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Estimating the Hausdorff dimensions of univoque sets for self-similar sets

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Abstract

An approach is given for estimating the Hausdorff dimension of the univoque set of a self-similar set. This sometimes allows us to get the exact Hausdorff dimensions of the univoque sets.

Keywords: Hausdorff dimension; Univoque set; Sets of k-codings; Self-similar sets

1. Introduction

Let $\{f_i\}_{i=1}^m$ be an iterated function system (IFS) of contractive similitudes on $\mathbb{R}^d$ defined as

$$f_i(x) = r_i R_i x + b_i, \quad i \in \Omega = \{1, \dots, m\},$$

where $0 < r_i < 1$ is the contractive ratio, R_i is a $d \times d$ orthogonal transformation and $x, b_i \in \mathbb{R}^d$. Then there exists a unique nonempty compact set $K \subseteq \mathbb{R}^d$ satisfying (cf. [9])

$$K = \bigcup_{i=1}^m f_i(K). \tag{1}$$

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The set K is called the self-similar set generated by the IFS $\{f_j\}_{j=1}^m$. The IFS $\{f_j\}_{j=1}^m$ is said to satisfy the open set condition (OSC) (cf. [9]) if there exists a non-empty bounded open set $V \subseteq \mathbb{R}^d$ such that

$$V \supseteq \bigcup_{i=1}^m f_i(V) \text{ with disjoint union on the right side.}$$

Under the open set condition, the Hausdorff dimension of K coincides with the similarity dimension, denoted by $\dim_S K$, which is the unique solution s of the equation $\sum_{j=1}^m r_j^s = 1$. For any $x \in K$, there exists a sequence $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ such that

$$x = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \bigcap_{n=1}^\infty f_{i_1} \circ \cdots \circ f_{i_n}(K).$$

Such sequence $(i_n)_{n=1}^\infty$ is called a coding of x. The attractor K defined by (1) may equivalently be defined to be the set of points in $\mathbb{R}^d$ which admit a coding, i.e., one can define a surjective projection map between the symbolic space $\Omega^{\mathbb{N}} = \{1, \ldots, m\}^{\mathbb{N}}$ and the self-similar set K by

$$\Pi((i_n)_{n=1}^\infty) := \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

A point $x \in K$ may have multiple codings. $x \in K$ is called a univoque point if it has only one coding. The set of univoque points is called the univoque set, denoted by U or U_1. Generally, for $k \in \mathbb{N}$ we set

$$U_k = \{x \in K : x \text{ has exact } k \text{ codings}\}.$$

The univoque set plays a pivotal role in studying the sets of multiple codings (cf. [4,5,10]), e.g., we have

$$\dim_H U_k \le \dim_H U \quad \text{for } k \ge 2, \tag{2}$$

since $U_k \subseteq \bigcup_{\mathbf{i}\in\Omega^*} f_{\mathbf{i}}(U)$ where, as usual, $\Omega^* = \bigcup_{n=1}^\infty \Omega^n$ and $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$ for $\mathbf{i} = i_1 \cdots i_n \in \Omega^n$. Therefore, it is crucial to find the Hausdorff dimension of the univoque set for self-similar sets. There are many papers about the Hausdorff dimension of U when K is an interval (cf. [1,2,6,7,12–14,19,20]). In geometry measure theory, the slicing problem has strong connection with the multiple representations, see [3,11,22,23]. To consider the dimension of set of points with unique representation is essentially a problem of finding the dimension of the univoque sets. Motivated by the multiple codings, Gu, Ye, and Xi [8] considered the geodesics on the higher-dimensional (bigger than 3) Sierpinski gasket, and they proved that the cardinality of the geodesics between two points in the Sierpinski gasket must be in $\{1, 2, 3, 4, 5, 6, 8\}$. For more results concerning geodesics, see [16,21,24–26].

In the present paper, we offer an approach to estimate $\dim_H U$ for general self-similar sets. Let M be a nonempty compact subset of $\mathbb{R}^d$ satisfying $f_i(M) \subseteq M$ for $1 \le i \le m$ (so $K \subseteq M$). Let (Recalling that $\Omega = \{1, \ldots, m\}$)

$$\mathcal{S}_1 = \{\mathbf{k} \in \Omega : f_{\mathbf{k}}(M) \cap f_{\mathbf{j}}(M) = \emptyset \text{ for all } \mathbf{j} \in \Omega \setminus \{\mathbf{k}\}\},$$
$$\mathcal{T}_1 = \Omega \setminus \mathcal{S}_1.$$

For positive integer i let

$$\begin{aligned} \mathcal{S}_{i+1} &= \{\mathbf{k} \in \mathcal{T}_i \times \Omega : f_{\mathbf{k}}(M) \cap f_{\mathbf{j}}(M) = \emptyset \text{ for all } \mathbf{j} \in (\mathcal{T}_i \times \Omega) \setminus \{\mathbf{k}\}\}, \\ \mathcal{T}_{i+1} &= (\mathcal{T}_i \times \Omega) \setminus \mathcal{S}_{i+1}. \end{aligned} \tag{3}$$

