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On small bases which admit points with two expansions

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Given two positive integers M and k, let $\mathcal{B}_k(M)$ be the set of bases $q > 1$ such that there exists a real number $x \in [0, M/(q-1)]$ having precisely k different q-expansions over the alphabet $\{0, 1, \ldots, M\}$. In this paper we consider $k = 2$ and investigate the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$. We prove that for $M = 2m$ the smallest base is

$$q_2(M) = \frac{m + 1 + \sqrt{m^2 + 2m + 5}}{2},$$

and for $M = 2m - 1$ the smallest base $q_2(M)$ is the largest positive root of

$$x^4 = (m-1)\,x^3 + 2m\,x^2 + m\,x + 1.$$

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Moreover, for $M = 2$ we show that $q_2(2)$ is also the smallest base of $\mathcal{B}_k(2)$ for all $k \geq 3$.

1. Introduction

Fix a positive integer M. For $q \in (1, M+1]$ the sequence $(d_i) = d_1 d_2 \dots$ with each $d_i \in \{0, 1, \dots, M\}$ is called a *q-expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i}.$$

Here the *alphabet* $\{0, 1, \dots, M\}$ will be fixed throughout the paper. Clearly, x has a q-expansion if and only if $x \in I_{q,M} := [0, M/(q-1)]$.

Since the pioneering work of Rényi [19] and Parry [18], representations of real numbers in non-integer bases have been widely studied in the past thirty years. Different from integer base expansions it is well known that almost every $x \in I_{q,M}$ has a continuum of q-expansions (cf. [20,5]). Moreover, for each $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1, M+1]$ and $x \in I_{q,M}$ such that x has precisely k different q-expansions (see, e.g., [9]). For $k = 1$ the unique q-expansions were extensively investigated. For example, Glendinning and Sidorov showed in [11] that for $M = 1$ when the base q is close to $M + 1$ the set of $x \in I_{q,M}$ with a unique q-expansion has positive Hausdorff dimension (for $M > 1$, see e.g., [16]). De Vries and Komornik [7] investigated the topological properties of the unique q-expansions. Recently, Komornik et al. [13] studied the measure theoretical aspects of the unique q-expansions. For more information on the unique q-expansions we refer the readers to [14,8,15], and the surveys [12,20].

Inspired by the papers of Sidorov [21] and Baker [3] we consider the following sets. For $k \in \mathbb{N} \cup \{\aleph_0\}$, let

$$\mathcal{B}_k(M) := \{q \in (1, M+1] : \text{there exists } x \in I_{q,M} \text{ having precisely } k \text{ different } q\text{-expansions}\}.$$

For $M = 1$ Sidorov [21] determined the smallest base $q_2(1) \approx 1.71064$ of $\mathcal{B}_2(1)$, and proved that the set $\mathcal{B}_2(1)$ contains an interval. Later in [4] Baker and Sidorov considered the smallest base of $\mathcal{B}_k(1)$ for $k \geq 3$ and showed that they are all equal to $q_f(1) \approx 1.75488$. Note that the golden ratio $q_G = (1 + \sqrt{5})/2$ is the smallest base of $\mathcal{B}_{\aleph_0}(1)$ (see Lemma 2.2 below). Recently, Baker [3] showed that the second smallest base of $\mathcal{B}_{\aleph_0}(1)$ is $q_{\aleph_0}(1) \approx 1.64541$. Hence, he concluded that for any $q \in (q_G, q_{\aleph_0}(1))$ each point $x \in I_{q,1}$ either has a unique q-expansion or has a continuum of q-expansions. Based on the ideas of [3] the first and the third authors showed in [22] that $q_2(1)$ does not belong to $\mathcal{B}_{\aleph_0}(1)$, and deduced that $\mathcal{B}_{\aleph_0}(1)$ is not a closed set. Therefore

$$q_2(1) \in \mathcal{B}_1(1) \cap \mathcal{B}_2(1) \cap \mathcal{B}_{2^{\aleph_0}}(1) \quad \text{and} \quad q_2(1) \notin \mathcal{B}_{\aleph_0}(1) \cup \bigcup_{k=3}^{\infty} \mathcal{B}_k(1). \tag{1.1}$$

Then it is natural to ask "what can we say about the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$ for a general integer $M \geq 1$?" In the following theorem we determine the smallest base $q_2(M)$ for any $M \geq 1$.

Theorem 1.1.

(a) *If $M = 2m$, then the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$ is given by*

$$q_2(M) = \frac{m + 1 + \sqrt{m^2 + 2m + 5}}{2};$$

(b) *If $M = 2m - 1$, then the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$ is the largest positive root of*

$$x^4 = (m-1)x^3 + 2m\,x^2 + m\,x + 1.$$

(c) *For any $m \in \mathbb{N}$ the smallest base $q_2(2m)$ is a Pisot number, and $q_2(2m-1)$ is a Perron number. Moreover, $q_2(M) = M/2 + r(M)$ with*

$$\lim_{m\to\infty} r(2m) = 1 \quad \text{and} \quad \lim_{m\to\infty} r(2m-1) = \frac{3}{2}.$$

By Theorem 1.1 we give the numerical calculations of $q_2 = q_2(M)$ for $M = 1, 2, \ldots, 7$.

M	1	2	3	4	5	6	7
$q_2 \approx$	1.71064	2.41421	2.75965	3.30278	3.80320	4.23607	4.83469

When $M = 1$ by (1.1) it follows that each $x \in I_{q_2(1),1}$ has a unique $q_2(1)$-expansion, precisely two different $q_2(1)$-expansions, or a continuum of $q_2(1)$-expansions. One may expect that this will also happen for the base $q_2(M)$ for $M > 1$. However, our next result shows that this is not true for all $M \geq 1$. In the following theorem we show that for $M = 2$ and for any $k = 1, 2, \ldots, \aleph_0$ or $2^{\aleph_0}$ there exists x having precisely k different $q_2(2)$-expansions.

Theorem 1.2. *Let $M = 2$. Then*

$$q_2(2) = 1 + \sqrt{2} \in \mathcal{B}_{2^{\aleph_0}}(2) \cap \mathcal{B}_{\aleph_0}(2) \cap \bigcap_{k=1}^{\infty} \mathcal{B}_k(2).$$

Furthermore, $q_2(2)$ is the smallest element of $\mathcal{B}_k(2)$ for any $k = 2, 3, \ldots$.

The paper is arranged in the following way. In Section 2 we will explicitly describe the set of unique q-expansions for small bases q with alphabet $\{0, 1, \ldots, M\}$. This is helpful to find the smallest base $q_2(M)$ which admits two expansions. The proof of Theorem 1.1 will be given in Section 3 for even M and in Section 4 for odd M. In the final section we will prove Theorem 1.2 and end the paper with some questions.

2. Unique expansions

In this section we recall some basic properties of the unique expansions. For $q \in (1, M+1]$ let $\mathcal{U}_q$ be the *univoque* set of $x \in I_{q,M}$ having a unique q-expansion, and let $\mathcal{U}'_q$ be the set of corresponding expansions. In order to characterize the univoque set $\mathcal{U}_q$ we need some notation from symbolic dynamics (see, e.g., [17]).

Let $\{0, 1, \ldots, M\}^\infty$ be the set of sequences $(d_i) = d_1 d_2 \ldots$ with each element $d_i \in \{0, 1, \ldots, M\}$. For two words $\mathbf{c} = c_1 \ldots c_m$ and $\mathbf{d} = d_1 \ldots d_n$ we denote their concatenation by $\mathbf{cd} = c_1 \ldots c_m d_1 \ldots d_n$. Accordingly, for $k \in \mathbb{N}$ we denote by $\mathbf{c}^k$ the concatenation of $\mathbf{c}$ with itself k times, and denote by $\mathbf{c}^\infty$ the concatenation of $\mathbf{c}$ with itself infinitely many times. In this paper we will use lexicographical order "$\prec, \preccurlyeq, \succ$" or "$\succcurlyeq$" between sequences in $\{0, 1, \ldots, M\}^\infty$. For example, for two sequences $(c_i), (d_i) \in \{0, 1, \ldots, M\}^\infty$ we say $(c_i) \prec (d_i)$ if there exists $n \in \mathbb{N}$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Moreover, we write $(c_i) \preccurlyeq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. For a sequence $(d_i) \in \{0, 1, \ldots, M\}^\infty$ we denote by

$$\overline{(d_i)} = (M - d_1)(M - d_2) \ldots$$

the *reflection* of (d_i).

