\frac{r^{k+2}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{k+1}})}}\psi_{j'_{r^{k+1}}}\\ &+\cdots+\sum_{j'_{r^{n-2}}\in V'_{r^{n-2}}}\frac{r^{n-1}}{\sqrt{s_{n-1}(i)s_{n-1}(j'_{r^{n-2}})}}\psi_{j'_{r^{n-2}}},\end{aligned} \tag{15}$$

holds for $\lambda \neq 1+\frac{r}{2}$.

From the results above, we draw that $\frac{2(1-\lambda)^2(1+qr)-r}{2(1-\lambda)+r}$ is an eigenvalue of P_{n-1}. Then, it implies that $\frac{2\lambda+(2-2q)r-\lambda^2(1+qr)+4qr\lambda}{2(1-\lambda)+r}$ is an eigenvalue of L_{n-1} from $L_n = I_n - P_n$, and $\psi_0 = (\psi_i)^T_{i\in V^n_{\text{old}}}$ is one associated eigenvector. By using Eq. (13), it shows explicitly a bijection between V^n_{old} and V^n_{new}, then ψ leads to be entirely determined. Besides that, $m_{L_{n-1}}(\frac{2\lambda+(2-2q)r-\lambda^2(1+qr)+4qr\lambda}{2(1-\lambda)+r}) \geq m_{L_{n-1}}(\lambda)$.

Suppose now that $m_{L_{n-1}}(\frac{2\lambda+(2-2q)r-\lambda^2(1+qr)+4qr\lambda}{2(1-\lambda)+r}) \geq m_{L_{n-1}}(\lambda)$. This means that there should exist an extra eigenvector associated to $\frac{2\lambda+(2-2q)r-\lambda^2(1+qr)+4qr\lambda}{2(1-\lambda)+r}$ without a corresponding eigenvector in L_n. But Eq. (13) provides this extra eigenvector with an associated eigenvector of P_n, and this contradicts our assumption. Therefore,

$$m_{L_{n-1}}\left[\frac{2\lambda+(2-2q)r-\lambda^2(1+qr)+4qr\lambda}{2(1-\lambda)+r}\right] = m_{L_{n-1}}(\lambda) \tag{16}$$

holds for $\lambda \neq 1+\frac{r}{2}$. □

Notice that if $\lambda = 1$ is an eigenvalue of L_n, $\lambda = 2$ is its corresponding eigenvalue of L_{n-1}. When $\lambda = 1+\frac{r}{2}$. Case (i): if $r = 1$, all new added verties in V^n_{new} can be determined by Eq. (13) and Eqs. (13) and (14) hold for any old vertex, thus $m_{L_n}(1+\frac{r}{2}) = N_{n-1}$. Case (ii): if $r \neq 1$, it's contradictory to evaluate Eqs. (13) and (14), then the eigenvalue has no multiplicity. Case (i) has been discussed in detail in Ref. 1, so here we follow mainly the case of $r \neq 1$.

Lemma 3.2. *Let λ be any eigenvalue of the normalized Laplacian matrix L_{n-1}, $n > 0$, such that $\lambda \neq 2$ and let*

$$f_1(x) = \frac{1+2qr+x+\sqrt{(1+2qr+x)^2-2(1+qr)(2+r)x}}{2(1+qr)}$$

and

$$f_2(x) = \frac{1 + 2qr + x - \sqrt{(1+2qr+x)^2 - 2(1+qr)(2+r)x}}{2(1+qr)}.$$

Then here is the result that $f_1(\lambda)$ and $f_2(\lambda)$ are eigenvalues of L_n, and their multiplicities, respectively denoted by $m_{L_n}[f_1(\lambda)]$ and $m_{L_n}[f_2(\lambda)]$, are the same as the multiplicity $m_{L_{n-1}}(\lambda)$ of the eigenvalue λ of L_{n-1}, that is $m_{L_n}[f_1(\lambda)] = m_{L_n}[f_2(\lambda)] = m_{L_{n-1}}(\lambda)$.