We have that both $\mathcal{S}_i$ and $\mathcal{T}_i$ are subsets of Ω^i, and that $\mathcal{S}_k = \mathcal{T}_k = \emptyset$ for $k > i$ if $\mathcal{T}_i = \emptyset$ for some i. Note that $\mathcal{S}_{i+1}$ may also be empty even if $\mathcal{T}_i \neq \emptyset$. Let

$$\Gamma = \bigcup_{i \geq 1} \mathcal{S}_i. \tag{4}$$

From the definition of $\mathcal{S}_i$ it follows that sets $f_{\mathbf{k}}(M), \mathbf{k} \in \Gamma$ are pairwise disjoint. It is clear that Γ becomes largest when M is taken as K. To help the readers have a better understanding of the definitions of $\mathcal{S}_i$ and $\mathcal{T}_i$, we provide some examples in Section 3. In Example 3.1, we offer the sets $S_i, T_i, 1 \leq i \leq 3$. Other $S_i, T_i, i \geq 4$ can be obtained inductively. An $\mathbf{i} \in \Omega^{\mathbb{N}}$ is said to begin with Γ if $\mathbf{i}|k \in \Gamma$ for some $k \in \mathbb{N}$. Let

$$V = \{\mathbf{i} \in \Omega^{\mathbb{N}} : \mathbf{i} \text{ does not begin with } \Gamma\}. \tag{5}$$

Each $\mathbf{k} \in \Gamma$ is a word of finite length, i.e., $\mathbf{k} \in \Omega^*$. The concatenation of an infinite sequence from Γ is just an element of $\Omega^{\mathbb{N}}$. Thus the product set $\Gamma^{\mathbb{N}}$ can be identified as the subset of $\Omega^{\mathbb{N}}$ and so $\Pi\left(\Gamma^{\mathbb{N}}\right) \subseteq \Pi\left(\Omega^{\mathbb{N}}\right) = K$. In this paper we obtain

Theorem 1.1. *Let Γ and V be defined by (4) and (5) respectively. Then*

$$\dim_H U = \max\{\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right), \dim_H \Pi(V \cap \Pi^{-1}(U))\}.$$

Let s be determined by

$$\sum_{\mathbf{i} \in \Gamma} r_{\mathbf{i}}^s = 1.$$

Then we have $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s$ which will be proved in Lemma 2.1. Hence

Corollary 1.2. *We have $\dim_H U \geq s$ and the equality holds if and only if $\dim_H \Pi(V \cap \Pi^{-1}(U)) \leq s$.*

The OSC plays an important role in determining the Hausdorff dimension of a self-similar set. Let us recall that K is generated by the IFS $\{f_i\}_{i=1}^m$ in (1). The following fact is obvious:

$$0 < \mathcal{H}^s(U) = \mathcal{H}^s(K) < \infty \quad \text{if } \{f_i\}_{i=1}^m \text{ satisfies the OSC,} \tag{6}$$

where s is given by $\sum_{i=1}^m r_i^s = 1$. In fact, we have $U = K \setminus \left(K^* \cup \bigcup_{\mathbf{i} \in \Omega^*} f_{\mathbf{i}}(K^*)\right)$ with $K^* = \bigcup_{i \neq j}(f_i(K) \cap f_j(K))$ and the OSC implies that $\mathcal{H}^s(f_i(K) \cap f_j(K)) = 0$ for any $i \neq j$ (see [18]).

From (6) it follows that $\dim_H U = \dim_H K = \dim_S K$ if the IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition. We shall show that under some extra condition the inverse is also true. An IFS $\{f_i\}_{i=1}^m$ is said to have an exact overlap if there exist distinct $\mathbf{i}, \mathbf{j} \in \Omega^*$ such that $f_{\mathbf{i}} = f_{\mathbf{j}}$. The notion of "generalized finite type condition" appeared in the following lemma which was posed by Lau and Ngai in [15]. This condition is a little complicated. It is weaker than the open set condition. Roughly speaking, when the IFS has some overlaps that are not so complex, i.e. the overlaps in each level, when generating the overlapping self-similar set, appear regularly. Therefore, we are able to define the so-called "type". The key of generalized finite type condition is that there are only finite "types" in all the levels. From the symbolic perspective, we transform the full shift into some sofic shift or subshift of finite type. Subsequently, we are allowed to calculate the dimension of the associated attractor. With the definition of generalized finite type condition, we have the following result.

Theorem 1.3. *Let K be the self-similar set generated by the IFS $\{f_i\}_{i=1}^m$. Suppose that $\{f_i\}_{i=1}^m$ satisfies the generalized finite type condition. Then $\{f_i\}_{i=1}^m$ satisfies the open set condition if and only if $\dim_H U = \dim_S K$.*

This paper is organized as follows. In Section 2, we give the proofs of Theorems 1.1 and 1.3. Section 3 is devoted to some examples.