For $q \in (1, M+1]$ let $\alpha(q) = (\alpha_i(q))$ be the *quasi-greedy* q-expansion of 1 (cf. [6]), i.e., the lexicographically largest infinite q-expansion of 1. Here an expansion (d_i) is called *infinite* if (d_i) does not end with a string of zeros. The following lexicographical characterization of $\mathcal{U}'_q$ was essentially due to Parry [18] (see also, [1]).

Lemma 2.1. *Let $q \in (1, M+1]$. Then an expansion $(d_i) \in \mathcal{U}'_q$ if and only if*

$$\begin{cases} d_{n+1} d_{n+2} \ldots \prec \alpha(q) & \text{whenever} \quad d_n < M, \\ d_{n+1} d_{n+2} \ldots \succ \overline{\alpha(q)} & \text{whenever} \quad d_n > 0. \end{cases}$$

Moreover, the map $q \to \alpha(q)$ is strictly increasing from $(1, M+1]$ onto the set of infinite expansions (γ_i) satisfying

$$\gamma_{i+1} \gamma_{i+2} \ldots \preccurlyeq \gamma_1 \gamma_2 \cdots \quad \text{for all} \quad i \geq 0.$$

For $M \geq 1$ we recall from [2] that the *generalized golden ratio* $p_1 = p_1(M)$ admits the quasi-greedy expansion

$$\alpha(p_1(M)) = \begin{cases} m^\infty & \text{if} \quad M = 2m, \\ (m\,(m-1))^\infty & \text{if} \quad M = 2m-1. \end{cases} \tag{2.1}$$

The following lemma for the q-expansions of $x \in I_{q,M}$ with $q \in (1, p_1]$ was established in [2,10].

Lemma 2.2. *If $q \in (1, p_1)$, then any $x \in (0, M/(q-1))$ has a continuum of q-expansions. If $q = p_1$, then any $x \in (0, M/(q-1))$ either has a continuum of q-expansions, or has countably many q-expansions.*

Recall that $\mathcal{B}_2(M)$ is the set of bases $q \in (1, M+1]$ for which there exists $x \in I_{q,M}$ having precisely two q-expansions. Observe that for each $q \in (1, M+1]$ the endpoints of the interval $I_{q,M}$ always have a unique q-expansion. Then by Lemma 2.2 it follows that the smallest base $q_2(M)$ is strictly larger than p_1. In the next two sections we will show that $q_2(M) \le p_2$, where $p_2 = p_2(M)$ admits the quasi-greedy expansion

$$\alpha(p_2(M)) = \begin{cases} ((m+1)(m-1))^\infty & \text{if} \quad M = 2m, \\ (mm(m-1)(m-1))^\infty & \text{if} \quad M = 2m-1. \end{cases} \tag{2.2}$$

Note that if a point $x \in I_{q,M}$ has two q-expansions, then the tail of each expansion of x must belong to $\mathcal{U}_q'$. So, it is necessary to give a detailed description of the set $\mathcal{U}_q'$ for $p_1 < q \le p_2$.

First we consider $M = 2m$. The following proposition for $\mathcal{U}_q'$ was implicitly shown in [16, Lemma 4.12].

Proposition 2.3. *If $M = 2m$, then for $p_1 < q \le p_2$ we have*

$$\mathcal{U}_q' = \left\{0^\infty, \overline{0^\infty}\right\} \cup \bigcup_{k=0}^{\infty} \bigcup_{u=0}^{m} \left\{0^k u m^\infty, \overline{0^k u m^\infty}\right\}.$$

Proof. First we consider the "$\supseteq$" part. Note that $q > p_1$. Then by (2.1) and Lemma 2.1 it follows that

$$\alpha(q) \succ \alpha(p_1) = m^\infty.$$

Therefore, the "$\supseteq$" part can be verified by using Lemma 2.1.

Now we consider the "$\subseteq$" part. Take $(d_i) \in \mathcal{U}_q'$ with $q \in (p_1, p_2]$. By symmetry we may assume that $d_1 \le m$. Apart from the trivial case that $(d_i) = 0^\infty$ let $n \ge 1$ be the smallest integer such that $d_n > 0$. Now we split the proof into the following two cases: (I) $n = 1$; (II) $n > 1$.

Case (I). $n = 1$. Then $0 < d_n \le m$. Note by (2.2) and Lemma 2.1 that

$$\alpha(q) \preccurlyeq \alpha(p_2) = ((m+1)(m-1))^\infty. \tag{2.3}$$

Then by Lemma 2.1 it follows that

$$d_{n+1} \in \{m-1, m, m+1\}.$$

We claim that $d_{n+1}d_{n+2}\ldots = m^\infty$. This can be verified by the following observations.

- If $d_{n+1} = m-1$, then by using $d_n > 0$ and (2.3) in Lemma 2.1 it follows that

  $$d_{n+1}d_{n+2}\ldots \succ \overline{\alpha(q)} \succcurlyeq \overline{\alpha(p_2)} = ((m-1)(m+1))^\infty,$$

  which implies $d_{n+2} \geq m+1$.

  On the other hand, by using $d_{n+1} = m-1 < M$ and (2.3) in Lemma 2.1 we obtain

  $$d_{n+2}d_{n+3}\ldots \prec \alpha(q) \preccurlyeq \alpha(p_2) = ((m+1)(m-1))^\infty.$$

  Therefore, $d_{n+2} = m+1$.
- If $d_{n+1} = m+1$, then by using $d_n < M$ and (2.3) in Lemma 2.1 it follows that

  $$d_{n+1}d_{n+2}\cdots \prec \alpha(q) \preccurlyeq \alpha(p_2) = ((m+1)(m-1))^\infty,$$

  which implies $d_{n+2} \leq m-1$.

  On the other hand, by using $d_{n+1} = m+1 > 0$ and (2.3) in Lemma 2.1 it follows that

  $$d_{n+2}d_{n+3}\ldots \succ \overline{\alpha(q)} \succcurlyeq \overline{\alpha(p_2)} = ((m-1)(m+1))^\infty.$$

  Therefore, $d_{n+2} = m-1$.

By the above arguments we conclude that if $d_{n+1} = m-1$ then $(d_i) = d_1((m-1)(m+1))^\infty$, and if $d_{n+1} = m+1$ then $(d_i) = d_1((m+1)(m-1))^\infty$. This leads to a contradiction with Lemma 2.1 and (2.3). Therefore,

$$(d_i) = d_1 m^\infty \quad \text{with} \quad 0 < d_1 \leq m. \tag{2.4}$$

Case (II). $n > 1$. Since $d_{n-1} = 0$, we have by using (2.3) in Lemma 2.1 that $d_n \in \{1, \cdots, m+1\}$. If $d_n = m+1$, then by the same arguments as in Case I it follows that

$$(d_i) = 0^{n-1}((m+1)(m-1))^\infty,$$

leading to a contradiction with Lemma 2.1 and (2.3). Then $0 < d_n \leq m < M$. In a similar way as in Case I we conclude that

$$(d_i) = 0^{n-1} d_n m^\infty \quad \text{with} \quad 0 < d_n \leq m. \tag{2.5}$$

Therefore, by (2.4) and (2.5) we establish the "$\subseteq$" part. □

Now we turn to the case $M = 2m - 1$. The following characterization of the set $\mathcal{U}'_q$ was implicitly given in [11, Proposition 13].

Proposition 2.4. *If $M = 2m - 1$, then for $p_1 < q \le p_2$ we have*

$$\mathcal{U}'_q = \{0^\infty, \overline{0^\infty}\} \cup \bigcup_{k=0}^{\infty} \bigcup_{u=0}^{m-1} \{0^k u(m(m-1))^\infty, 0^k u((m-1)m)^\infty\}$$
$$\cup \bigcup_{k=0}^{\infty} \bigcup_{u=0}^{m-1} \left\{\overline{0^k u(m(m-1))^\infty}, \overline{0^k u((m-1)m)^\infty}\right\}.$$

Proof. For $m = 1$ the proposition was established by Glendinning and Sidorov [11]. In the following we assume $m \ge 2$.

The "$\supseteq$" part can be easily verified by using Lemma 2.1 and (2.1). Then it suffices to prove the "$\subseteq$" part.

Take $(d_i) \in \mathcal{U}'_q$ with $q \in (p_1, p_2]$. By symmetry we assume $d_1 \le m - 1$. Excluding the trivial case that $(d_i) = 0^\infty$ let $n \ge 1$ be the smallest integer such that $d_n > 0$. We split the proof into the following two cases: (I) $n = 1$; (II) $n > 1$.

