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Multiple expansions of real numbers with digits set $\{0, 1, q\}$

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Abstract
For $q > 1$ we consider expansions in base q with digits set $\{0, 1, q\}$. Let $\mathcal{U}_q$ be the set of points which have a unique q-expansion. For $k = 2, 3, \ldots, \aleph_0$ let $\mathcal{B}_k$ be the set of bases $q > 1$ for which there exists x having precisely k different q-expansions, and for $q \in \mathcal{B}_k$ let $\mathcal{U}_q^{(k)}$ be the set of all such x's which have exactly k different q-expansions. In this paper we show that

$$\mathcal{B}_{\aleph_0} = [2, \infty) \quad \text{and} \quad \mathcal{B}_k = (q_c, \infty) \quad \text{for any} \quad k \geq 2,$$

where $q_c \approx 2.32472$ is the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Moreover, we show that for any integer $k \geq 2$ and any $q \in \mathcal{B}_k$ the Hausdorff dimensions of $\mathcal{U}_q^{(k)}$ and $\mathcal{U}_q$ are the same, i.e.,

$$\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q \quad \text{for any} \quad k \geq 2.$$

Finally, we conclude that the set of points having a continuum of q-expansions has full Hausdorff dimension.

Keywords Unique expansion · Multiple expansion · Countable expansion · Hausdorff dimension

Mathematics Subject Classification Primary 11A63; Secondary 10K50 · 11K55 · 37B10

1 Introduction

Expansions in non-integer bases were pioneered by Rényi [18] and Parry [16]. Unlike integer base expansions, for a given $\beta \in (1, 2)$, it is well-known that typically a real number $x \in I_\beta := [0, 1/(\beta - 1)]$ has a continuum of β-expansions with digits set $\{0, 1\}$ (cf. [2,19]), i.e., for Lebesuge almost every $x \in I_\beta$ there exist a continuum of zero-one sequences (x_i) such that $x = \sum_{i=1}^{\infty} x_i/\beta^i$. However, there still exist $x \in I_\beta$ having a unique β-expansion (cf. [5,10,13]). Denote by $\mathcal{U}_\beta$ the set of all $x \in I_\beta$ with a unique β-expansion. De Vries and Komornik [3] investigated the topological properties of $\mathcal{U}_\beta$. Komornik et al. [12] considered the Hausdorff dimension of $\mathcal{U}_\beta$, and concluded that the dimension function $\beta \mapsto \dim_H \mathcal{U}_\beta$

Dedicated to Michel Dekking on the occasion of his 70th birthday.

Extended author information available on the last page of the article

behaves like a Devil's staircase. Interestingly, for any $k = 2, 3, \ldots$ or $\aleph_0$ Erdős et al. [6,7] showed that there exist $\beta \in (1, 2)$ and $x \in I_\beta$ such that x has precisely k different β-expansions. For more information on expansions in non-integer bases we refer to [1,21,23], and the surveys [4,11,20].

In this paper we consider expansions with digits set $\{0, 1, q\}$. Given $q > 1$, the infinite sequence (d_i) is called a *q-expansion* of x, if

$$x = ((d_i))_q := \sum_{i=1}^{\infty} \frac{d_i}{q^i}, \quad d_i \in \{0, 1, q\} \quad \text{for all } i \geq 1.$$

We emphasize that the *digits set* $\{0, 1, q\}$ also depends on the base q.

For $q > 1$ let E_q be the set of points which have a q-expansion. Then E_q is the attractor of the *iterated function system* (IFS)

$$\phi_d(x) = \frac{x+d}{q}, \quad d \in \{0, 1, q\}.$$

So, E_q is the non-empty compact set satisfying $E_q = \bigcup_{d \in \{0,1,q\}} \phi_d(E_q)$ (cf. [8]). Observe that $\phi_0(E_q) \cap \phi_1(E_q) \neq \emptyset$ for any $q > 1$. Then E_q is a *self-similar set with overlaps*. Ngai and Wang [15] gave the Hausdorff dimension of E_q:

$$\dim_H E_q = \frac{\log q^*}{\log q} \quad \text{for any} \quad q > q^*, \tag{1.1}$$

where $q^* = (3 + \sqrt{5})/2$. Yao and Li [22] considered all possible IFSs generating the set E_q. Zou et al. [24] considered the set of points in E_q which have a unique q-expansion. In this paper, we investigate the set of points in E_q having multiple q-expansions.

For $k = 1, 2, \ldots, \aleph_0$ or $2^{\aleph_0}$, let

$$\mathcal{B}_k := \left\{q \in (1, \infty) : \exists\, x \in E_q \text{ with precisely } k \text{ different } q\text{-expansions}\right\}.$$

Accordingly, for $q \in \mathcal{B}_k$ let

$$\mathcal{U}_q^{(k)} := \left\{x \in E_q : x \text{ has precisely } k \text{ different } q\text{-expansions}\right\}.$$

For simplicity, we write $\mathcal{U}_q := \mathcal{U}_q^{(1)}$ for the set of $x \in E_q$ having a unique q-expansion, and denote by $\mathcal{U}_q'$ the set of all q-expansions corresponding to elements of $\mathcal{U}_q$.

In this paper we will describe the sizes of the sets $\mathcal{B}_k$ and $\mathcal{U}_q^{(k)}$. Our first result is on the set $\mathcal{B}_k$ for $k = 1, 2, \ldots, \aleph_0$ or $2^{\aleph_0}$. Clearly, when $k = 1$ we have $\mathcal{B}_1 = (1, \infty)$, since 0 always has a unique q-expansion for any $q > 1$. When $k = 2, 3, \ldots, \aleph_0$ or $2^{\aleph_0}$ we have the following

Theorem 1 *Let $q_c \approx 2.32472$ be the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Then*

$$\mathcal{B}_{2^{\aleph_0}} = (1, \infty), \quad \mathcal{B}_{\aleph_0} = [2, \infty), \quad \mathcal{B}_k = (q_c, \infty) \ \textit{for any} \ k \geq 2.$$

By Theorem 1 it follows that for $q \in [2, q_c]$, any $x \in E_q$ can only have a unique q-expansion, countably infinitely many q-expansions, or a continuum of q-expansions.