2. Proof of Theorems 1.1 and 1.3

Denote by $\mathbf{ij}$ the concatenation of $\mathbf{i}, \mathbf{j} \in \Omega^*$ and $\mathbf{i}^k$ stands for the concatenation of $\mathbf{i}$ with itself k times. By $|\mathbf{i}|$ we denote the length of $\mathbf{i} \in \Omega^*$. For $\mathbf{i} = i_1 \cdots i_k \in \Omega^*$ we denote by $[\mathbf{i}]$ the cylinder set based on $\mathbf{i}$, i.e., $[\mathbf{i}] = \{(x_i) \in \Omega^{\mathbb{N}} :\ x_i = i_i \text{ for } 1 \le i \le k\}$. For an $\mathbf{i} = (i_k)_{k\ge1} \in \Omega^{\mathbb{N}}$ let $\mathbf{i}|p = i_1 \cdots i_p$. For $\mathbf{i} = i_1 \cdots i_k \in \Omega^*$ denote $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_k}$ and $r_{\mathbf{i}} = \prod_{\ell=1}^k r_{i_\ell}$.

Lemma 2.1. *Let $\Gamma \subseteq \Omega^*$ be given by (4). Then $\Pi\left(\Gamma^{\mathbb{N}}\right) \subseteq U$ and $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s$ where s is determined by $\sum_{\mathbf{i}\in\Gamma} r_{\mathbf{i}}^s = 1$.*

Proof. Note that by the definition of Γ we have
(I) $[\mathbf{i}], \mathbf{i} \in \Gamma$ are pairwise disjoint;
(II) $f_{\mathbf{i}}(K) \cap \Pi\left(\Omega^{\mathbb{N}} \setminus [\mathbf{i}]\right) = \emptyset$ for each $\mathbf{i} \in \Gamma$.

First we show that $\Pi\left(\Gamma^{\mathbb{N}}\right) \subseteq U$. For an $x \in \Pi\left(\Gamma^{\mathbb{N}}\right)$ let $x = \Pi((x_k)_{k\ge1})$ with $(x_k)_{k\ge1} \in \Gamma^{\mathbb{N}}$. Suppose that $(y_k)_{k\ge1} \in \Omega^{\mathbb{N}}$ satisfies that $x = \Pi((y_k)_{k\ge1})$. We claim that $(y_k)_{k\ge1} = (x_k)_{k\ge1}$. On the contrary, let ℓ be the smallest integer such that $y_\ell \ne x_\ell$. Let $(x_k)_{k\ge1} = (\mathbf{i}_k)_{k\ge1}$ with $\mathbf{i}_k \in \Gamma$. Let γ be smallest integer such that $\ell \le |\mathbf{i}_1 \cdots \mathbf{i}_\gamma|$. Then

$$\Pi((x_k)_{k\ge\delta}) = \Pi((y_k)_{k\ge\delta}) \text{ where } \delta = |\mathbf{i}_1 \cdots \mathbf{i}_\gamma| - |\mathbf{i}_\gamma| + 1.$$

However, $\Pi((x_k)_{k\ge\delta}) \in f_{\mathbf{i}_\gamma}(K), \mathbf{i}_\gamma \in \Gamma$ and $(y_k)_{k\ge\delta} \notin [\mathbf{i}_\gamma]$. This leads to a contradiction to the fact $f_{\mathbf{i}_\gamma}(K) \cap \Pi\left(\Omega^{\mathbb{N}} \setminus [\mathbf{i}_\gamma]\right) = \emptyset$.

In what follows we prove that $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s$. If Γ is finite, then $\Pi\left(\Gamma^{\mathbb{N}}\right)$ is a self-similar set generated by the IFS $\{f_{\mathbf{i}} :\ \mathbf{i} \in \Gamma\}$. This IFS satisfies the OSC since $K \supseteq \bigcup_{\mathbf{i}\in\Gamma} f_{\mathbf{i}}(K)$ with disjoint union. Thus $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s$.

In the following we assume that Γ is infinite. Denote $\Gamma_k = \{\mathbf{i} \in \Gamma : |\mathbf{i}| \le k\}$, $k \in \mathbb{N}$. Then Γ_k is finite (we assume k is big enough such that $\Gamma_k \ne \emptyset$). Thus

$$\dim_H \Pi\left(\Gamma_k^{\mathbb{N}}\right) = s_k \quad \text{where} \sum_{\mathbf{i}\in\Gamma_k} r_{\mathbf{i}}^{s_k} = 1.$$

Therefore, $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) \ge \sup_k s_k = \lim_{k\to\infty} s_k = s$ where the last equality can be obtained by the equation $\sum_{\mathbf{i}\in\Gamma} r_{\mathbf{i}}^s = 1$ and $\Gamma = \bigcup_{k\ge1} \Gamma_k$.