Case (I). $n = 1$. Then $0 < d_n \le m - 1$. Note by (2.2) and Lemma 2.1 that

$$\alpha(q) \preccurlyeq \alpha(p_2) = (mm(m-1)(m-1))^\infty. \tag{2.6}$$

By Lemma 2.1 it follows that $d_{n+1} \in \{m-1, m\}$. We claim that $d_{n+1}d_{n+2}\dots$ equals $(m(m-1))^\infty$ or its reflection $((m-1)m)^\infty$.

- If $d_{n+1}d_{n+2} = (m-1)(m-1)$, then by using $d_n > 0$ and (2.6) in Lemma 2.1 it follows that

  $$d_{n+1}d_{n+2}\dots \succ \overline{\alpha(q)} \succcurlyeq \overline{\alpha(p_2)} = ((m-1)(m-1)mm)^\infty,$$

  which implies $d_{n+3}d_{n+4} \succcurlyeq mm$.
  On the other hand, by using $d_{n+2} = m - 1 < M$ and (2.6) in Lemma 2.1 we have

  $$d_{n+3}d_{n+4}\dots \prec \alpha(q) \preccurlyeq \alpha(p_2) = (mm(m-1)(m-1))^\infty.$$

  Therefore, $d_{n+3}d_{n+4} = mm$.
- If $d_{n+1}d_{n+2} = mm$, then by using $d_n < M$ and (2.6) in Lemma 2.1 it follows that

  $$d_{n+1}d_{n+2}\dots \prec \alpha(q) \preccurlyeq \alpha(p_2) = (mm(m-1)(m-1))^\infty,$$

  which implies $d_{n+3}d_{n+4} \preccurlyeq (m-1)(m-1)$.
  On the other hand, by using $d_{n+2} = m > 0$ and (2.6) in Lemma 2.1 it gives that

  $$d_{n+3}d_{n+4}\dots \succ \overline{\alpha(q)} \succcurlyeq \overline{\alpha(p_2)} = ((m-1)(m-1)mm)^\infty.$$

  Therefore, $d_{n+3}d_{n+4} = (m-1)(m-1)$.

Hence, by the above arguments it follows that if $d_{n+1}d_{n+2} = mm$ then $(d_i) = d_1(mm(m-1)(m-1))^\infty$, and if $d_{n+1}d_{n+2} = (m-1)(m-1)$ then $(d_i) = d_1((m-1)(m-1)mm)^\infty$. This leads to a contradiction with Lemma 2.1 and (2.6). Therefore,

$$(d_i) = d_1(m(m-1))^\infty \quad \text{or} \quad d_1((m-1)m)^\infty, \tag{2.7}$$

where $0 < d_1 \le m-1$.

Case (II). $n > 1$. Then by using $d_{n-1} = 0 < M$ in Lemma 2.1 it follows that

$$d_n, d_{n+1} \in \{1, \cdots, m\}.$$

If $d_n = m$, then $d_n > 0$, and by using (2.6) in Lemma 2.1 it follows that $d_{n+1} \ge m-1$. By the same arguments as in Case I it follows that

$$(d_i) = 0^{n-1}(m(m-1))^\infty. \tag{2.8}$$

If $0 < d_n < m \le M$, then by a similar way as in Case (I) we conclude that

$$(d_i) = 0^{n-1}d_n(m(m-1))^\infty \quad \text{or} \quad 0^{n-1}d_n((m-1)m)^\infty, \tag{2.9}$$

where $0 < d_n < m$.

Therefore, by (2.7)–(2.9) we prove the "$\subseteq$" part. $\square$

To find the smallest base of $\mathcal{B}_2(M)$ we still need the following geometrical explanation of expansions in non-integer bases. For $k \in \{0, 1, \ldots, M\}$ and $q \in (1, M+1]$ let

$$f_k(x) = \frac{x+k}{q}.$$

Then the interval $I_{q,M} = [0, M/(q-1)]$ can be written as

$$I_{q,M} = \bigcup_{k=0}^{M} f_k(I_{q,M}) = \bigcup_{k=0}^{M} \left[\frac{k}{q}, \frac{M}{q(q-1)} + \frac{k}{q}\right]. \tag{2.10}$$

Note that $q \le M+1$. This implies that the subintervals $f_k(I_{q,M})$ and $f_{k+1}(I_{q,M})$ are overlapped for each $k \in \{0, 1, \ldots, M-1\}$.

Take a point $x \in I_{q,M}$ with a q-expansion $(x_i(q))$, i.e., $x = \sum_{i=1}^\infty x_i(q)/q^i$. If $\sum_{i=1}^\infty x_{j+i}(q)/q^i \in f_{k_1}(I_{q,M}) \cap f_{k_2}(I_{q,M})$ for some $j \ge 0$ and $k_1 < k_2$, then x has at least two q-expansions: one begins with $x_1(q)\cdots x_j(q)k_1$, and the other begins with $x_1(q)\cdots x_j(q)k_2$. Therefore, a point $x \in I_{q,M}$ has a unique q-expansion $(x_i(q))$ if and only if

$$\sum_{i=1}^\infty \frac{x_{j+i}(q)}{q^i} \notin \bigcup_{k_1<k_2} f_{k_1}(I_{q,M}) \cap f_{k_2}(I_{q,M}) \quad \text{for any } j \ge 0.$$

Observe that the fundamental intervals $f_0(I_{q,M}), f_1(I_{q,M}), \ldots, f_M(I_{q,M})$ are located from the left to right. $f_0(I_{q,M})$ is the most left one, and $f_M(I_{q,M})$ is the most right one. Then by (2.10) and the definition of p_1 in (2.1) one can easily verify the following lemma.

Lemma 2.5. *Let $q > p_1$. Then $f_{k_1}(I_{q,M}) \cap f_{k_2}(I_{q,M}) \cap f_{k_3}(I_{q,M}) = \emptyset$ for any $k_1 < k_2 < k_3$.*

Clearly, by Lemma 2.5 it follows that for $q > p_1$ each $x \in I_{q,M}$ belongs to at most two fundamental intervals of $\{f_k(I_{q,M}) : k = 0, \ldots, M\}$. This is a very useful property which is helpful to find the smallest base of $\mathcal{B}_2(M)$ in the remaining part of the paper.

3. Smallest base of $\mathcal{B}_2(M)$ with $M = 2m$

In this section we will determine the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$ for $M = 2m$ and will prove the first statement of Theorem 1.1. For $q > 1$ and an expansion $(d_i) \in \{0, 1, \ldots, M\}^\infty$ we set

$$((d_i))_q := \sum_{i=1}^{\infty} \frac{d_i}{q^i}.$$

By (2.1) and (2.2) it follows that

$$p_1 = m + 1 \quad \text{and} \quad p_2 = \frac{m + 1 + \sqrt{m^2 + 6m + 1}}{2}. \tag{3.1}$$

By Proposition 2.3 and Lemma 2.5 it follows that

$$x = (100m^\infty)_{p_2} = (0\overline{(m-1)m^\infty})_{p_2}$$

has exactly two p_2-expansions, i.e., $p_2 \in \mathcal{B}_2(M)$. This implies

$$q_2(M) \in \mathcal{B}_2(M) \cap (p_1, p_2].$$

In the following lemma we give a characterization of the set $\mathcal{B}_2(M) \cap (p_1, p_2]$.

Lemma 3.1. *Let $M = 2m$ and $q \in (p_1, p_2]$. Then $q \in \mathcal{B}_2(M)$ if and only if q is a root of*

$$(10^k u m^\infty)_q = (0\overline{0^j v m^\infty})_q, \tag{3.2}$$

for some $k, j = 0, 1, \cdots$ and $u, v \in \{0, \cdots, m\}$.

Proof. First we prove the sufficiency. Take $q \in (p_1, p_2]$. Suppose that $(10^k u m^\infty)_q = (0\overline{0^j v m^\infty})_q$ for some $k, j = 0, 1, \ldots$ and $u, v \in \{0, \ldots, m\}$. Then

$$x := (10^k u m^\infty)_q = (0\overline{0^j v m^\infty})_q$$

has at least two different q-expansions. Let (x_i) be a q-expansion of x. Then $x_1 \in \{0, 1\}$ by Lemma 2.5. When $x_1 = 1$, by Proposition 2.3 it yields that $qx - 1 = (0^k um^\infty)_q$ has a unique q-expansion. When $x_1 = 0$, by Proposition 2.3 we also have that $qx = (\overline{0^j vm^\infty})_q$ has a unique q-expansion. Thus x has exactly two different q-expansions, and so $q \in \mathcal{B}_2(M)$.