When $k = 1$, the following theorem for the *univoque set* $\mathcal{U}_q = \mathcal{U}_q^{(1)}$ was proven in [24].

Theorem 1.1 (i) *If $q \in (1, q_c]$, then $\mathcal{U}_q = \{0, q/(q-1)\}$.*
(ii) *If $q \in (q_c, q^*)$, then $\mathcal{U}_q$ contains a continuum of points.*
(iii) *If $q \in [q^*, \infty)$, then* $\dim_H \mathcal{U}_q = \log q_c / \log q$.

Our second result complements Theorem 1.1, and shows that there is no difference between the Hausdorff dimensions of $\mathcal{U}_q^{(k)}$ and $\mathcal{U}_q$.

Theorem 2 (i) $\dim_H \mathcal{U}_q > 0$ *if and only if* $q > q_c$.
(ii) *For any integer* $k \geq 2$ *and any* $q \in \mathcal{B}_k$ *we have* $\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q$.

As a result of Theorem 2 it follows that q_c is indeed the *critical base*, in the sense that $\mathcal{U}_q^{(k)}$ has positive Hausdorff dimension if $q > q_c$, while $\mathcal{U}_q^{(k)}$ has zero Hausdorff dimension if $q \leq q_c$. In fact, by Theorems 1 and 1.1 (i) it follows that for $q \leq q_c$ the set $\mathcal{U}_q = \{0, q/(q-1)\}$ and $\mathcal{U}_q^{(k)} = \emptyset$ for any integer $k \geq 2$.

Our final result focuses on the sizes of $\mathcal{U}_q^{(\aleph_0)}$ and $\mathcal{U}_q^{(2^{\aleph_0})}$.

Theorem 3 (i) *Let* $q \in \mathcal{B}_{\aleph_0} \backslash (q_c, q^*)$. *Then* $\mathcal{U}_q^{(\aleph_0)}$ *is countably infinite.*
(ii) *For any* $q > 1$ *we have* $\dim_H \mathcal{U}_q^{(2^{\aleph_0})} = \dim_H E_q$.

Remark 1.2 In Lemma 5.5 we prove a stronger result of Theorem 3 (ii), and show that the Hausdorff measures of $\mathcal{U}_q^{(2^{\aleph_0})}$ and E_q are the same for any $q > 1$, i.e.,

$$\mathcal{H}^s\left(\mathcal{U}_q^{(2^{\aleph_0})}\right) = \mathcal{H}^s(E_q) \in (0, \infty),$$

where $s = \dim_H E_q$.

The rest of the paper is arranged as follows. In Sect. 2 we recall some properties of unique q-expansions. The proof of Theorem 1 for the sets $\mathcal{B}_k$ will be presented in Sect. 3, and the proofs of Theorems 2 and 3 for the sets $\mathcal{U}_q^{(k)}$ will be given in Sects. 4 and 5, respectively. Finally, in Sect. 6 we give some examples and end the paper with some questions.

2 Unique expansions

In this section we recall some properties of the univoque set $\mathcal{U}_q$ from [24]. Recall that

$$q_c \approx 2.32472 \quad \text{and} \quad q^* = \frac{3+\sqrt{5}}{2} \approx 2.61803, \tag{2.1}$$

where q_c is the appropriate root of the equation $x^3 - 3x^2 + 2x - 1 = 0$. Note that for $q \in (1, q^*]$ the attractor $E_q = [0, q/(q-1)]$ is an interval. However, for $q > q^*$ the attractor E_q is a Cantor set which contains neither interior nor isolated points.

Given $q > 1$, let $\{0, 1, q\}^{\mathbb{N}}$ be the set of all infinite sequences (d_i) over the alphabet $\{0, 1, q\}$. By a word $\mathbf{c}$ we mean a finite string of digits $\mathbf{c} = c_1 \ldots c_n$ with each digit $c_i \in \{0, 1, q\}$. For two words $\mathbf{c} = c_1 \ldots c_m$ and $\mathbf{d} = d_1 \ldots d_n$, we denote by $\mathbf{cd} = c_1 \ldots c_m d_1 \ldots d_n$ their concatenation. For a positive integer k we write $\mathbf{c}^k = \mathbf{c} \cdots \mathbf{c}$ for the k-fold concatenation of $\mathbf{c}$ with itself. Furthermore, we write $\mathbf{c}^\infty = \mathbf{cc} \cdots$ the infinite periodic sequence with periodic block $\mathbf{c}$. Throughout the paper we will use lexicographical ordering $\prec, \preccurlyeq, \succ$ and $\succcurlyeq$ between sequences. More precisely, for two sequences $(c_i), (d_i) \in \{0, 1, q\}^{\mathbb{N}}$ we say $(c_i) \prec (d_i)$ or $(d_i) \succ (c_i)$ if there exists an integer $n \geq 1$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Furthermore, we say $(c_i) \preccurlyeq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$.

Recall that $\mathcal{U}_q$ is the set of points in E_q with a unique q-expansion, and $\mathcal{U}'_q$ is the set of corresponding q-expansions. Then

$$\mathcal{U}'_q = \left\{(d_i) \in \{0, 1, q\}^{\mathbb{N}} : ((d_i))_q \in \mathcal{U}_q\right\}.$$

The following lexicographical characterization of $\mathcal{U}'_q$ for $q > q^*$ was established in [24, Lemma 3.1].

Lemma 2.1 *Let $q > q^*$. Then $(d_i) \in \mathcal{U}'_q$ if and only if*

$$\begin{cases} (d_{n+i}) \prec q0^\infty & \textit{if} \quad d_n = 0, \\ (d_{n+i}) \succ 1^\infty & \textit{if} \quad d_n = 1. \end{cases}$$

To describe $\mathcal{U}'_q$ for $q \in (1, q^*]$ we need the following notation. Let

$$\alpha(q) = (\alpha_i(q))$$

be the *quasi-greedy* q-expansion of $q - 1$, i.e., the lexicographically largest q-expansion of $q - 1$ with infinitely many non-zero digits. We emphasize that $\alpha(q)$ is well-defined for $q \in (1, q^*]$. By (2.1) and a direct calculation one can verify that

$$\alpha(q_c) = q_c 1^\infty, \quad \alpha(q^*) = (q^*)^\infty. \tag{2.2}$$

Note by Theorem 1.1 that for $q \in (1, q_c]$ we have $\mathcal{U}_q = \{0, q/(q-1)\}$, and then $\mathcal{U}'_q = \{0^\infty, q^\infty\}$. So, it suffices to consider $\mathcal{U}'_q$ for $q \in (q_c, q^*]$. The following lemma was obtained in [24, Lemmas 3.1 and 3.2].