Arbitrarily fix a $t > s$. For any $\delta > 0$ one can take a big integer n such that each set in $\{f_{\mathbf{i}}(K) : \mathbf{i} \in \Gamma^n\}$ has diameter less than δ. Note that

$$\sum_{\mathbf{i}\in\Gamma^n} |f_{\mathbf{i}}(K)|^t = |K|^t \left(\sum_{\mathbf{i}\in\Gamma} r_{\mathbf{i}}^t\right)^n \le |K|^t,$$

which implies that $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) \le t$. $\square$

Proof of Theorem 1.1. Note that

$$\Omega^{\mathbb{N}} = V \cup V^c = V \cup \Gamma^{\mathbb{N}} \cup \{\mathbf{ui} : \mathbf{u} \in \Gamma^*,\ \mathbf{i} \in V\}.$$

Thus

$$\Pi^{-1}(U) = \Gamma^{\mathbb{N}} \cup (V \cap \Pi^{-1}(U)) \cup \{\mathbf{uj} : \mathbf{u} \in \Gamma^*,\ \mathbf{j} \in V \cap \Pi^{-1}(U)\},$$

which implies the desired result. $\square$

Now we turn to proving Theorem 1.3. We need following

Lemma 2.2 (*[28, Theorem 2.1]*)**.** *An IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition if and only if it is of general finite type and has no exact overlaps.*

Proof of Theorem 1.3. The necessity follows from (6). We now prove the sufficiency. Note that $\{f_i\}_{i=1}^m$ is of general finite type. Thus by Lemma 2.2 it suffices to show that the IFS $\{f_i\}_{i=1}^m$ has no exact overlaps. Otherwise, there exist distinct $\mathbf{i}, \mathbf{j} \in \Omega^*$ such that $f_{\mathbf{i}} = f_{\mathbf{j}}$. Let K_1 be the self-similar set generated by the IFS $\{f_{\mathbf{k}} : \mathbf{k} \in \Omega^{|\mathbf{i}|}$ and $\mathbf{k} \neq \mathbf{i}\}$. Then $\dim_H K_1 \le \dim_H K < \dim_S K$. On the other hand, for any $x \in U$ its unique coding cannot contain the block $\mathbf{i}$ and so $x \in K_1$. Thus, $\dim_H U \le \dim_H K_1 \le \dim_H K < \dim_S K$, a contradiction! $\square$

3. Examples

The result in the following example was obtained in [27] by giving a lexicographical characterization of the unique codings. Now we reprove it by applying Theorem 1.1, which provides a quite different way from that in [27].

Example 3.1 (*see [27]*)**.** Let K be the self-similar set generated by the IFS

$$\{f_1(x) = \rho x,\ \ f_2(x) = \rho x + \rho,\ \ f_3(x) = \rho x + 1\} \text{ where } 0 < \rho < (3 - \sqrt{5})/2.$$

Then $\dim_H U = \frac{\log \lambda}{-\log \rho}$, where $\lambda \approx 2.3247$ is the appropriate solution of

$$x^3 - 3x^2 + 2x - 1 = 0.$$

Proof. First one can check that $f_1 \circ f_3 = f_2 \circ f_1$. Take $M = [0, (1-\rho)^{-1}]$. Then

$$f_1(M) \cap f_2(M) = [0, \rho/(1-\rho)] \cap [\rho, (2\rho - \rho^2)/(1-\rho)] = [\rho, \rho/(1-\rho)]$$

and

$$f_1(M) \cap f_3(M) = f_2(M) \cap f_3(M) = \emptyset.$$

(see Fig. 1).

By the definitions of $\mathcal{S}_i$ and $\mathcal{T}_i$, it is easy to check that

$$\mathcal{S}_1 = \{3\}, \mathcal{S}_2 = \{23\}, \mathcal{S}_3 = \{123, 223\},$$

and

$$\mathcal{T}_1 = \{1, 2\}, \mathcal{T}_2 = \{11, 12, 13, 21, 22\},$$

$$\mathcal{T}_3 = \{111, 112, 113, 121, 122, 131, 132, 133, 211, 212, 213, 221, 222\}.$$

For $k \ge 3$ the set $\mathcal{S}_k$ becomes a bit complicated. However, it is not so difficult to find out that $|\mathcal{S}_k| = k - 1$ by noting that $f_1 \circ f_3 = f_2 \circ f_1$, where $|A|$ denotes the cardinality of set A. Let

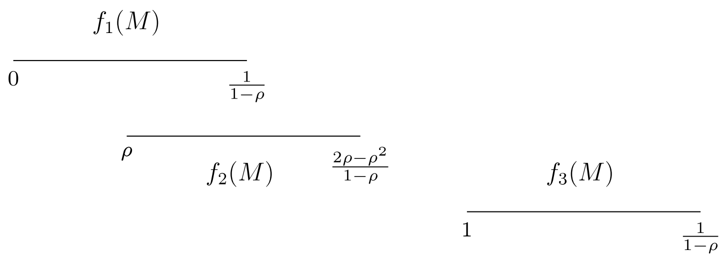

Fig. 1. The location of $f_i(M), i = 1, 2, 3$.