Proof. This is a direct consequence of Lemma 3.1. □

The above results indicate that the majority of eigenvalues of L_n can be obtained from the spectrum of L_{n-1}, except eigenvalue 2. And the rest of this spectrum consists of 1*s*. Furthermore, it shows that each eigenvalue of L_{n-1} gives rise to two eigenvalues of L_n. Therefore, $f_1(\lambda)$ (or $f_2(\lambda)$) has the same number of linearly independent eigenvectors as that of L_{n-1}. Moreover, the eigenvectors of eigenvalues of L_n are linearly independent because L_n is real and symmetric.

Definition 3.3. Let $U = \{u_1, u_2, \ldots, u_k\}$ be any finite multiset of real numbers. The multiset $R^{-1}(U)$ is defined as

$$R^{-1}(U) = \{f_1(u_1), f_2(u_1), f_1(u_2), f_2(u_2), \ldots, f_1(u_k)f_2(u_k)\}. \tag{17}$$

Theorem 3.4. *The spectrum σ_n of L_n is*

$$\sigma_n = \begin{cases} R^{-1}\Big(\sigma_0 \backslash \{2\} \backslash \Big\{1 + \dfrac{r}{2}\Big\}\Big) \cup \{\underbrace{1, 1, \ldots, 1}_{m_{L_1(1)}}\}, \quad \text{if } n = 1, \\ \qquad \text{where } m_{L_1(1)} = |E_0| - |N_0| + 2m_{L_1(2)} + 2m_{L_1(1+\frac{r}{2})}, \\ R^{-1}(\sigma_{n-1}) \cup \{\underbrace{1, 1, \ldots, 1}_{m_{L_n(1)}}\}, \quad \text{if } n > 1. \\ \qquad \text{where } m_{L_n(1)} = \dfrac{(1+2q)^{n-1}+1}{2}|E_0| - |N_0|. \end{cases}$$

Proof. Lemma 3.2 implies that from the eigenvalues of generation $n-1$, one can yield the eigenvalues of the next generation n with the exception of eigenvalue 2. Thus, there exists an eigenvalue that cannot be derived from L_{n-1}, it must be eigenvalue equal to 2. Therefore, $f_1(\lambda)$ (or $f_2(\lambda)$) has the same number of linearly independent eigenvectors as that of L_{n-1}. Because L_{n-1} is a real and symmetrical matrix, each eigenvalue of L_{n-1} has linearly independent eigenvectors. It is the same with either of its child eigenvalues in L_n. Then, the spectrum of L_n inherited $2|N_{n-1}| - 2m_{L_{n-1}}(2) - 2m_{L_{n-1}(1+\frac{r}{2})}$ eigenvalues from L_{n-1}. The rest of the spectrum for and L_n consists of 1*s*. Then it can be explicitly determined that

$$\begin{aligned} m_{L_n(1)} &= |N_n| - (2|N_{n-1}| - 2m_{L_{n-1}}(2) - 2m_{L_{n-1}(1+\frac{r}{2})}) \\ &= \frac{(1+2q)^{n-1}+1}{2}|E_0| - |N_0| + 2m_{L_{n-1}(2)} + 2m_{L_{n-1}(1+\frac{r}{2})}. \end{aligned}$$

For $n = 1$, we obtain

$$\begin{aligned} m_{L_1(1)} &= \frac{(1+2q)^0+1}{2}|E_0| - |N_0| + 2m_{L_1(2)} + 2m_{L_1(1+\frac{r}{2})} \\ &= |E_0| - |N_0| + 2m_{L_0(2)} + 2m_{L_0(1+\frac{r}{2})}. \end{aligned}$$

For $n > 1$, we have $m_{L_n(2)} = m_{L_n(1+\frac{r}{2})} = 0$. Then

$$m_{L_n(1)} = \frac{(1+2q)^{n-1}+1}{2}|E_0| - |N_0|.$$

So from Eq. (17), the theorem is proved. □

4. Applications of the Normalized Laplacian Spectra of $\tau_q^n(G)$

In this section, we apply the obtained eigenvalues and their multiplicities to determine relevant invariants related to the structure of graphs. Then we derive accurately closed-form expressions for the graph's Kemeny's constant and multiplicative Kirchhoff index. It's revealed that they're all influenced only by generation n, weight r and some invariants of the initial graph.