Lemma 2.2 *Let $q \in (q_c, q^*]$. Then*

$$A_q \subseteq \mathcal{U}'_q \subseteq B_q,$$

where A_q is the set of sequences $(d_i) \in \{0, 1, q\}^{\mathbb{N}}$ satisfying

$$\begin{cases} (d_{n+i}) \prec 1\alpha(q) & \textit{if} \quad d_n = 0, \\ 1^\infty \prec (d_{n+i}) \prec \alpha(q) & \textit{if} \quad d_n = 1, \\ (d_{n+i}) \succ 0q^\infty & \textit{if} \quad d_n = q, \end{cases} \tag{2.3}$$

and B_q is the set of sequences $(d_i) \in \{0, 1, q\}^{\mathbb{N}}$ satisfying the first two inequalities in (2.3).

For $q > 1$ let $\Phi : \{0, 1, q\}^{\mathbb{N}} \to \{0, 1, 2\}^{\mathbb{N}}$ be defined by

$$\Phi((d_i)) = (d'_i),$$

where $d'_i = d_i$ if $d_i \in \{0, 1\}$, and $d'_i = 2$ if $d_i = q$. Clearly, Φ is bijective and strictly increasing. The following lemma was given in [24, Lemma 3.2].

Lemma 2.3 *The map $q \to \Phi(\alpha(q))$ is strictly increasing in* $(1, q^*]$.

By (2.2) and Lemma 2.3 it follows that for any $q \in (q_c, q^*)$ we have $q1^\infty \prec \alpha(q) \prec q^\infty$.

3 Proof of Theorem 1

In this section we will investigate the set $\mathcal{B}_k$ of bases $q > 1$ in which there exists $x \in E_q$ having k different q-expansions. Excluding the trivial case for $k = 1$ that $\mathcal{B}_1 = (1, \infty)$ we consider $\mathcal{B}_k$ for $k = 2, 3, \ldots, \aleph_0$ or $2^{\aleph_0}$.

The following lemma was established in [24, Theorem 4.1] and [9, Theorem 1.1].

Lemma 3.1 *Let $q \in (1, 2)$.*

(i) *If $q \in (1, 2)$, then any $x \in E_q$ has either a unique q-expansion, or a continuum of q-expansions.*
(ii) *If $q = 2$, then any $x \in E_q$ can only have a unique q-expansion, countably infinitely many q-expansions, or a continuum of q-expansions.*

For $q > 1$ we recall that $\phi_d(x) = (x + d)/q$ for $d \in \{0, 1, q\}$. Let

$$S_q := \big(\phi_0(E_q) \cap \phi_1(E_q)\big) \cup \big(\phi_1(E_q) \cap \phi_q(E_q)\big). \tag{3.1}$$

Then S_q is associated with the *switch region*, since any $x \in S_q$ has at least two q-expansions. More precisely, any $x \in \phi_0(E_q) \cap \phi_1(E_q)$ has at least two q-expansions: one begins with the digit 0 and one begins with the digit 1. Accordingly, any $x \in \phi_1(E_q) \cap \phi_q(E_q)$ also has at least two q-expansions: one starts with the digit 1 and one starts with the digit q. We point out that the union in (3.1) is disjoint if $q > 2$. In particular, for $q > q^*$ the intersection $\phi_1(E_q) \cap \phi_q(E_q) = \emptyset$.

For $x \in E_q$ let $\Sigma(x)$ be the set of all q-expansions of x, i.e.,

$$\Sigma(x) := \Big\{(d_i) \in \{0, 1, q\}^{\mathbb{N}} : ((d_i))_q = x\Big\},$$

and denote its cardinality by $|\Sigma(x)|$.

We recall from [1] that a point $x \in S_q$ is called a *q-null infinite point* if x has an expansion $(d_i) \in \{0, 1, q\}^{\mathbb{N}}$ such that whenever

$$x_n := (d_{n+1}d_{n+2}\ldots)_q \in S_q,$$

one of the following quantities is infinity, and the other two are finite:

$$\left|\Sigma(\phi_0^{-1}(x_n))\right|, \quad \left|\Sigma(\phi_1^{-1}(x_n))\right| \quad \text{and} \quad \left|\Sigma(\phi_q^{-1}(x_n))\right|.$$

Then any q-null infinite point has countably infinitely many q-expansions.

First we consider the set $\mathcal{B}_{\aleph_0}$, which is based on the following characterization (cf. [1,23]).

Lemma 3.2 *$q \in \mathcal{B}_{\aleph_0}$ if and only if S_q contains a q-null infinite point.*

Lemma 3.3 $\mathcal{B}_{\aleph_0} = [2, \infty)$.

Proof By Lemma 3.1 we have $\mathcal{B}_{\aleph_0} \subseteq [2, \infty)$ and $2 \in \mathcal{B}_{\aleph_0}$. So, it suffices to prove $(2, \infty) \subseteq \mathcal{B}_{\aleph_0}$.