$\Gamma = \bigcup_{k\geq 1} \mathcal{S}_k$. Thus by Lemma 2.1

$$\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s, \text{ where } \rho^s + \rho^{2s} + \sum_{k=3}^{\infty}(k-1)\rho^{ks} = 1.$$

It is an easy exercise to check that $s = \frac{\log \lambda}{-\log \rho}$ where $\lambda \approx 2.3247$ is the appropriate solution of $x^3 - 3x^2 + 2x - 1 = 0$.

Now we show that $\dim_H \Pi(V) \leq s$, where $V = \{\mathbf{i} \in \Omega^{\mathbb{N}} : \mathbf{i} \text{ does not begin with } \Gamma\}$ is as that in Theorem 1.1. By the geometric structure of K one can see that for each positive integer k, the set $\Pi(V)$ can be covered by 2^k many number of intervals of length $\rho^k(\rho-1)^{-1}$. Thus

$$\mathcal{H}^s_{\rho^k(\rho-1)^{-1}}(\Pi(V)) \leq (1-\rho)^{-s}(2\rho^s)^k \to 0 \text{ as } k \to \infty$$

since $2\rho^s < 1$. Thus, $\dim_H U = \frac{\log \lambda}{-\log \rho}$ by Theorem 1.1. $\square$

Example 3.2. Take $0 < \lambda < (3-\sqrt{5})/2$. Let K be the self-similar set generated by the IFS $\{f_1, \ldots, f_5\}$ where

$$f_i(x, y) = (\lambda x, \lambda y) + (a_i, b_i)$$

with $(a_1, b_1) = (0, 0)$, $(a_2, b_2) = (1-\lambda, 0)$, $(a_3, b_3) = (1-\lambda, 1-\lambda)$, $(a_4, b_4) = (0, 1-\lambda)$ and $(a_5, b_5) = (\lambda(1-\lambda), (1-\lambda)^2)$. Then $\dim_H U = s \approx \dfrac{\log 4.61347}{-\log \lambda}$ where $\lambda^{3s} - 2\lambda^{2s} + 5\lambda^s - 1 = 0$.

Proof. First one can check that $f_4 \circ f_2 = f_5 \circ f_4$. Among the squares $f_i([0, 1]^2)$, $1 \leq i \leq 5$, only $f_4([0, 1]^2) \cap f_5([0, 1]^2) \neq \emptyset$ (see Fig. 2). Thus $\mathcal{S}_1 = \{1, 2, 3\}$ and $\mathcal{S}_2 = \{41, 43, 51, 52, 53\}$. As in above example, for $k \geq 3$ the set $\mathcal{S}_k$ becomes a bit complicated. However, it is not so difficult to find out that $|\mathcal{S}_k| = 3k-1$ by noting that $f_4 \circ f_2 = f_5 \circ f_4$. Let $\Gamma = \bigcup_{k\geq 1} \mathcal{S}_k$. Thus by Lemma 2.1 we have $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s \approx \dfrac{\log 4.61347}{-\log \lambda}$ where

$$3\lambda^s + 5\lambda^{2s} + \sum_{k=3}^{\infty}(3k-1)\lambda^{ks} = 1,$$

which is equivalent to $\lambda^{3s} - 2\lambda^{2s} + 5\lambda^s - 1 = 0$.

Now we show that $\dim_H(\Pi(V \cap \Pi^{-1}(U))) \leq s$, where

$$V = \{\mathbf{i} \in \Omega^{\mathbb{N}} : \mathbf{i} \text{ does not begin with } \Gamma\}$$

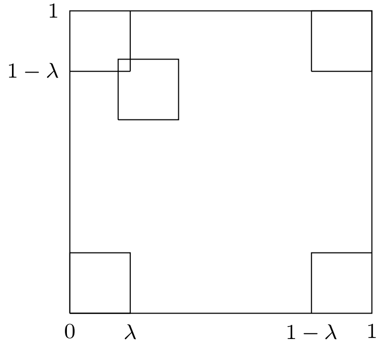

Fig. 2. The locations of squares $f_i([0,1]^2), i = 1, 2, 3, 4, 5$.

is as that in Theorem 1.1. By the geometric structure of K one can see that for each positive integer k, the set $(\Pi(V \cap \Pi^{-1}(U)))$ can be covered by 2^k many number of squares with diameter $\sqrt{2}\lambda^k$. Thus

$$\mathcal{H}^s_{\sqrt{2}\lambda^k}(K_\alpha) \leq 2^k \sqrt{2}^s \lambda^{sk} \to 0 \text{ as } k \to \infty$$

since $2\lambda^s < 1$. Thus, $\dim_H U = s$ by Theorem 1.1. □

In the above we change the map f_5 by letting

$$(a_5, b_5) = (\lambda - \lambda^{u+1}, 1 - 2\lambda + \lambda^{u+1}) \text{ with } u \in \mathbb{N},$$

where we require that $\lambda^{u+1} - 3\lambda + 1 > 0$. Then $\dim_H U$ can be also obtained by the same way as in Example 3.2 and so $\dim_H U_k, k \geq 2$ can be obtained as well. In fact, we have