4.1. *Kemeny's constant*

Definition 4.1. Given a graph G, the Kemeny's constant $K(G)$, or average hitting time, is the expected number of steps required for the transition from a starting vertex i to a destination vertex, which is chosen randomly according to a stationary distribution of unbiased random walks on G (see Ref. 23 for more details).

According to Ref. 24, Kemeny's constant can be expressed in terms of the spectrum $\sigma_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{|N_n|}^{(n)}\}$ of the normalized Laplacian matrix of G[25,26] and Kemeny's constant can be computed as

$$K(\tau_q^n(G)) = \sum_{k=2}^{|N_n|} \frac{1}{\lambda_k^{(n)}}, \tag{18}$$

where $0 = \lambda_1^{(n)} < \lambda_2^{(n)} \le \cdots \le \lambda_{|N_n|}^{(n)} \le 2$ are the eigenvalues of L_n.

Theorem 4.1. *The general expression of Kemeny's constant for* $\tau_q^n(G)$ *is*

$$\begin{aligned} K(\tau_q^n(G)) = 2^n\left(\frac{1+2qr}{2+r}\right)^n K(G) &+ (2q+1)^n(2+r) \\ &\times \frac{2^n(1+2qr)^n - (2q+1)^n(2+r)^n}{(2qr-4q-r)(2q+1)^n(2+r)^n}|E_0| \\ &+ (r|N_0|+2)\frac{2^n(1+2qr)^n - (2+r)^n}{(4q-1)r(2+r)^n}. \end{aligned}$$

Proof. From Lemma 3.2, each eigenvalue $\lambda_k^{(n-1)}$ in σ_{n-1} gives rise to two eigenvalues $f_1(\lambda_k^{(n-1)})$ and $f_2(\lambda_k^{(n-1)})$ in σ_n, which obey the relations

$$f_1(\lambda_k^{(n-1)}) + f_2(\lambda_k^{(n-1)}) = \frac{1 + 2qr + \lambda_k^{(n-1)}}{1 + qr}$$

and

$$f_1(\lambda_k^{(n-1)}) \cdot f_2(\lambda_k^{(n-1)}) = \frac{(2 + r)\lambda_k^{(n-1)}}{2(1 + qr)}.$$

Then,

$$\frac{1}{f_1(\lambda_k^{(n-1)})} + \frac{1}{f_2(\lambda_k^{(n-1)})} = \frac{f_1(\lambda_k^{(n-1)}) + f_2(\lambda_k^{(n-1)})}{f_1(\lambda_k^{(n-1)}) \cdot f_2(\lambda_k^{(n-1)})} = \frac{2 + 4qr}{(2 + r)\lambda_k^{(n-1)}} + \frac{2}{2 + r}.$$

From Eq. (18) and Theorem 3.4, one has

$$\begin{aligned} K(\tau_q^n(G)) &= \sum_{k=2}^{|N_n|} \frac{1}{\lambda_k^{(n)}} \\ &= \sum_{k=2}^{|N_{n-1}|} \left(\frac{1}{f_1(\lambda_k^{(n-1)})} + \frac{1}{f_2(\lambda_k^{(n-1)})} \right) + \frac{1}{1} \times m_{L_n(1)} \\ &= \frac{2 + 4qr}{2 + r} K(\tau_q^{n-1}(G)) + (|N_{n-1}| - 1) \cdot \frac{2}{2 + r} \\ &\quad + \frac{(1 + 2q)^{n-1} + 1}{2} |E_0| - |N_0| \\ &= \frac{2 + 4qr}{2 + r} K(\tau_q^{n-1}(G)) + (1 + 2q)^{n-1}|E_0| - \left(\frac{r}{2 + r}|N_0| + \frac{2}{2 + r} \right), \end{aligned}$$

where $\lambda_k^{(i)}$ represents the eigenvalue of $L_i, i = 0, 1, \ldots, n$.