Take $q \in (2, \infty)$. Note that $0 = (0^\infty)_q$ and $q/(q-1) \in (q^\infty)_q$ belong to $\mathcal{U}_q$. We claim that

$$x = (0q^\infty)_q$$

is a q-null infinite point. Note that $(10^\infty)_q = (0q0^\infty)_q$. Then by the words substitution $10 \sim 0q$ it follows that all expansions 1^k0q^∞, $k \ge 0$, are q-expansions of x, i.e.,

$$\bigcup_{k=0}^{\infty}\left\{1^k0q^\infty\right\} \subseteq \Sigma(x).$$

This implies that $|\Sigma(x)| = \infty$. Furthermore, since $q > 2$, the union in (3.1) is disjoint. This implies

$$x = (0q^\infty)_q = (10q^\infty)_q \in \phi_0(E_q) \cap \phi_1(E_q) \backslash \phi_q(E_q).$$

Then $\phi_0^{-1}(x) = (q^\infty)_q \in \mathcal{U}_q$, $\phi_1^{-1}(x) = x$ and $\phi_q^{-1}(x) \notin E_q$, i.e.,

$$|\Sigma(\phi_0^{-1}(x))| = 1, \quad |\Sigma(\phi_1^{-1}(x))| = \infty, \quad |\Sigma(\phi_q^{-1}(x))| = 0.$$

By iteration it follows that x is a q-null infinite point. Hence, by Lemma 3.2 we have $q \in \mathcal{B}_{\aleph_0}$, and therefore $(2, \infty) \subseteq \mathcal{B}_{\aleph_0}$. □

Now we turn to describe the set $\mathcal{B}_k$. By Lemma 3.1 it follows that $\mathcal{B}_k \subseteq (2, \infty)$ for any $k \ge 2$. First we consider $\mathcal{B}_2$ and need the following

Lemma 3.4 *Let $q > 2$. Then $q \in \mathcal{B}_2$ if and only if either*

$$(0(a_i))_q = (1(b_i))_q \quad \textit{for some} \quad (a_i), (b_i) \in \mathcal{U}_q',$$

or

$$(1(c_i))_q = (q(d_i))_q \quad \textit{for some} \quad (c_i), (d_i) \in \mathcal{U}_q'.$$

Proof First we prove the necessary condition. Take $q \in \mathcal{B}_2$. Suppose $x \in E_q$ has two different q-expansions, say

$$((a_i))_q = x = ((b_i))_q.$$

Then there exists a least integer $k \ge 1$ such that $a_k \ne b_k$. Then

$$(a_k a_{k+1} \ldots)_q = (b_k b_{k+1} \ldots)_q \in S_q \quad \text{and} \quad (a_{k+i}), (b_{k+i}) \in \mathcal{U}_q'. \tag{3.2}$$

Since $q > 2$, it gives that the union in (3.1) is disjoint. Then the necessity follows by (3.2).

To prove the sufficiency, without loss of generality, we assume $(0(a_i))_q = (1(b_i))_q$ with $(a_i), (b_i) \in \mathcal{U}_q'$. Note by $q > 2$ that the union in (3.1) is disjoint. Then

$$(0(a_i))_q = (1(b_i)) \in \phi_0(E_q) \cap \phi_1(E_q) \backslash \phi_q(E_q).$$

This implies that x has exactly two different q-expansions. So, $q \in \mathcal{B}_2$. □

Recall from (2.2) that $q_c \approx 2.32472$ and $q^* = (3 + \sqrt{5})/2$ admit the quasi-greedy expansions $\alpha(q_c) = q_c1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. In the following lemma we describe the set $\mathcal{B}_2$.

Lemma 3.5 $\mathcal{B}_2 = (q_c, \infty)$.

Proof First we show that $\mathcal{B}_2 \subseteq (q_c, \infty)$. By Lemma 3.1 it suffices to prove that any $q \in (2, q_c]$ is not contained in $\mathcal{B}_2$. Take $q \in (2, q_c]$. By Theorem 1.1 we have $\mathcal{U}_q' = \{(0^\infty), (q^\infty)\}$. Then by Lemma 3.4 it follows that if $q \in \mathcal{B}_2 \cap (2, q_c]$ then q must satisfy one of the following equations

$$(0q^\infty)_q = (10^\infty)_q \quad \text{or} \quad (1q^\infty)_q = (q0^\infty)_q.$$

This is impossible since neither equation has a solution in $(2, q_c]$. Hence, $\mathcal{B}_2 \subseteq (q_c, \infty)$.

Now we turn to prove $(q_c, \infty) \subseteq \mathcal{B}_2$. By Lemmas 2.1 and 3.4, one can verify that for any $q > q^*$ the number

$$x = (0q0^\infty)_q = (10^\infty)_q$$

has precisely two different q-expansions. This implies that $(q^*, \infty) \subseteq \mathcal{B}_2$.

For $q \in (q_c, q^*]$, one has by (2.2) that $\alpha(q_c) = q_c 1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. Then by Lemma 2.3 there exists an integer $m \geq 0$ such that

$$\alpha(q) \succ q1^m q0^\infty.$$

Hence, by Lemmas 2.2 and 3.4 one can verify that

$$y = (0q(1^{m+1}q)^\infty)_q = (10(1^{m+1}q)^\infty)_q$$

has precisely two different q-expansions. So, $(q_c, q^*] \subseteq \mathcal{B}_2$, and the proof is complete. $\square$

Lemma 3.6 *$\mathcal{B}_k = (q_c, \infty)$ for any $k \geq 3$.*

Proof First we prove $\mathcal{B}_k \subseteq \mathcal{B}_2$ for any $k \geq 3$. By Lemma 3.1 it follows that $\mathcal{B}_k \subseteq (2, \infty)$. Take $q \in \mathcal{B}_k$ with $k \geq 3$. Suppose $x \in E_q$ has exactly k different q-expansions. Since $q > 2$, the union in (3.1) is disjoint. This implies that there exists a word $d_1 \dots d_n$ such that

$$\phi_{d_1}^{-1} \circ \cdots \circ \phi_{d_n}^{-1}(x)$$

has exactly two different q-expansions. So, $q \in \mathcal{B}_2$. Hence, $\mathcal{B}_k \subseteq \mathcal{B}_2$ for any $k \geq 3$.

Now we prove $\mathcal{B}_2 \subseteq \mathcal{B}_k$ for any $k \geq 3$. Note by Lemma 3.5 that $\mathcal{B}_2 = (q_c, \infty)$. Then it suffices to prove $(q_c, \infty) \subseteq \mathcal{B}_k$. First we prove $(q^*, \infty) \subseteq \mathcal{B}_k$. Take $q \in (q^*, \infty)$. We claim that for any $k \geq 1$,

$$x_k = (0q^{k-1}(1q)^\infty)_q$$

has precisely k different q-expansions. We will prove this by induction on k.