Example 3.3. Suppose that $\lambda \in (0,1), u \in \mathbb{N}$ satisfy $\lambda^{u+1} - 3\lambda + 1 > 0$. Let K be the self-similar set generated by the IFS $\{f_1, \ldots, f_5\}$ where

$$f_i(x, y) = (\lambda x, \lambda y) + (a_i, b_i)$$

with $(a_1, b_1) = (0, 0)$, $(a_2, b_2) = (1 - \lambda, 0)$, $(a_3, b_3) = (1 - \lambda, 1 - \lambda)$, $(a_4, b_4) = (0, 1 - \lambda)$ and $(a_5, b_5) = (\lambda - \lambda^{u+1}, 1 - 2\lambda + \lambda^{u+1})$. Then

$$\dim_H U_{k+1} = \dim_H U \text{ for any } k \in \mathbb{N}.$$

Proof. By (2) we only need to show that $\dim_H U_{k+1} \geq \dim_H U$. This will be done by showing

$$f_{42^{uk}1}(U) \subseteq U_{k+1} \text{ for each } k \geq 1.$$

Now arbitrarily fix a point $c \in U$ with the unique coding (c_i). We prove that $f_{42^{uk}1}(c) \in U_{k+1}, k \geq 1$ by induction.

Let $k = 1$. Note that $x_1 = f_{42^u 1}(c) = f_{54^u 1}(c) \in f_{42^u}([0,1]^2) = f_{54^u}([0,1]^2)$ and

$$f_{\mathbf{i}}([0,1]^2) \cap f_{42^u}([0,1]^2) = \emptyset \text{ for all } \mathbf{i} \in \{1, 2, 3, 4, 5\}^{u+1} \setminus \{42^u, 54^u, 54^{u-1}5\}.$$

Hence any coding (d_i) of x_1 has to begin with 42^u, 54^u or $54^{u-1}5$. We claim that (d_i) cannot begin with $54^{u-1}5$. Otherwise, we have $f_{41}(c) = f_{51}(c) \in f_{41}([0,1]^2) \cap f_{51}([0,1]^2) = \emptyset$. On the other hand, we have $\pi(\sigma^{u+1}(d_i)) = \pi(1(c_i)) = f_1(c) \in U$ where σ is the left shift on $\Omega^{\mathbb{N}}$. Thus, (d_i) has to be $42^u 1(c_i)$ or $54^u 1(c_i)$, i.e., $x_1 \in U_2$.

Suppose that $x_k = f_{42^{uk}1}(c) = \pi(42^{uk}1(c_i)) \in U_{k+1}$. Let (d_i) be a coding of $x_{k+1} := f_{42^{u(k+1)}1}(c)$. As before we know that (d_i) has to begin with 42^u, 54^u or $54^{u-1}5$, and so (d_i) has to begin with 42^{u-1} or 54^{u-1}. Note that

$$x_{k+1} = f_{42^u}(\pi(2^{uk}1(c_i))) = f_{54^u}(\pi(2^{uk}1(c_i))) = f_{54^{u-1}}(\pi(42^{uk}1(c_i))) = f_{54^{u-1}}(x_k).$$

For the case that (d_i) begins with 42^{u-1} we have that $(d_i) = 42^{uk}1(c_i)$ since $\pi(\sigma^u(d_i)) = \pi(2^{uk+1}1(c_i)) \in U$. For the case that (d_i) begins with 54^{u-1} we have that (d_i) has exactly $k+1$ many choices since

$$\pi((d_i)_{i>u}) = \pi(\sigma^u(d_i)) = \pi(42^{uk}(c_i)) = x_k \in U_{k+1}.$$

Hence we complete the proof. $\square$

In the last example we try to describe Γ by a way which was developed in [15,17].

Example 3.4. Let K be the self-similar set generated by the IFS

$$\left\{ f_1(x) = \frac{x}{4},\ f_2(x) = \frac{x}{4} + \frac{9}{17},\ f_3(x) = \frac{x+3}{4} \right\}$$

Then $\dim_H U = s$, where s is the unique solution of the following equation:

$$\frac{1}{4^s} + \frac{1}{4^{2s}} + \sum_{n=2}^{\infty}(a_n + c_n)\frac{1}{4^{(n+1)s}} = 1,$$

where a_n, c_n for $n \geq 2$ are determined by

$$\begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{7}$$

Proof. We take $M = [0, 1]$ (one can check that $f_i(M) \subseteq M$ for $i \in \Omega := \{1, 2, 3\}$) and label it by T_1. Its offspring are

$$f_1(M) = [0, 1/4], \quad f_2(M) = [9/17, 53/68] \text{ and } \quad f_3(M) = [3/4, 1].$$