From the above recursive relation, we can obtain

$$\begin{aligned} K(\tau_q^n(G)) = 2^n \left(\frac{1 + 2qr}{2 + r} \right)^n K(G) &+ (2q + 1)^n (2 + r) \\ &\times \frac{2^n (1 + 2qr)^n - (2q + 1)^n (2 + r)^n}{(2qr - 4q - r)(2q + 1)^n (2 + r)^n} |E_0| \\ &+ (r|N_0| + 2) \frac{2^n (1 + 2qr)^n - (2 + r)^n}{(4q - 1) r (2 + r)^n}. \end{aligned}$$

□

4.2. *Multiplicative Kirchhoff index*

Definition 4.2.1. If we replace each edge of a simple connected graph G by a unit resistor, we obtain an electrical network G^* associated with G. The resistance distance r_{ij} between vertices i and j of G is equal to the effective resistance between the two corresponding vertices of G^*.[27]

Definition 4.2.2. The multiplicative Kirchhoff index of G is defined as

$$Kf^*(G) = \sum_{i<j} s_i s_j r_{ij}, \quad i, j = 1, 2, \ldots, |N_n|.$$

It is known that $Kf^*(G)$ can be expressed in terms of the spectrum $\sigma_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{|N_n|}^{(n)}\}$ of the normalized Laplacian matrix of G.[28] Thus,

$$Kf^*(G) = 2|E_0| \sum_{k=2}^{|N_0|} \frac{1}{\lambda_k},$$

where $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{|N_0|} \le 2$.

And from Ref. 28, for $n \ge 0$, it has

$$Kf^*(\tau_q^n(G)) = 2|E_n| \sum_{k=2}^{|N_n|} \frac{1}{\lambda_k^{(n)}}, \tag{19}$$

where $0 = \lambda_1^{(n)} < \lambda_2^{(n)} \le \cdots \le \lambda_{|N_n|}^{(n)} \le 2$ are the eigenvalues of L_n.

Obviously, it has

$$Kf^*(\tau_q^n(G)) = 2|E_n| K(\tau_q^n(G)). \tag{20}$$

This equation reflects the fact that, for a connected graph, the resistance distance can be related to random walks.[29]

Theorem 4.2. *The general expression of the multiplicative Kirchhoff indices* $Kf^*(\tau_q^n(G))$ *is*

$$\begin{aligned} Kf^*(\tau_q^n(G)) = {} & 2^n \left(\frac{1+2qr}{2+r}\right)^n (2q+1)^n Kf^*(G) + (4+2r) \\ & \times \frac{2^n(1+2qr)^n - (2q+1)^{2n}(2+r)^n}{(2qr-4q-r)(2q+1)^n(2+r)^n} \\ & \cdot (2q+1)^n|E_0|^2 + (2r|N_0|+4)(2q+1)^n \\ & \times \frac{2^n(1+2qr)^n - (2+r)^n}{(4q-1)r(2+r)^n}|E_0|. \end{aligned}$$

Proof. We use Eq. (20) and Theorem 4.1 to obtain

$$\begin{aligned} Kf^*(\tau_q^n(G)) &= 2|E_n| \sum_{k=2}^{|N_n|} \frac{1}{\lambda_k^{(n)}} \\ &= \frac{2+4qr}{2+r}(2q+1)Kf^*(\tau_q^{n-1}(G)) + 2(2q+1)^{2n-1}|E_0|^2 \\ &\quad - 2\left(\frac{r}{2+r}|N_0| + \frac{2}{2+r}\right)(2q+1)^n|E_0|, \end{aligned}$$

where $\lambda_k^{(i)}$ represents the eigenvalue of $L_i, i = 0, 1, \ldots, n$.