For $k = 1$ one can easily check by using Lemma 2.1 that $x_1 = (0(1q)^\infty)_q \in \mathcal{U}_q$. Suppose x_k has exactly k different q-expansions. Now we consider x_{k+1}, which can be written as

$$x_{k+1} = (0q^k(1q)^\infty)_q = (10q^{k-1}(1q)^\infty)_q.$$

By Lemma 2.1 we have $q^k(1q)^\infty \in \mathcal{U}'_q$. Moreover, by the induction hypothesis $(0q^{k-1}(1q)^\infty)_q = x_k$ has exactly k different q-expansions. Then x_{k+1} has at least $k+1$ different q-expansions. On the other hand, since $q > q^* > 2$, the union in (3.1) is disjoint. Then

$$x_{k+1} \in \phi_0(E_q) \cap \phi_1(E_q) \backslash \phi_q(E_q).$$

This implies that x_{k+1} indeed has $k+1$ different q-expansions. By induction this proves the claim, and hence $(q^*, \infty) \subseteq \mathcal{B}_k$ for all $k \geq 3$.

It remains to prove $(q_c, q^*] \subseteq \mathcal{B}_k$. Take $q \in (q_c, q^*]$. By (2.2) and Lemma 2.3 there exists an integer $m \geq 0$ such that

$$\alpha(q) \succ q1^m q0^\infty. \tag{3.3}$$

We claim that

$$y_k = (0q^{k-1}(1^{m+1}q)^\infty)_q$$

has exactly k different q-expansions. Again, this will be proven by induction on k.

If $k = 1$, then by using (3.3) in Lemma 2.2 it gives that $y_1 = (0(1^{m+1}q)^\infty)_q$ has a unique q-expansion. Suppose y_k has exactly k different q-expansions. Now we consider

$$y_{k+1} = (0q^k(1^{m+1}q)^\infty)_q = (10q^{k-1}(1^{m+1}q)^\infty)_q.$$

By (3.3) and Lemma 2.2 it yields that $q^k(1^{m+1}q)^\infty \in \mathcal{U}'_q$. Furthermore, by the induction hypothesis $(0q^{k-1}(1^{m+1}q)^\infty)_q = y_k$ has exactly k different q-expansions. This implies that y_{k+1} has at least $k+1$ different q-expansions. On the other hand, note that $q > q_c > 2$, and therefore the union in (3.1) is disjoint. So, $y_{k+1} \in \phi_0(E_q) \cap \phi_1(E_q) \backslash \phi_q(E_q)$, which implies that y_{k+1} indeed has $k+1$ different q-expansions. By induction this proves the claim, and then $(q_c, q^*] \subseteq \mathcal{B}_k$ for all $k \geq 3$. This completes the proof. □

Proof of Theorem 1 By Lemmas 3.3, 3.5 and 3.6 it suffices to prove $\mathcal{B}_{2^{\aleph_0}} = (1, \infty)$. This can be verified by observing that

$$x = ((100)^\infty)_q \in \mathcal{U}_q^{(2^{\aleph_0})}$$

for any $q > 1$, because by the word substitution $10 \sim 0q$ one can show that x indeed has a continuum of different q-expansions. □

4 Proof of Theorem 2

For $q > 1$ and $k \in \mathbb{N}$ we recall that $\mathcal{U}_q^{(k)}$ is the set of $x \in [0, q/(q-1)]$ having precisely k different q-expansions. In this section we are going to investigate the Hausdorff dimension of $\mathcal{U}_q^{(k)}$. First we show that $q_c \approx 2.32472$ is the critical base for $\mathcal{U}_q$.

Lemma 4.1 *Let $q > 1$. Then $\dim_H \mathcal{U}_q > 0$ if and only if $q > q_c$.*

Proof The necessity follows from Theorem 1.1 (i). For the sufficiency we take $q \in (q_c, \infty)$. If $q > q^*$, then by Theorem 1.1 (iii) we have

$$\dim_H \mathcal{U}_q = \frac{\log q_c}{\log q} > 0.$$

So it remains to prove $\dim_H \mathcal{U}_q > 0$ for any $q \in (q_c, q^*]$.

Take $q \in (q_c, q^*]$. Recall from (2.2) that $\alpha(q_c) = q_c 1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. Then by Lemma 2.3 there exists an integer $m \geq 0$ such that $\alpha(q) \succ q1^m q0^\infty$. Whence, by Lemma 2.2 one can verify that all sequences in

$$\Delta'_m := \prod_{i=1}^{\infty} \{q1^{m+1}, 1^{m+2}\}$$

excluding those ending with 1^∞ belong to $\mathcal{U}'_q$. This implies that

$$\dim_H \mathcal{U}_q \geq \dim_H \Delta_m(q), \tag{4.1}$$

where $\Delta_m(q) := \left\{((d_i))_q : (d_i) \in \Delta'_m\right\}$. Note that $\Delta_m(q)$ is a self-similar set generated by the IFS

$$f_1(x) = \frac{x}{q^{m+2}} + (q1^{m+1}0^\infty)_q, \quad f_2(x) = \frac{x}{q^{m+2}} + (1^{m+2}0^\infty)_q,$$

which satisfies the open set condition (cf. [8]). Therefore, by (4.1) we conclude that

$$\dim_H \mathcal{U}_q \geq \dim_H \Delta_m(q) = \frac{\log 2}{(m+2)\log q} > 0.$$

□

In the following we will consider the Hausdorff dimension of $\mathcal{U}_q^{(k)}$ for any $k \geq 2$, and prove $\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q$. The upper bound of $\dim_H \mathcal{U}_q^{(k)}$ is easy.