Now we consider the necessity. Take $q \in (p_1, p_2] \cap \mathcal{B}_2(M)$. Then there exists $x \in I_{q,M}$ having exactly two different q-expansions (a_i) and (b_i), i.e.,

$$((a_i))_q = x = ((b_i))_q. \tag{3.3}$$

Let $n \geq 1$ be the least integer such that $a_n \neq b_n$. Without loss of generality we assume $a_n > b_n$. Then by (3.3) it follows that

$$(a_n a_{n+1} \cdots)_q = (b_n b_{n+1} \cdots)_q \quad \text{and} \quad (a_{n+i}),\ (b_{n+i}) \in \mathcal{U}'_q.$$

By Lemma 2.5 we have $a_n = b_n + 1$, and therefore

$$\frac{1}{q} = \frac{1}{q} \sum_{k=1}^{\infty} \frac{b_{n+k} - a_{n+k}}{q^k} \leq \frac{b_{n+1} - a_{n+1}}{q^2} + \sum_{k=3}^{\infty} \frac{2m}{q^k}.$$

This, together with $q > p_1 = m + 1$, implies that $a_{n+1} < b_{n+1}$. Hence,

$$(1a_{n+1}a_{n+2}\ldots)_q = (0b_{n+1}b_{n+2}\ldots)_q, \tag{3.4}$$

where $a_{n+1} < b_{n+1}$ and $(a_{n+i}), (b_{n+i}) \in \mathcal{U}'_q$.

Now we claim that (a_{n+i}) can not be of the form $\overline{0^j vm^\infty}$, and (b_{n+i}) can not be of the form $0^k um^\infty$ for any $k, j = 0, 1, \ldots, \infty$ and $u, v \in \{0, 1, \ldots, m\}$.

- If (a_{n+i}) is of the form $\overline{0^j vm^\infty}$, then by (3.4) it follows that

  $$(Ma_{n+1}a_{n+2}\ldots)_q = ((M-1)b_{n+1}b_{n+2}\ldots)_q$$

  has at least two q-expansions. However, by Proposition 2.3 we know that $(Ma_{n+1}a_{n+2}\ldots)_q$ has a unique q-expansion, leading to a contradiction.
- If (b_{n+i}) is of the form $0^k um^\infty$, then by (3.4) it gives that

  $$(1a_{n+1}a_{n+2}\ldots)_q = (0b_{n+1}b_{n+2}\ldots)_q$$

  has at least two q-expansions. This also leads to a contradiction, since by Proposition 2.3 we know that $(0b_{n+1}b_{n+2}\cdots)_q$ should have a unique q-expansion.

Therefore, by (3.4) and Proposition 2.3 it follows that $q \in (p_1, p_2]$ satisfies

$$(10^k um^\infty)_q = (0\overline{0^j vm^\infty})_q$$

for some $k, j = 0, 1, \ldots, \infty$ and $u, v \in \{0, 1, \ldots, m\}$. So, we will finish the proof by showing that $k, j \neq \infty$. Since the proof for $j \neq \infty$ is similar, here we only prove $k \neq \infty$.

Suppose on the contrary that $k = \infty$. Then $1 = ((2m)^j(2m - v)m^\infty)_q$ for some $j = 0, 1, \ldots, \infty$ and $v \in \{0, 1, \ldots, m\}$. By Lemma 2.1 it follows that

$$\alpha(q) = (2m)^j(2m - v)m^\infty.$$

Then for $j = 0$ and $v = m$ we have $\alpha(q) = m^\infty = \alpha(p_1)$, and for any other parameters j and v we have $\alpha(q) \succ ((m+1)(m-1))^\infty = \alpha(p_2)$. By Lemma 2.1 it follows that $q = p_1$ or $q > p_2$, leading to a contradiction with our hypothesis that $q \in (p_1, p_2]$. □

Observe that the smallest base $q_2(M)$ belongs to the interval $(p_1, p_2]$. Furthermore, by Lemma 3.1 to find the smallest base $q_2(M)$ it suffices to investigate the solutions of countably many equations as in (3.2). If the parameters k, j, u and v satisfy (3.7) (see below), then by Lemma 3.3 it follows that Equation (3.2) has a unique root in (p_1, ∞), say $q_{k,j,u,v}$. Moreover, we will show in Lemma 3.4 that these bases $(q_{k,j,u,v})$ are monotonic w.r.t. the parameters k, j, u and v. This implies that we are able to determine the smallest base $q_2(M)$ among them.

First we need to show that these bases $q_{k,j,u,v}$ are well-defined. Note that $q_{k,j,u,v} \in (p_1, \infty)$ is a zero of the following function

$$\begin{aligned} f_{k,j,u,v}(q) &= (q^3 - q^2)\big((10^k u m^\infty)_q - (0\overline{0^j v m^\infty})_q\big) \\ &= -q - 2mq + q^2 + q^{-k}(m - u + uq) + q^{-j}(m - v + vq). \end{aligned} \tag{3.5}$$

Then the uniqueness of $q_{k,j,u,v}$ follows by the following lemma which says that the function $f_{k,j,u,v}$ is monotonic in (p_1, ∞).

Lemma 3.2. *Given $k, j \geq 0$ and $u, v \in \{0, 1, \ldots, m\}$, the function $f_{k,j,u,v}$ is strictly increasing in (p_1, ∞).*

Proof. Differentiating $f_{k,j,u,v}$ in (3.5) it gives

$$\begin{aligned} f'_{k,j,u,v}(q) = -1 - 2m + 2q + q^{-k}\left(u - ku + \frac{ku - km}{q}\right) \\ + q^{-j}\left(v - jv + \frac{jv - jm}{q}\right). \end{aligned}$$

Since $q > p_1 = m + 1$, we have $-1 - 2m + 2q > 1$. In order to guarantee the positivity of $f'_{k,j,u,v}(q)$, by symmetry it suffices to prove

$$q^{-k}\left(u - ku + \frac{ku - km}{q}\right) \geq -\frac{1}{2} \tag{3.6}$$

for any $k \geq 0$ and $u \in \{0, \ldots, m\}$.

Clearly, the inequality (3.6) holds for $k = 0$ or 1. For $k \geq 2$ we deduce by using $q > p_1 = m + 1$ that

$$1 - k + \frac{k}{q} \leq 1 - k + \frac{k}{m+1} \leq 1 - \frac{k}{2} \leq 0,$$

and therefore

$$\begin{aligned} q^{-k}\left(u - ku + \frac{ku - km}{q}\right) &= -\frac{km}{q^{k+1}} + \frac{u}{q^k}\left(1 - k + \frac{k}{q}\right) \\ &\geq -\frac{km}{q^{k+1}} + \frac{m}{q^k}\left(1 - k + \frac{k}{q}\right) \\ &= \frac{m(1-k)}{q^k} > \frac{1-k}{q^{k-1}} \geq -\frac{1}{2}, \end{aligned}$$

where the last inequality follows by $2(k-1) \leq 2^{k-1} \leq q^{k-1}$ for any $k \geq 2$. This establishes (3.6). □

By Lemma 3.2 it follows that the function $f_{k,j,u,v}$ has at most one zero in (p_1, ∞). In the following lemma we characterize the parameters k, j, u and v such that the function $f_{k,j,u,v}$ indeed has a unique zero in (p_1, ∞).

Lemma 3.3. *The equation $f_{k,j,u,v}(q) = 0$ has a unique root in (p_1, ∞) if and only if the parameters k, j, u and v satisfy*

$$k, j \geq 0, \quad u, v \in \{0, 1, \ldots, m\}, \quad \textit{and} \quad \frac{u+1}{(m+1)^{k+1}} + \frac{v+1}{(m+1)^{j+1}} < 1. \tag{3.7}$$

Proof. By Lemma 3.2 and the continuity of $f_{k,j,u,v}$ it follows that the equation $f_{k,j,u,v}(q) = 0$ has a unique root in (p_1, ∞) if and only if

$$f_{k,j,u,v}(p_1) < 0.$$

Observe by using (3.1) in (3.5) that

$$f_{k,j,u,v}(p_1) = m(m+1)\left(\frac{u+1}{(m+1)^{k+1}} + \frac{v+1}{(m+1)^{j+1}} - 1\right).$$

This prove the lemma. □

Lemma 3.3 implies that if the parameters k, j, u and v satisfy (3.7) then the root $q_{k,j,u,v} \in (p_1, \infty)$ is well-defined. In the following lemma we will investigate the monotonicity of the bases $(q_{k,j,u,v})$ w.r.t. these parameters k, j, u and v.