From the recursive relation above, we can obtain

$$\begin{aligned} Kf^*(\tau_q^n(G)) = 2^n\left(\frac{1+2qr}{2+r}\right)^n(2q+1)^nKf^*(G)+(4+2r) \\ \times\frac{2^n(1+2qr)^n-(2q+1)^{2n}(2+r)^n}{(2qr-4q-r)(2q+1)^n(2+r)^n} \\ \cdot\,(2q+1)^n|E_0|^2+(2r|N_0|+4)(2q+1)^n \\ \times\frac{2^n(1+2qr)^n-(2+r)^n}{(4q-1)r(2+r)^n}|E_0|. \end{aligned}$$

□

5. Simulation Example

In this section, we will give an example to demonstrate the effectiveness of the proposed approach. For $k=0$, initial graph $\tau_q^0(G)$ is considered as a 3-node complete graph, where $|N_0|=|E_0|=3$. For $n\geq 1$, $\tau_q^n(G)$ is obtained from $\tau_q^{n-1}(G)$ by performing the q-triangulation operation on $\tau_q^{n-1}(G)$. Figure 1 illustrates the first several iterations for pseudofractal networks for a particular case of $q=1$.

For $\tau_q^0(G)$, its diagonal degree matrix and adjacency matrix are, respectively, denoted as

$$S_0=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

and

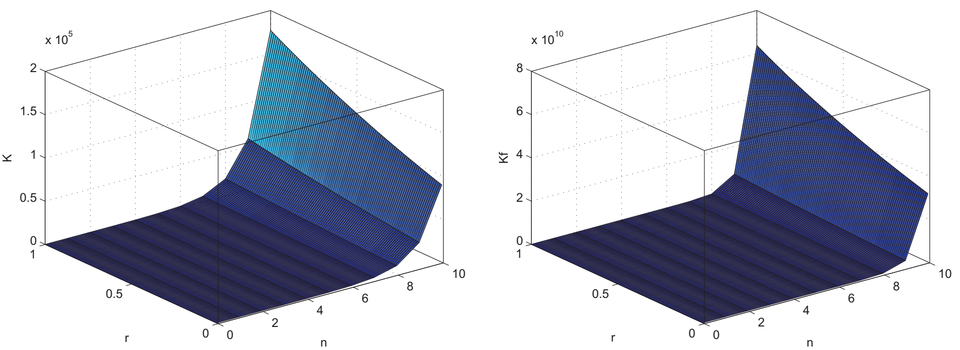
$$W_0=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}.$$

For $P_n=S_n^{-\frac{1}{2}}W_nS_n^{-\frac{1}{2}}$ or the (i,j)th entry of P_n is $P_n(i,j)=\frac{w_n(i,j)}{\sqrt{s_n(i)s_n(j)}}$, then