Lemma 4.2 *Let $q > 1$. Then $\dim_H \mathcal{U}_q^{(k)} \leq \dim_H \mathcal{U}_q$ for any $k \geq 2$.*

Proof Recall that $\phi_d(x) = (x+d)/q$ for $d \in \{0, 1, q\}$. Then the lemma follows by observing that for any $k \geq 2$,

$$\mathcal{U}_q^{(k)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1 \cdots d_n \in \{0,1,q\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n}(\mathcal{U}_q),$$

and the countable stability of Hausdorff dimension. □

For the lower bound of $\dim_H \mathcal{U}_q^{(k)}$ we need more. By Lemmas 4.1 and 4.2 it follows that

$$\dim_H \mathcal{U}_q^{(k)} = 0 = \dim_H \mathcal{U}_q \quad \text{for any } q \leq q_c.$$

So, it suffices to consider $q > q_c$. Let

$$F'_q(1) := \left\{(d_i) \in \mathcal{U}'_q : d_1 = 1\right\}$$

be the *follower set* in $\mathcal{U}'_q$ generated by the word 1, and let $F_q(1)$ be the set of $x \in E_q$ which have a q-expansion in $F'_q(1)$, i.e., $F_q(1) = \{((d_i))_q : (d_i) \in F'_q(1)\}$.

Lemma 4.3 *Let $q > q_c$. Then $\dim_H \mathcal{U}_q^{(k)} \geq \dim_H F_q(1)$ for any $k \geq 1$.*

Proof For $k \geq 1$ and $q > q_c$ let

$$\Lambda_q^k := \left\{((d_i))_q : d_1 \ldots d_k = 0q^{k-1}, (d_{k+i}) \in F'_q(1)\right\}.$$

Then $\Lambda_q^k = \phi_0 \circ \phi_q^{k-1}(F_q(1))$, and therefore $\dim_H \Lambda_q^k = \dim_H F_q(1)$. So it suffices to prove $\Lambda_q^k \subseteq \mathcal{U}_q^{(k)}$. Arbitrarily take

$$x_k = \left(0q^{k-1}(c_i)\right)_q \in \Lambda_q^k \quad \text{with} \quad (c_i) \in F'_q(1).$$

We will prove by induction on k that x_k has exactly k different q-expansions.

For $k = 1$, by Lemmas 2.1 and 2.2 it follows that $x_1 = (0(c_i))_q \in \mathcal{U}_q$. Suppose $x_k = (0q^{k-1}(c_i))_q$ has precisely k different q-expansions. Now we consider x_{k+1}, which can be expanded as

$$x_{k+1} = \left(0q^k(c_i)\right)_q = (10q^{k-1}(c_i))_q.$$

By Lemmas 2.1 and 2.2 we have $q^k(c_i) \in \mathcal{U}'_q$, and by the induction hypothesis it yields that $(0q^{k-1}(c_i))_q = x_k$ has k different q-expansions. This implies that x_{k+1} has at least $k+1$ different q-expansions. On the other hand, since $q > q_c > 2$, it gives that the union in (3.1) is disjoint. So, $x_{k+1} \in \phi_0(E_q) \cap \phi_1(E_q) \backslash \phi_q(E_q)$, which implies that x_{k+1} indeed has $k+1$ different q-expansions.

By induction this proves $x_k \in \mathcal{U}_q^{(k)}$ for all $k \geq 1$. Since x_k was taken arbitrarily from Λ_q^k, we conclude that $\Lambda_q^k \subseteq \mathcal{U}_q^{(k)}$ for any $k \geq 1$. The proof is complete. □

Lemma 4.4 *Let $q > q_c$. Then* $\dim_H F_q(1) \geq \dim_H \mathcal{U}_q$.

Proof First we consider $q > q^*$. By Lemma 2.1 one can show that $\mathcal{U}'_q$ is contained in an irreducible sub-shift of finite type X'_A over the states $\{0, 1, q\}$ with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.2}$$

Moreover, the complement set $X'_A \backslash \mathcal{U}'_q$ contains all sequences ending with 1^∞. This implies that

$$\dim_H \mathcal{U}_q = \dim_H X_A(q), \tag{4.3}$$

where $X_A(q) := \left\{((d_i))_q : (d_i) \in X'_A\right\}$. Note that $X_A(q)$ is a graph-directed set satisfying the open set condition (cf. [24, Theorem 3.4]), and the sub-shift of finite type X'_A is irreducible. Then by (4.3) it follows that

$$\dim_H \mathcal{U}_q = \dim_H X_A(q) = \dim_H F_q(1).$$

Now we consider $q \in (q_c, q^*]$. By Lemma 2.2 it follows that

$$\mathcal{U}'_q \subseteq \left\{q^\infty\right\} \cup \bigcup_{k=0}^{\infty} \left\{q^k 0^\infty\right\} \cup \bigcup_{k=0}^{\infty} \bigcup_{m=0}^{\infty} \left\{q^k 0^m F'_q(1)\right\},$$

where

$$q^k 0^m F'_q(1) := \left\{(d_i) : d_1 \ldots d_{k+m} = q^k 0^m, (d_{k+m+i}) \in F'_q(1)\right\}.$$

This implies that $\dim_H \mathcal{U}_q \leq \dim_H F_q(1)$. □

Proof of Theorem 2 The theorem follows directly by Lemmas 4.1–4.4. □

5 Proof of Theorem 3

In this section we will consider the set $\mathcal{U}_q^{(\aleph_0)}$ which consists of all $x \in E_q$ having countably infinitely many q-expansions.

Lemma 5.1 *For any $q \in \mathcal{B}_{\aleph_0}$ the set $\mathcal{U}_q^{(\aleph_0)}$ contains infinitely many points.*

Proof Let $q \in \mathcal{B}_{\aleph_0}$. By Theorem 1 we have $q \in [2, \infty)$. Then it suffices to show that for any $k \geq 1$,

$$z_k := (0^k q^\infty)_q$$

is a q-null infinite points, and thus $z_k \in \mathcal{U}_q^{(\aleph_0)}$.

If $q > 2$, then by the proof of Lemma 3.3 it yields that $z_1 = (0q^\infty)_q$ is a q-null infinite point. Moreover, note that $z_k = \phi_0^{k-1}(z_1) \notin S_q$ for any $k \geq 2$. This implies that all of these points $z_k, k \geq 1$, are q-null infinite points. So, $\{z_k : k \geq 1\} \subseteq \mathcal{U}_q^{(\aleph_0)}$.