Lemma 3.4.

(1). *The sequence* $(q_{k,j,u,v})$ *is strictly increasing w.r.t. the parameters* k *and* j*;*
(2). *The sequence* $(q_{k,j,u,v})$ *is strictly decreasing w.r.t. the parameters* u *and* v*.*

Proof. First we prove (1). Since the proof for j is analogous, we only give the proof for the parameter k.

Fix $j \geq 0$ and $u, v \in \{0, 1, \cdots, m\}$. We write $q_k = q_{k,j,u,v}$. Then by (3.5) we have

$$\begin{aligned} f_{k+1,j,u,v}(q_{k+1}) &= -q_{k+1} - 2mq_{k+1} + q_{k+1}^2 + q_{k+1}^{-k-1}(m - u + q_{k+1}u) \\ &\quad + q_{k+1}^{-j}(m - v + q_{k+1}v) \\ &< -q_{k+1} - 2mq_{k+1} + q_{k+1}^2 + q_{k+1}^{-k}(m - u + q_{k+1}u) \\ &\quad + q_{k+1}^{-j}(m - v + q_{k+1}v) \\ &= f_{k,j,u,v}(q_{k+1}), \end{aligned}$$

where the strict inequality holds since $m - u + q_{k+1}u > 0$. This, together with $f_{k+1,j,u,v}(q_{k+1}) = 0 = f_{k,j,u,v}(q_k)$, implies that

$$f_{k,j,u,v}(q_k) < f_{k,j,u,v}(q_{k+1}).$$

Therefore, by Lemma 3.2 it follows that $q_k < q_{k+1}$.

Now we turn to prove (2). The proof for v is similar. Here we only give the proof for the parameter u. Fix $k, j \geq 0$ and $v \in \{0, 1, \ldots, m\}$. For simplicity we denote by $q_u = q_{k,j,u,v}$. Then by (3.5) it follows that

$$\begin{aligned} f_{k,j,u+1,v}(q_{u+1}) &= -q_{u+1} - 2mq_{u+1} + q_{u+1}^2 + q_{u+1}^{-k}\big(m - (u+1) + q_{u+1}(u+1)\big) \\ &\quad + q_{u+1}^{-j}(m - v + q_{u+1}v) \\ &> -q_{u+1} - 2mq_{u+1} + q_{u+1}^2 + q_{u+1}^{-k}(m - u + q_{u+1}u) \\ &\quad + q_{u+1}^{-j}(m - v + q_{u+1}v) \\ &= f_{k,j,u,v}(q_{u+1}). \end{aligned}$$

Observe that $f_{k,j,u+1,v}(q_{u+1}) = 0 = f_{k,j,u,v}(q_u)$. This implies that $f_{k,j,u,v}(q_u) > f_{k,j,u,v}(q_{u+1})$. By Lemma 3.2 we conclude that $q_u > q_{u+1}$. □

Now we investigate the set $\mathcal{B}_2(M) \cap (p_1, p_2]$, and determine the smallest base $q_2(M)$ for $M = 2m$.

Proposition 3.5. *Let* $M = 2m$*. Then*

$$\mathcal{B}_2(M) \cap (p_1, p_2] = \{q_{1,0,u,m-1} : u = 0, 1, \cdots, m-1\}.$$

Furthermore, the smallest base of $\mathcal{B}_2(M)$ *is*

$$q_2(M) = q_{1,0,m-1,m-1} = \frac{m+1+\sqrt{m^2+2m+5}}{2}.$$

Proof. By Lemma 3.1 it suffices to investigate the parameters k, j, u and v such that

$$p_1 < q_{k,j,u,v} \le p_2.$$

By Lemmas 3.2, 3.3 and by (3.5) it follows that $q_{k,j,u,v} \in \mathcal{B}_2(M) \cap (p_1, p_2]$ if and only if the parameters k, j, u and v satisfy

$$f_{k,j,u,v}(p_1) < 0, \quad f_{k,j,u,v}(p_2) \ge 0,$$

which is equivalent to that the parameters k, j, u and v satisfy (3.7) and

$$m(1-p_2) + \frac{m-u+up_2}{p_2^k} + \frac{m-v+vp_2}{p_2^j} \ge 0. \tag{3.8}$$

Note by (3.5) that $f_{k,j,u,v}(q) = f_{j,k,v,u}(q)$. Then we may assume $k \ge j$.

If $m = 1$, then by (3.5) and Lemma 3.4 one can verify that $q_{k,j,u,v} \in (p_1, p_2]$ if and only if

$$(k, j, u, v) \in \{(2,1,1,1), (2,0,1,0), (1,1,1,0), (1,1,0,1), (1,0,0,0)\}.$$

Note that $q_{2,1,1,1} = q_{2,0,1,0} = q_{1,1,1,0} = q_{1,1,0,1} = q_{1,0,0,0} = 1+\sqrt{2}$. Therefore,

$$\mathcal{B}_2 \cap (p_1, p_2] = \left\{1+\sqrt{2}\right\}.$$

In the following we will assume $m \ge 2$. First we show that $j = 0$. Note by (3.5) that

$$q_{1,1,m,m} = 2m > p_2.$$

Then by Lemma 3.4 we have $j = 0$ as required. So, by (3.7) we have $v \le m-1$. Moreover, one can check that $q_{2,0,m,m-1} = p_2$. By Lemma 3.4 this implies that $q_{2,0,u,m-1} > p_2$ for $u < m$, and that $q_{k,0,u,m-1} > p_2$ for $k \ge 3$. Note that $q_{1,0,0,m-1} = p_2$. Hence, it suffices to consider $k \le 1$.

If $k = j = 0$, then by (3.5) we have

$$q_{0,0,u,v} = 2m - u - v,$$

which can not fall into the interval $(p_1, p_2]$, since by (3.1) we have

$$m+1 = p_1 < p_2 = \frac{m+1+\sqrt{(m+1)^2+4m}}{2} < m+2.$$

If $k = 1$, $j = 0$, then by (3.5) we have

$$q_{1,0,u,v} = \frac{2m - v + \sqrt{(2m-v)^2 + 4(m-u)}}{2}.$$

Furthermore, one can check that $q_{1,0,u,v} \in (p_1, p_2]$ if and only if $v = m-1$ and $0 \le u \le m-1$.

Hence, by Lemma 3.4 it follows that

$$q_2(2m) = q_{1,0,m-1,m-1} = \frac{m+1+\sqrt{m^2+2m+5}}{2}. \qquad \square$$

4. Smallest base of $\mathcal{B}_2(M)$ with $M = 2m - 1$

In this section we are going to investigate the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$ for $M = 2m-1$, and prove the second statement of Theorem 1.1. The idea is similar to the case for $M = 2m$ as in Section 3. But here we need more effort for the reason that the unique expansions described in Proposition 2.4 for $M = 2m - 1$ are more complicated than those in Proposition 2.3 for $M = 2m$.

Let $M = 2m - 1$. Recall from (2.1) and (2.2) that

$$p_1 = \frac{m + \sqrt{m^2 + 4m}}{2}, \tag{4.1}$$

and $p_2 \in (p_1, \infty)$ satisfies

$$p_2^3 = (m+1)p_2^2 - p_2 + m. \tag{4.2}$$

By Proposition 2.4 and Lemma 2.5 it follows that the number

$$(10^4((m-1)m)^\infty)_{p_2} = (0\overline{(m-1)((m-1)m)^\infty})_{p_2}$$

has exactly two p_2-expansions, i.e., $p_2 \in \mathcal{B}_2(M)$. This implies that $q_2(M) \in \mathcal{B}_2(M) \cap (p_1, p_2]$.

Similar to Lemma 3.1 we characterize the set $\mathcal{B}_2(M) \cap (p_1, p_2]$ for $M = 2m - 1$.