Then (see Fig. 3)

$$\mathcal{S}_1 = \{1\} \text{ and } \mathcal{T}_1 = \{2, 3\}.$$

Note that the offspring of $f_1(M)$ have the same geometric location as the offspring of M. So $f_1(M)$ is labeled by T_1 as well. We label $f_2(M)$ and $f_3(M)$ by T_2 and T_3, respectively. Thus one can simply denote M and its offspring as follows:

$$(M, T_1) \to (f_1(M), T_1) + (f_2(M), T_2) + (f_3(M), T_3).$$

Now let us calculate $f_{\mathbf{i}}(M) : \mathbf{i} \in \mathcal{T}_1 \times \Omega$:

$$f_{21}(M) = \left[\frac{9}{17}, \frac{9}{17} + \frac{1}{16}\right], \qquad f_{22}(M) = \left[\frac{45}{68}, \frac{45}{68} + \frac{1}{16}\right],$$
$$f_{23}(M) = \left[\frac{195}{272}, \frac{195}{272} + \frac{1}{16}\right]$$
$$f_{31}(M) = \left[\frac{3}{4}, \frac{3}{4} + \frac{1}{16}\right], \qquad f_{32}(M) = \left[\frac{60}{68}, \frac{60}{68} + \frac{1}{16}\right], \quad f_{33}(M) = \left[\frac{15}{16}, 1\right].$$

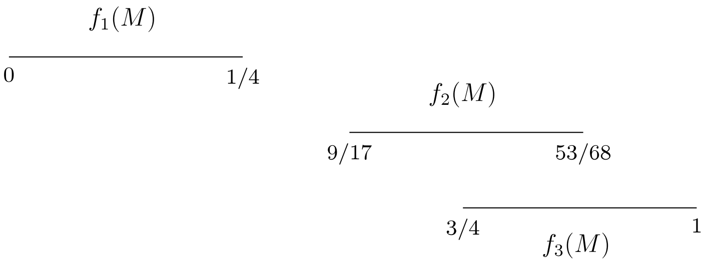

Fig. 3. The location of $f_i(M), i = 1, 2, 3$.

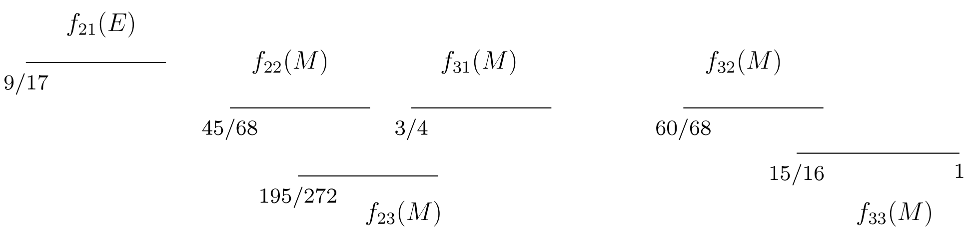

Fig. 4. The location of $f_{\mathbf{i}}(M), \mathbf{i} \in \mathcal{T}_1 \times \Omega$.

Thus we have (see Fig. 4)

$$\mathcal{S}_2 = \{21\} \text{ and } \mathcal{T}_2 = (\mathcal{T}_1 \times \Omega) \setminus \mathcal{S}_2 = \{22, 23, 31, 32, 33\}.$$

By the same argument as above the offspring $f_{21}(M)$ of $f_2(M)$ has label T_1, while the other two offspring $f_{22}(M)$, $f_{23}(M)$ of $f_2(M)$ will obtain new labels T_4, T_5, respectively. This can be simply denoted by

$$(f_2(M), T_2) \to (f_{21}(M), T_1) + (f_{22}(M), T_4) + (f_{23}(M), T_5). \tag{8}$$

Similarly, for the $f_3(M)$ and its offspring we have

$$(f_3(M), T_3) \to (f_{31}(M), T_6) + (f_{32}(M), T_2) + (f_{33}(M), T_3). \tag{9}$$

Since one knows what will happen for the offspring of $f_{32}(M)$ and $f_{33}(M)$, let us continue to calculate the offspring of $f_{\mathbf{i}}(M) : \mathbf{i} \in \mathcal{T}_2 \setminus \{32, 33\} = \{22, 23, 31\}$:

$$f_{221}(M) = \left[\frac{45}{68}, \frac{45}{68} + \frac{1}{64}\right], \qquad f_{222}(M) = \left[\frac{189}{272}, \frac{189}{272} + \frac{1}{64}\right],$$

$$f_{223}(M) = \left[\frac{771}{1088} \frac{771}{1088} + \frac{1}{64}\right]$$

$$f_{231}(M) = \left[\frac{195}{272}, \frac{195}{272} + \frac{1}{64}\right], \qquad f_{232}(M) = \left[\frac{51}{68}, \frac{51}{68} + \frac{1}{64}\right],$$