If $q = 2$, then by using the substitutions

$$0q \sim 10, \quad 0q^\infty = 1^\infty = q0^\infty,$$

one can also show that z_k is a q-null infinite point. In fact, all of the q-expansions of $z_k = (0^k q^\infty)_q$ are of the form

$$0^k q^\infty, \quad 0^{k-1}1^\infty, \quad 0^{k-1}1^m 0q^\infty \quad \text{and} \quad 0^{k-1}1^{m-1}q0^\infty,$$

where $m \geq 1$. Therefore, $z_k \in \mathcal{U}_q^{(\aleph_0)}$ for any $k \geq 1$. □

By Lemma 5.1 it follows that $\mathcal{U}_q^{(\aleph_0)}$ is at least countably infinite for any $q \in \mathcal{B}_{\aleph_0} = [2, \infty)$. In the following lemma we show that $\mathcal{U}_q^{(\aleph_0)}$ is indeed countably infinite if $q \geq q^*$.

Lemma 5.2 *Let $q \geq q^*$. Then $\mathcal{U}_q^{(\aleph_0)}$ is at most countable.*

Proof Let $x \in \mathcal{U}_q^{(\aleph_0)}$. Then x has a q-expansion (d_i) such that

$$|\Sigma(x_n)| = \infty \quad \text{for infinitely many } n \in \mathbb{N},$$

where $x_n := ((d_{n+i}))_q$. This implies that (d_i) can not end in $\mathcal{U}_q'$.

Note by the proof of Lemma 4.4 that $\mathcal{U}_q' \subseteq X_A'$, where X_A' is a sub-shift of finite type over the state $\{0, 1, q\}$ with adjacency matrix A defined in (4.2). Moreover, $X_A' \backslash \mathcal{U}_q'$ is at most countable (cf. [24, Theorem 3.4]). Note that the expansion (d_i) of $x \in \mathcal{U}_q^{(\aleph_0)}$ does not end in $\mathcal{U}_q'$. Then it suffices to prove that the sequence (d_i) must end in X_A'.

Suppose on the contrary that (d_i) does not end in X_A'. Then by (4.2) the word $0q$ or 10 occurs infinitely many times in (d_i). Using the word substitution $0q \sim 10$ this implies that $x = ((d_i))_q$ has a continuum of q-expansions, leading to a contradiction with $x \in \mathcal{U}_q^{(\aleph_0)}$. □

Furthermore, we can prove that $\mathcal{U}_q^{(\aleph_0)}$ is also countably infinite for $q \in [2, q_c]$.

Lemma 5.3 *Let $q \in [2, q_c]$. Then $\mathcal{U}_q^{(\aleph_0)}$ is at most countable.*

Proof Take $q \in [2, q_c]$. By Theorems 1 and 1.1 it follows that any $x \in E_q$ with $|\Sigma(x)| < \infty$ must belong to $\mathcal{U}_q = \{0, q/(q-1)\}$. Suppose $x \in \mathcal{U}_q^{(\aleph_0)}$. Then there exists a word $d_1 \ldots d_n$ such that

$$\phi_{d_1}^{-1} \circ \cdots \circ \phi_{d_n}^{-1}(x) \in \mathcal{U}_q.$$

This implies that the set $\mathcal{U}_q^{(\aleph_0)}$ is at most countable, since

$$\mathcal{U}_q^{(\aleph_0)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1 \ldots d_n \in \{0,1,q\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n} \left(\mathcal{U}_q\right).$$

□

When $q \in (q_c, q^*)$, one might expect that $\mathcal{U}_q^{(\aleph_0)}$ is also countably infinite. Unfortunately, we are not able to prove this. Instead, we show that the Hausdorff dimension of $\mathcal{U}_q^{(\aleph_0)}$ is strictly smaller than $\dim_H E_q = 1$.

Lemma 5.4 *For $q \in (q_c, q^*)$ we have* $\dim_H \mathcal{U}_q^{(\aleph_0)} \le \dim_H \mathcal{U}_q < 1$.

Proof Take $q \in (q_c, q^*)$. Note that

$$\mathcal{U}_q^{(\aleph_0)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1 \ldots d_n \in \{0,1,q\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n}(\mathcal{U}_q).$$

By using the countable stability of Hausdorff dimension this implies that $\dim_H \mathcal{U}_q^{(\aleph_0)} \le \dim_H \mathcal{U}_q$. In the following it suffices to prove $\dim_H \mathcal{U}_q < 1$.

Note that $\mathcal{U}'_q \subseteq X'_A$, where X'_A is the sub-shift of finite type over the state $\{0, 1, q\}$ with adjacency matrix A defined in (4.2). Then

$$\mathcal{U}_q \subseteq X_A(q) = \left\{((d_i))_q : (d_i) \in X'_A\right\}.$$

Note that $X_A(q)$ is a graph-directed set (cf. [14]). This implies that

$$\dim_H \mathcal{U}_q \le \dim_H X_A(q) \le \frac{\log q_c}{\log q} < 1.$$

□

At the end of this section we investigate the set $\mathcal{U}_q^{(2^{\aleph_0})}$ which consists of all points having a continuum of q-expansions, and show that $\mathcal{U}_q^{(2^{\aleph_0})}$ has full Hausdorff measure.

Lemma 5.5 *For any $q > 1$ we have*

$$\mathcal{H}^{\dim_H E_q}\left(\mathcal{U}_q^{(2^{\aleph_0})}\right) = \mathcal{H}^{\dim_H E_q}(E_q) \in (0, \infty).$$

Proof Clearly, for $q \in (1, q^*]$ we have $E_q = [0, q/(q-1)]$, and then $\mathcal{H}^{\dim_H E_q}(E_q) \in (0, \infty)$. Moreover, for $q > q^*$ we have by (1.1) that $\dim_H E_q = \log q^* / \log q$, and the set E_q has positive and finite Hausdorff measure (cf. [15]). Therefore,

$$0 < \mathcal{H}^{\dim_H E_q}(E_q) < \infty \quad \text{for any} \quad q > 1. \tag{5.1}$$

First we prove the lemma for $q \le q^*$. By Theorems 1 and 1.1 it follows that for any $q \in (1, q^*]$,

$$\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q < 1 = \dim_H E_q \quad \text{for any} \quad k \ge 2.$$

Moreover, by Lemmas 5.2–5.4 we have $\dim_H \mathcal{U}_q^{(\aleph_0)} < 1$. Observe that

$$E_q = \mathcal{U}_q^{(2^{\aleph_0})} \cup \mathcal{U}_q^{(\aleph_0)} \cup \bigcup_{k=1}^{\infty} \mathcal{U}_q^{(k)} \quad \text{for any } q > 1. \tag{5.2}$$

Therefore, by (5.1) and (5.2) we have $\mathcal{H}^{\dim_H E_q}(\mathcal{U}_q^{(2^{\aleph_0})}) = \mathcal{H}^{\dim_H E_q}(E_q) \in (0, \infty)$.