Lemma 4.1. *Let $M = 2m - 1$ and $q \in (p_1, p_2]$. Then $q \in \mathcal{B}_2(M)$ if and only if q satisfies one of the following equations:*

$$(10^{k_1} u_1 (m(m-1))^\infty)_q = (0\overline{0^{j_1} v_1 (m(m-1))^\infty})_q \tag{4.3}$$

$$(10^{k_2} u_2 ((m-1)m)^\infty)_q = (0\overline{0^{j_2} v_2 ((m-1)m)^\infty})_q \tag{4.4}$$

$$(10^{k_3} u_3 (m(m-1))^\infty)_q = (0\overline{0^{j_3} v_3 ((m-1)m)^\infty})_q \tag{4.5}$$

$$(10^{k_4} u_4 ((m-1)m)^\infty)_q = (0\overline{0^{j_4} v_4 (m(m-1))^\infty})_q \tag{4.6}$$

for some parameters $k_i, j_i = 0, 1, \cdots$, and $u_i, v_i \in \{0, 1, \cdots, m-1\}$, where $i = 1, 2, 3, 4$.

Proof. The sufficiency follows by Proposition 2.4 and Lemma 2.5. In the following we prove the necessity.

Take $q \in (p_1, p_2] \cap \mathcal{B}_2(M)$. Then there exists $x \in I_{q,M}$ having exactly two different q-expansions (a_i) and (b_i), i.e.,

$$((a_i))_q = x = ((b_i))_q. \tag{4.7}$$

Let $n \geq 1$ be the least integer such that $a_n \neq b_n$. Without loss of generality we assume $a_n > b_n$. Then by (4.7) it follows that

$$(a_n a_{n+1} \cdots)_q = (b_n b_{n+1} \cdots)_q \quad \text{and} \quad (a_{n+i}),\ (b_{n+i}) \in \mathcal{U}_q'$$

By Lemma 2.5 we have $a_n = b_n + 1$, and therefore

$$1 = (b_{n+1} b_{n+2} \cdots)_q - (a_{n+1} a_{n+2} \cdots)_q \tag{4.8}$$

where $(a_{n+i}),\ (b_{n+i}) \in \mathcal{U}_q'$.

Now we claim that (a_{n+i}) can be neither of the form $\overline{0^j v(m(m-1))^\infty}$ nor of the form $\overline{0^j v((m-1)m)^\infty}$, and (b_{n+i}) can be neither the form $0^k u(m(m-1))^\infty$ nor of the form $0^k u((m-1)m)^\infty$, where $k, j = 0, 1, 2, \cdots, \infty$ and $u, v \in \{0, 1, \cdots, m-1\}$.

- If (a_{n+i}) is of the form $\overline{0^j v(m(m-1))^\infty}$ or $\overline{0^j v((m-1)m)^\infty}$ with $j = 0, 1, \cdots, \infty$ and $v \in \{0, 1, \cdots, m-1\}$, then by (4.8) and (2.1) it follows that

$$\begin{aligned} 1 = (b_{n+1} b_{n+2} \cdots)_q - (a_{n+1} a_{n+2} \cdots)_q &\leq ((2m-1)^\infty)_q - ((m(m-1))^\infty)_q \\ &< ((m(m-1))^\infty)_{p_1} = 1, \end{aligned}$$

  leading to a contradiction.
- If (b_{n+i}) is of the form $0^k u(m(m-1))^\infty$ or $0^k u((m-1)m)^\infty$ with $k = 0, 1, \cdots, \infty$ and $u \in \{0, 1, \cdots, m-1\}$, then by (4.8) and (2.1) it follows that

$$\begin{aligned} 1 = (b_{n+1} b_{n+2} \cdots)_q - (a_{n+1} a_{n+2} \cdots)_q &\leq ((m(m-1))^\infty)_q \\ &< ((m(m-1))^\infty)_{p_1} = 1, \end{aligned}$$

  again leading to a contradiction.

Therefore, by Proposition 2.4 it follows that (a_{n+i}) is of the form

$$0^k u(m(m-1))^\infty \quad \text{or} \quad 0^k u((m-1)m)^\infty,$$

and (b_{n+i}) is of the form $\overline{0^j v(m(m-1))^\infty}$ or $\overline{0^j v((m-1)m)^\infty}$, where $k, j = 0, 1, \cdots, \infty$ and $u, v \in \{0, 1, \cdots, m-1\}$. Hence, to finish the proof it suffices to prove $k, j \neq \infty$. Since the proof for $j \neq \infty$ is similar, we only prove $k \neq \infty$.

Suppose on the contrary that $k = \infty$. Then $(a_{n+i}) = 0^\infty$. Note that (b_{n+i}) is of the form $\overline{0^j v(m(m-1))^\infty}$ or $\overline{0^j v((m-1)m)^\infty}$ with $j = 0, 1, \cdots, \infty$ and $v \in \{0, 1, \cdots, m-1\}$. Then by (4.8) and Lemma 2.1 it follows that

$$\alpha(q) = b_{n+1}b_{n+2}\cdots.$$

This implies that

$$\alpha(q) \preccurlyeq (m(m-1))^\infty = \alpha(p_1) \quad \text{or} \quad \alpha(q) \succ (mm(m-1)(m-1))^\infty = \alpha(p_2).$$

Then by Lemma 2.1 we have $q \notin (p_1, p_2]$, leading to a contradiction with the hypothesis that $q \in (p_1, p_2]$. $\square$

Remark 4.2. We point out that (4.5) and (4.6) are equivalent. In fact, if q is a root of (4.5) for some $k_3, j_3 \geq 0$ and $u_3, v_3 \in \{0, 1, \cdots, m-1\}$, then by reflection we have

$$(10^{j_3} v_3((m-1)m)^\infty)_q = (0\overline{0^{k_3} u_3(m(m-1))^\infty})_q.$$

This corresponds to (4.6) with $(k_4, j_4, u_4, v_4) = (j_3, k_3, v_3, u_3)$.

By Lemma 4.1 and Remark 4.2 to find the smallest base $q_2(M)$ for $M = 2m - 1$ it suffices to investigate the appropriate roots of Equations (4.3)–(4.5). In a similar way as in Section 3 we will show that these roots also have the monotonicity. This allows us to determine the smallest base $q_2(M)$. In this direction we split the proof into the following three subsections according to Equations (4.3)–(4.5).

4.1. Solutions of Equation (4.3)

Given $k, j \geq 0$ and $0 \leq u, v \leq m-1$, in terms of Equation (4.3) we define the function

$$\begin{aligned} g^{(1)}_{k,j,u,v}(q) &= (q^3 - q)\big((10^k u(m(m-1))^\infty)_q - (0\overline{0^j v(m(m-1))^\infty})_q\big) \\ &= (q+1)(q-2m) + q^{-k-1}(m-1-u+mq+uq^2) \\ &\quad + q^{-j-1}(m-1-v+mq+vq^2). \end{aligned} \tag{4.9}$$

By Lemma 4.1 we only need to consider the zeros of $g^{(1)}_{k,j,u,v}$ in (p_1, ∞). In the following lemma we show that the function $g^{(1)}_{k,j,u,v}$ is monotonic.

Lemma 4.3. *For any $k, j \geq 0$ and $u, v \in \{0, 1, \cdots, m-1\}$ the function $g^{(1)}_{k,j,u,v}$ is strictly increasing in (p_1, ∞).*

Proof. In terms of (4.9) and by symmetry it suffices to prove that

$$h_{k,u}(q) = \frac{(q+1)(q-2m)}{2} + q^{-k-1}(m-1-u+mq+uq^2)$$

has a positive derivative in (p_1, ∞) for any $k \geq 0$ and $u \in \{0, 1, \cdots, m-1\}$. Differentiating $h_{k,u}$ it yields that

$$\begin{aligned} h'_{k,u}(q) = q - m + \frac{1}{2} + uq^{-k}\left(\frac{k+1}{q^2} + 1 - k\right) \\ + q^{-k}\left(-(k+1)\frac{m-1}{q^2} - \frac{km}{q}\right). \end{aligned} \tag{4.10}$$

Then by using $q > p_1$ and $p_1^2 = mp_1 + m$ in (4.10) one can show that $h'_{k,u}(q) > 0$ for $k = 0, 1$ and 2.