$$P_0=\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}.$$

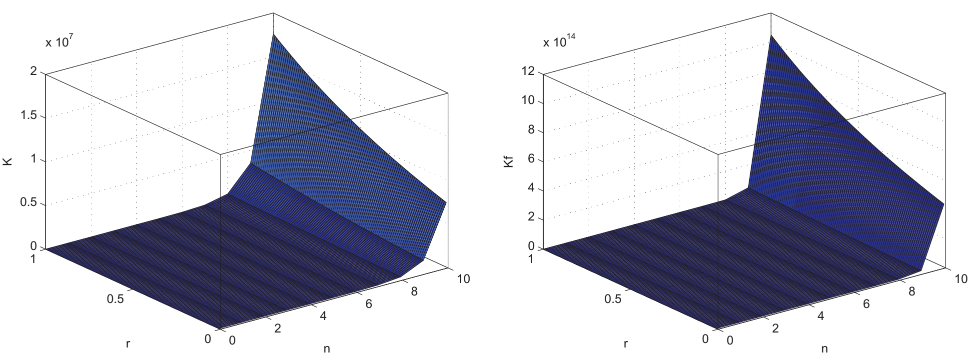
By evaluation, the eigenvalues of P_0 are 1, $-\frac{1}{2}$ and $-\frac{1}{2}$. Then, from $L_n=I_n-P_n$, the eigenvalues of P_0 are 0, $\frac{3}{2}$ and $\frac{3}{2}$. Hence, by Eqs. (18) and (19), Kemeny's constant and multiplicative Kirchhoff index for $\tau_q^0(G)$ are $K(\tau_q^0(G))=\frac{4}{3}$ and $Kf^*(\tau_q^0(G))=8$. Then, by Theorems 4.1 and 4.2, we obtain the following exact solutions to Kemeny's constant $K(\tau_q^n(G))$ and multiplicative Kirchhoff index $Kf^*(\tau_q^n(G))$ for $\tau_q^n(G)$,

$$K(\tau_q^n(G)) = \frac{2^{n+2}}{3}\left(\frac{1+2qr}{2+r}\right)^n + (2q+1)^n(6+3r) \times \frac{2^n(1+2qr)^n - (2q+1)^n(2+r)^n}{(2qr-4q-r)(2q+1)^n(2+r)^n} + (3r+2)\,\frac{2^n(1+2qr)^n - (2+r)^n}{(4q-1)r(2+r)^n}.$$



(a) (b)

Fig. 2. (a) Kemeny's constant $K(\tau_q^n(G))$ for generation $0 \le n \le 10$ when $q = 1$; (b) Multiplicative Kirchhoff index $Kf^*(\tau_q^n(G))$ for generation $0 \le n \le 10$ when $q = 1$.



(a) (b)

Fig. 3. (a) Kemeny's constant $K(\tau_q^n(G))$ for generation $0 \le n \le 10$ when $q = 1$; (b) Multiplicative Kirchhoff index $Kf^*(\tau_q^n(G))$ for generation $0 \le n \le 10$ when $q = 2$.

and

$$Kf^*(\tau_q^n(G)) = 2^{n+3}\left(\frac{1+2qr}{2+r}\right)^n (2q+1)^n + (2q+1)^n(36+18r) \times \frac{2^n(1+2qr)^n - (2q+1)^{2n}(2+r)^n}{(2qr-4q-r)(2q+1)^n(2+r)^n} + (18r+12)(2q+1)^n \frac{2^n(1+2qr)^n-(2+r)^n}{(4q-1)r(2+r)^n}.$$

As examples in the following, for $q = 1$ and $q = 2$, the simulation figures about Kemeny's constant $K(\tau_q^n(G))$ and multiplicative Kirchhoff index $Kf^*(\tau_q^n(G))$ for generation $0 \leq n \leq 10$ are, respectively, shown by Figs. 2 and 3.

6. Conclusion

In conclusion, we have considered the spectra of the normalized Laplacian matrix of $\tau_q^n(G)$ for a class of weight-driven graphs, whose strength and edge weight follow power-law distribution, which is observed in various real-world systems. We have determined all the eigenvalues and their multiplicities of the normalized Laplacian matrix for the graphs. Moreover, we have applied the obtained eigenvalues in deriving the closed-form expressions about their Kemeny's constant and multiplicative Kirchhoff index. As a simulation example, we finally provided analytical formulas for some related quantities of iterated q-triangulations for a 3-node complete graph, and obtained exact expressions for such quantities corresponding to pseudofractal networks. Furthermore, various structural and dynamical properties of a network are also relevant to the spectra of other matrices. Future works should include determining the spectra for other matrices' weighted networks.

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