Now we consider $q > q^*$. By Theorems 1.1 (iii), 2 and (1.1) it follows that

$$\dim_H \mathcal{U}_q^{(k)} = \frac{\log q_c}{\log q} < \frac{\log q^*}{\log q} = \dim_H E_q$$

for any $k \ge 1$. Moreover, by Lemma 5.2 we have $\dim_H \mathcal{U}_q^{(\aleph_0)} = 0$. Again, by (5.1) and (5.2) it follows that $\mathcal{H}^{\dim_H E_q}(\mathcal{U}_q^{(2^{\aleph_0})}) = \mathcal{H}^{\dim_H E_q}(E_q) \in (0, \infty)$. This completes the proof. □

Proof of Theorem 3 The theorem follows by Lemmas 5.1–5.3 and 5.5. □

6 Examples and final remarks

In this section we consider some examples. The first example is an application of Theorems 1–3 to expansions with deleted digits set.

Example 6.1 Let $q = 3$. We consider q-expansions with digits set $\{0, 1, 3\}$. This is a special case of expansions with deleted digits (cf. [17]). Then

$$E_3 = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{3^i} : d_i \in \{0, 1, 3\} \right\}.$$

By Theorems 1.1 and 2 we have

$$\dim_H \mathcal{U}_3^{(k)} = \dim_H \mathcal{U}_3 = \frac{\log q_c}{\log 3} \approx 0.767877$$

for any $k \geq 2$. This means that the set $\mathcal{U}_3^{(k)}$ consisting of all points in E_3 with precisely k different triadic expansions has the same Hausdorff dimension $\log q_c / \log 3$ for any integer $k \geq 1$. Moreover, by Theorem 3 it follows that $\mathcal{U}_3^{(\aleph_0)}$ is countably infinite, and

$$\dim_H \mathcal{U}_3^{(2^{\aleph_0})} = \dim_H E_3 = \frac{\log q^*}{\log 3} \approx 0.876036.$$

Theorem 1.1 gives a uniform formula for the Hausdorff dimension of $\mathcal{U}_q$ for $q \in [q^*, \infty)$. Excluding the trivial case for $q \in (1, q_c]$ that $\mathcal{U}_q = \{0, q/(q-1)\}$, it would be interesting to ask whether the Hausdorff dimension of $\mathcal{U}_q$ can be determined for $q \in (q_c, q^*)$. In the following we give an example for which the Hausdorff dimension of $\mathcal{U}_q$ can be explicitly calculated.

Example 6.2 Let $q = 1 + \sqrt{2} \in (q_c, q^*)$. Then

$$(q0^\infty)_q = (1qq0^\infty)_q \quad \text{and} \quad \alpha(q) = (q1)^\infty.$$

Moreover, the quasi-greedy q-expansion of $q-1$ with alphabet $\{0, q-1, q\}$ is $q(q-1)^\infty$. Therefore, by Lemmas 3.1 and 3.2 of [24] it follows that $\mathcal{U}_q'$ is the set of sequences $(d_i) \in \{0, 1, q\}^\infty$ satisfying

$$\begin{cases} d_{n+1}d_{n+2}\cdots \prec (1q)^\infty & \text{if} \quad d_n = 0, \\ 1^\infty \prec d_{n+1}d_{n+2}\cdots \prec (q1)^\infty & \text{if} \quad d_n = 1, \\ d_{n+1}d_{n+2}\cdots \succ 01^\infty & \text{if} \quad d_n = q. \end{cases}$$

Let X_A' be the sub-shift of finite type over the states

$$\{00, 01, 11, 1q, q0, q1, qq\}$$

with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then one can verify that $\mathcal{U}_q' \subseteq X_A'$, and $X_A' \backslash \mathcal{U}_q'$ contains all sequences ending with 1^∞ or $(1q)^\infty$. This implies that

$$\dim_H \mathcal{U}_q = \dim_H X_A(q),$$

where $X_A(q) = \left\{((d_i))_q : (d_i) \in X_A'\right\}$. Note that $X_A(q)$ is a graph-directed set satisfying the open set condition (cf. [14]). Then by Theorem 2 we have

$$\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q = \frac{h(X_A')}{\log q} \approx 0.691404.$$

Furthermore, by the word substitution $q00 \sim 1qq$ and in a similar way as in the proof of Lemma 5.2 one can show that $\mathcal{U}_q^{(\aleph_0)}$ is countably infinite. Finally, by Theorem 3 we have $\dim_H \mathcal{U}_q^{(2^{\aleph_0})} = \dim_H E_q = 1$.

Question 1. Can we give a uniform formula for the Hausdorff dimension of $\mathcal{U}_q$ for $q \in (q_c, q^*)$?

In beta expansions we know that the dimension function of the univoque set has a Devil's staircase behavior (cf. [12]).

Question 2. Does the dimension function $D(q) := \dim_H \mathcal{U}_q$ have a Devil's staircase behavior in the interval (q_c, q^*)?

By Theorem 3 one has that $\mathcal{U}_q^{(\aleph_0)}$ is countable for any $q \in \mathcal{B}_2 \backslash (q_c, q^*)$. Moreover, in Lemma 5.4 we show that $\dim_H \mathcal{U}_q^{(\aleph_0)} \leq \dim_H \mathcal{U}_q < 1$ for any $q \in (q_c, q^*)$. In view of Example 6.2 we ask the following

Question 3. Does there exist a $q \in (q_c, q^*)$ such that $\mathcal{U}_q^{(\aleph_0)}$ has positive Hausdorff dimension?

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