If $k \geq 3$, then by using $q > p_1$ we have $(k+1)/q^2 + 1 - k \leq 0$. Moreover, one can show that the function

$$\phi(k) = q^{-k}\left(\frac{km}{q} + (m-1)(k-1)\right)$$

satisfies $\phi(k+1) < \phi(k)$ for any $k \geq 2$. Therefore, by using $q > p_1$ and $p_1^2 = mp_1 + m$ in (4.10) it follows that

$$\begin{aligned} h'_{k,u}(q) &\geq q - m + \frac{1}{2} - q^{-k}\left(\frac{km}{q} + (m-1)(k-1)\right) \\ &\geq q - m + \frac{1}{2} - q^{-2}\left(\frac{2m}{q} + (m-1)\right) \\ &\geq p_1 - m + \frac{1}{2} - \frac{m-1}{p_1^2} - \frac{2m}{p_1^3} \\ &= p_1^{-3}\left(\frac{1}{2}p_1^3 + p_1 - m\right) > 0. \qquad \square \end{aligned}$$

Lemma 4.3 implies that the function $g^{(1)}_{k,j,u,v}$ has at most one zero in (p_1, ∞). In the following lemma we characterize those parameters k, j, u and v such that $g^{(1)}_{k,j,u,v}$ has a (unique) zero in (p_1, ∞).

Lemma 4.4. *The equation $g^{(1)}_{k,j,u,v}(q) = 0$ has a unique root in (p_1, ∞) if and only if the parameters k, j, u and v satisfy*

$$k, j \geq 0, \quad u, v \in \{0, 1, \ldots, m-1\} \quad \textit{and} \quad \frac{u+1}{mp_1^k} + \frac{v+1}{mp_1^j} < 1. \tag{4.11}$$

Proof. By Lemma 4.3 and the continuity of $g^{(1)}_{k,j,u,v}$ it follows that the equation $g^{(1)}_{k,j,u,v}(q) = 0$ has a unique root in (p_1, ∞) if and only if

$$g^{(1)}_{k,j,u,v}(p_1) < 0.$$

Note by using $p_1^2 = mp_1 + m$ in (4.9) that

$$g^{(1)}_{k,j,u,v}(p_1) = \frac{p_1 - 1}{p_1}\left(-1 + \frac{u+1}{mp_1^k} + \frac{v+1}{mp_1^j}\right).$$

This establishes the lemma. □

By Lemma 4.4 it follows that if the parameters k, j, u and v satisfy (4.11) then the Equation (4.3) has a unique root in (p_1, ∞), say $q^{(1)}_{k,j,u,v}$. In a similar way as in Lemma 3.4 one can verify the following monotonicity of the sequence $(q^{(1)}_{k,j,u,v})$.

Lemma 4.5.

(1). *The sequence* $(q^{(1)}_{k,j,u,v})$ *is strictly increasing w.r.t. the parameters* k *and* j*;*
(2). *The sequence* $(q^{(1)}_{k,j,u,v})$ *is strictly decreasing w.r.t. the parameters* u *and* v*.*

In the following lemma we show that no bases in $(p_1, p_2] \cap \mathcal{B}_2(M)$ satisfy Equation (4.3).

Lemma 4.6. *Let* $M = 2m - 1$*. Then Equation* (4.3) *has no solutions in* $(p_1, p_2]$*.*

Proof. By Lemmas 4.3 and 4.4 it suffices to prove that no parameters (k, j, u, v) satisfy both (4.11) and $g^{(1)}_{k,j,u,v}(p_2) \geq 0$. Note by (4.9) that $g^{(1)}_{k,j,u,v}(q) = g^{(1)}_{j,k,v,u}(q)$. Then we may assume that $k \geq j$. Therefore, the lemma follows by observing the following three cases.

Case I. $k \geq j \geq 1$. Then by Lemma 4.5 it suffices to prove

$$q^{(1)}_{1,1,m-1,m-1} > p_2,$$

or equivalently, $g^{(1)}_{1,1,m-1,m-1}(p_2) < 0$. This follows by using (4.2) in (4.9) that

$$\begin{aligned} g^{(1)}_{1,1,m-1,m-1}(p_2) &= {p_2}^{-1}\left({p_2}^3 - (2m-1){p_2}^2 - 2p_2 + 2m\right) \\ &\leq {p_2}^{-1}\left((2-m){p_2}^2 - 2(p_2 - m)\right) < 0. \end{aligned}$$

Case II. $k > j = 0$. Then by Lemma 4.5 it suffices to prove that $q^{(1)}_{k,0,m-1,m-1} \leq p_1$ for all $k \geq 1$, and $q^{(1)}_{1,0,m-2,m-1} = q^{(1)}_{1,0,m-1,m-2} > p_2$. By (4.9) and (4.1) one can show that

$$g^{(1)}_{k,0,m-1,m-1}(p_1) = \frac{m}{p_1^k} + \frac{m-1}{p_1^{k-1}} > 0.$$

By Lemma 4.3 this implies $q^{(1)}_{k,0,m-1,m-1} < p_1$.

Moreover, by using (4.2) in (4.9) it follows that

$$g^{(1)}_{1,0,m-1,m-2}(p_2) = g^{(1)}_{1,0,m-2,m-1}(p_2) = -\frac{(1+m-p_2)(p_2^2-1)}{p_2} < 0.$$

Therefore, $q^{(1)}_{1,0,m-2,m-1} > p_2$.

Case III. $k = j = 0$. Then by (4.9) it follows that

$$q^{(1)}_{0,0,u,v} = \frac{2m-u-v-2}{2} + \frac{\sqrt{(2m-u-v)^2-4}}{2}.$$

By using (4.1) and (4.2) one can show that $q^{(1)}_{0,0,u,v} \notin (p_1, p_2]$ for any $u, v \in \{0, 1, \cdots, m-1\}$. □

4.2. Solutions of Equation (4.4)

In this subsection we consider possible roots in (p_1, ∞) of Equation (4.4). Clearly, these roots are also the zeros of the function

$$\begin{aligned} g^{(2)}_{k,j,u,v}(q) &= (q^3 - q)\big((10^k u((m-1)m)^\infty)_q - (\overline{00^j v((m-1)m)^\infty})_q\big) \\ &= (q+1)(q-2m) + q^{-k-1}(m - q - u + mq + uq^2) \\ &\quad + q^{-j-1}(m - q - v + mq + vq^2). \end{aligned} \tag{4.12}$$

Similar to Lemmas 4.3–4.5 we can show analogous results for $g^{(2)}_{k,j,u,v}$. Hence the function $g^{(2)}_{k,j,u,v}$ has a unique zero $q^{(2)}_{k,j,u,v} \in (p_1, \infty)$ if the parameters k, j, u and v satisfy

$$k, j \geq 0, \quad u, v \in \{0, 1, \ldots, m-1\} \quad \text{and} \quad \frac{up_1 + u + p_1}{p_1^{k+2}} + \frac{vp_1 + v + p_1}{p_1^{j+2}} < 1.$$

In the following lemma we describe those bases $q^{(2)}_{k,j,u,v}$ in $\mathcal{B}_2(M) \cap (p_1, p_2]$.

Lemma 4.7. *Let $M = 2m - 1$. Then $q^{(2)}_{k,j,u,v} \in (p_1, p_2] \cap \mathcal{B}_2(M)$ if and only if*

$$k \in \{2, 3\}, \quad j = 0, \quad u \in \{0, 1, \cdots, m-1\} \quad \text{and} \quad v = m-1,$$

or symmetrically,

$$k = 0, \quad j \in \{2, 3\}, \quad u = m-1 \quad \text{and} \quad v \in \{0, 1, \ldots, m-1\}.$$

Proof. The proof is similar to Lemma 4.6. Note that $g^{(2)}_{k,j,u,v} = g^{(2)}_{j,k,v,u}$. Then $q^{(2)}_{k,j,u,v} = q^{(2)}_{j,k,v,u}$. By symmetry we may assume $k \geq j$.

First we show that $j = 0$. By the monotonicity it suffices to prove $q^{(2)}_{1,1,m-1,m-1} > p_2$. This can be verified by using (4.2) in (4.12) that

$$g^{(2)}_{1,1,m-1,m-1}(p_2) = {p_2}^{-2}\big((2-m){p_2}^3 - 3p_2(p_2 - m) - 2(p_2 - 1)\big) < 0.$$

Then $q^{(2)}_{1,1,m-1,m-1} > p_2$. Hence, $j = 0$ as required.

Now we claim that $k \leq 3$. Then it suffices to prove $q^{(2)}_{4,0,m-1,m-1} > p_2$. By using (4.2) in (4.12) it follows that

$$g^{(2)}_{4,0,m-1,m-1}(p_2) = \frac{1-p_2}{p_2^5} < 0.$$

Then $q^{(2)}_{4,0,m-1,m-1} > p_2$, and therefore $k \leq 3$.