$$f_{233}(M) = \left[\frac{831}{1088}, \frac{831}{1088} + \frac{1}{64}\right]$$

$$f_{311}(M) = \left[\frac{3}{4}, \frac{3}{4} + \frac{1}{64}\right], \qquad f_{312}(M) = \left[\frac{213}{272}, \frac{213}{272} + \frac{1}{64}\right],$$

$$f_{313}(M) = \left[\frac{51}{64}, \frac{52}{64}\right].$$

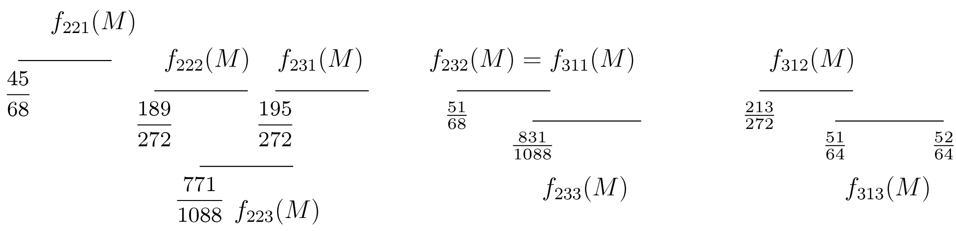

Fig. 5. The location of $f_{\mathbf{i}}(M), \mathbf{i} \in (\mathcal{T}_2 \setminus \{32, 33\}) \times \Omega$.

Thus we have (see Fig. 5)

$$\begin{aligned} &(f_{22}(M), T_4) \to (f_{221}(M), T_1) + (f_{222}(M), T_4) + (f_{223}(M), T_5) \\ &(f_{23}(M), T_5) \to (f_{231}(M), T_6) + (f_{232}(M), T_2) + (f_{233}(M), T_3) \\ &(f_{31}(M), T_6) \to (f_{311}(M), T_2) + (f_{312}(M), T_2) + (f_{313}(M), T_3) \end{aligned} \tag{10}$$

It is important to notice that no more labels occur in the above expression. Note that we have $f_{232} = f_{311}$. Thus $f_{232}(M)$ and $f_{311}(M)$ contribute nothing to Γ. Therefore, we replace (10) by

$$\begin{aligned} &(f_{22}(M), T_4) \to (f_{221}(M), T_1) + (f_{222}(M), T_4) + (f_{223}(M), T_5) \\ &(f_{23}(M), T_5) \to (f_{231}(M), T_6) + (f_{233}(M), T_3) \\ &(f_{31}(M), T_6) \to (f_{312}(M), T_2) + (f_{313}(M), T_3) \end{aligned} \tag{11}$$

It follows from (8), (9) and (11) that only T_2 and T_4 have contribution to Γ. Together with (3) one knows that the cardinality of $\mathcal{S}_{n+1}$ ($n \geq 2$) equals the number of T_2 and T_4 occurring in the nth generation offspring. By a_n, b_n, c_n, d_n and e_n we denote the number of T_2, T_3, T_4, T_5 and T_6 occurring in the nth generation offspring. By (8), (9) and (11) we have

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} := A \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{pmatrix}, \quad n \geq 2.$$

By (8) and (9) we have

$$a_2 = b_2 = c_2 = d_2 = e_2 = 1,$$

and so (7) is obtained. Therefore, we have $\dim_H \Pi\left(\Gamma^{\mathbb{N}}\right) = s$, where s is the unique solution of the following equation:

$$1 = \frac{1}{4^s} + \frac{1}{4^{2s}} + \sum_{n=2}^{\infty} |\mathcal{S}_{n+1}| \frac{1}{4^{(n+1)s}} = \frac{1}{4^s} + \frac{1}{4^{2s}} + \sum_{n=2}^{\infty} (a_n + c_n) \frac{1}{4^{(n+1)s}}.$$

In what follows we will show that $\dim_H \Pi(V \cap \Pi^{-1}(U)) \leq s$. One can check that the spectral radius of A is about $\lambda \approx 2.2775$. We claim that $\lambda < 4^s$. In fact, we have $4^s > 4^t \approx 2.4693$ where t is determined by

$$\frac{1}{4^t} + \frac{1}{4^{2t}} + \sum_{n=2}^{5} (a_n + c_n) \frac{1}{4^{(n+1)t}} = \frac{1}{4^t} + \frac{1}{4^{2t}} + \frac{2}{4^{3t}} + \frac{4}{4^{4t}} + \frac{9}{4^{5t}} + \frac{21}{4^{6t}} = 1.$$

Note that

$$\lim_{n\to\infty} \mathcal{H}^s_{4^{-n-2}}(\Pi(V \cap \Pi^{-1}(U))) \leq \overline{\lim_{n\to\infty}} (a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} + e_{n+1}) 4^{(-n-2)s} < \infty,$$

where the last inequality holds since all the $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}$ are bounded by $c\lambda^n$ for some $c > 0$, and the fact $\lambda < 4^s$. $\square$

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