Moreover, we claim that $k \notin \{0, 1\}$. By (4.12) it follows that

$$q^{(2)}_{0,0,u,v} = \frac{2m-u-v-2}{2} + \frac{\sqrt{(2m-u-v-2)^2 + 4(2m-u-v)}}{2},$$

and

$$q^{(2)}_{1,0,u,v} = \frac{2m-v-1}{2} + \frac{\sqrt{(2m-v-1)^2 + 4(m-u)}}{2}.$$

By (4.1) and (4.2) one can verify that

$$q^{(2)}_{0,0,u,v} \notin (p_1, p_2], \quad q^{(2)}_{1,0,u,v} \notin (p_1, p_2]$$

for any $u, v \in \{0, 1, \cdots, m-1\}$.

Therefore, $k \in \{2, 3\}$ and $j = 0$. By (4.1), (4.2) and (4.12) it follows that $q^{(2)}_{2,0,u,v} \in (p_1, p_2]$ if and only if $v = m - 1$. Furthermore, one can show that $q^{(2)}_{3,0,u,v} \in (p_1, p_2]$ if and only if $v = m - 1$. □

4.3. Solutions of Equation (4.5)

In this subsection we consider the possible roots in (p_1, ∞) of Equation (4.5). Given $k, j \geq 1$ and $0 \leq u, v < m$, by Lemma 4.1 it suffices to consider the zeros of the function

$$\begin{aligned} g^{(3)}_{k,j,u,v}(q) &= (q^3 - q)\big((10^k u(m(m-1))^\infty)_q - (\overline{00^j v((m-1)m)^\infty})_q\big) \\ &= (q+1)(q-2m) + q^{-k-1}(m-1-u+mq+uq^2) \\ &\quad + q^{-j-1}(m-q-v+mq+vq^2). \end{aligned} \tag{4.13}$$

Similar to Lemmas 4.3–4.5, one can prove analogous results for $g^{(3)}_{k,j,u,v}$. Hence, the function $g^{(3)}_{k,j,u,v}$ has a unique zero $q^{(3)}_{k,j,u,v}$ in (p_1, ∞), if the parameters k, j, u and v satisfy

$$k, j \geq 0, \quad u, v \in \{0, 1, \ldots, m-1\} \quad \text{and} \quad \frac{u+1}{mp_1^k} + \frac{vp_1 + p_1 + v}{p_1^{j+2}} < 1.$$

By using the monotonicity we determine the bases $(q^{(3)}_{k,j,u,v})$ in $\mathcal{B}_2(M) \cap (p_1, p_2]$.

Lemma 4.8. *Let $M = 2m - 1$. Then $q^{(3)}_{k,j,u,v} \in (p_1, p_2] \cap \mathcal{B}_2(M)$ if and only if*

$$k = 2, \quad j = 0, \quad u \in \{0, 1, \cdots, m-2\}, \quad v = m-1,$$

or

$$k = 3, \quad j = 0, \quad u \in \{0, 1, \cdots, m-1\}, \quad v = m-1,$$

or $(k, j, u, v) = (4, 0, m-1, m-1)$.

Proof. First we show that either $k = 0$ or $j = 0$. Then it suffices to prove $q^{(3)}_{1,1,m-1,m-1} > p_2$. By using (4.2) in (4.13) it follows that

$$g^{(3)}_{1,1,m-1,m-1}(p_2) = p_2^{-2}\big((2-m)p_2^3 - 3(p_2^2 - mp_2) - (p_2 - 1)\big) < 0.$$

Hence, we have either $k = 0$ or $j = 0$.

Now we claim $k \neq 0$. Suppose on the contrary that $k = 0$. Then $q^{(3)}_{0,j,u,v} \in (p_1, p_2]$ if and only if

$$g^{(3)}_{0,j,u,v}(p_1) < 0, \quad g^{(3)}_{0,j,u,v}(p_2) \geq 0. \tag{4.14}$$

By using (4.1) and (4.2) in (4.13) one can verify that no parameters j, u, v satisfy (4.14). Therefore, $k > j = 0$.

Finally, by using (4.1) and (4.2) in (4.13) it follows that $q^{(3)}_{k,0,u,v} \in (p_1, p_2]$ if and only if

$$k = 2, \quad j = 0, \quad u \in \{0, 1, \cdots, m-2\}, \quad v = m-1,$$

or

$$k = 3, \quad j = 0, \quad u \in \{0, 1, \cdots, m-1\}, \quad v = m-1,$$

or

$$(k, j, u, v) = (4, 0, m-1, m-1). \qquad \square$$

Now by Lemmas 4.6–4.8 we give a complete description of the set $\mathcal{B}_2(M)\cap(p_1,p_2]$ for $M=2m-1$, and determine the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$.

Proposition 4.9. *Let $M=2m-1$. Then*

$$(p_1,p_2]\cap\mathcal{B}_2(M)=\bigcup_{k=2}^{3}\left(\bigcup_{u=0}^{m-1}\left\{q^{(2)}_{k,0,u,m-1}\right\}\cup\bigcup_{u=0}^{m-2}\left\{q^{(3)}_{k,0,u,m-1}\right\}\right).$$

Furthermore, the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$ is

$$q_2(M)=q^{(2)}_{2,0,m-1,m-1},$$

the unique root in $(p_1,p_2]$ of $x^4=(m-1)x^3+2mx^2+mx+1$.

Proof. Note by (4.12) and (4.13) that $g^{(3)}_{k+1,0,m-1,m-1}=g^{(2)}_{k,0,0,m-1}$ for any $k\geq 0$. Then

$$q^{(3)}_{k+1,0,m-1,m-1}=q^{(2)}_{k,0,0,m-1}.$$

Hence, by Lemmas 4.6–4.8 it follows that

$$(p_1,p_2]\cap\mathcal{B}_2(M)=\bigcup_{k=2}^{3}\left(\bigcup_{u=0}^{m-1}\left\{q^{(2)}_{k,0,u,m-1}\right\}\cup\bigcup_{u=0}^{m-2}\left\{q^{(3)}_{k,0,u,m-1}\right\}\right).$$

Now we consider the smallest base $q_2(M)$. By the monotonicity it suffices to compare

$$s:=q^{(2)}_{2,0,m-1,m-1}\quad\text{and}\quad t:=q^{(3)}_{2,0,m-2,m-1}.$$

Note by (4.12) and (4.13) that

$$\begin{aligned}
g^{(3)}_{2,0,m-2,m-1}(t)&=(t+1)(t-2m)+t^{-3}\big(1+mt+(m-2)t^2\big)\\
&\qquad+t^{-1}\big(1+(m-1)(t^2+t)\big)\\
&<(t+1)(t-2m)+t^{-3}\big(1+(m-1)(t^2+t)\big)\\
&\qquad+t^{-1}\big(1+(m-1)(t^2+t)\big)\\
&=g^{(2)}_{2,0,m-1,m-1}(t).
\end{aligned}$$

This, together with $g^{(3)}_{2,0,m-2,m-1}(t)=0=g^{(2)}_{2,0,m-1,m-1}(s)$, implies that

$$g^{(2)}_{2,0,m-1,m-1}(s)<g^{(2)}_{2,0,m-1,m-1}(t).$$

By the monotonicity of the function $g^{(2)}_{2,0,m-1,m-1}$ it yields that $q^{(2)}_{2,0,m-1,m-1}<q^{(3)}_{2,0,m-2,m-1}$.

Hence, $q_2(M) = q^{(2)}_{2,0,m-1,m-1}$ is the unique root in $(p_1, p_2]$ of the equation

$$x^4 = (m-1)x^3 + 2mx^2 + mx + 1. \qquad \square$$

Recall from [17] that an algebraical integer greater than one is called a *Pisot number* if all of its Galois conjugates are strictly less than 1 in modulus. Accordingly, a *Salem number* is an algebraic integer larger than one such that all of its Galois conjugates in modulus do not exceed one, and at least one of its Galois conjugates indeed has modulus one. Moreover, a *Perron number* is an algebraic integer greater than one such that all of its Galois conjugates are strictly less than itself.

In the following proposition we investigate the asymptotic and algebraic properties of $q_2(M)$.

Proposition 4.10.

- $q_2(M) = \frac{M}{2} + r(M)$ *with*

$$\lim_{m\to\infty} r(2m) = 1 \quad \textit{and} \quad \lim_{m\to\infty} r(2m-1) = \frac{3}{2}.$$

- $q_2(2m)$ *is a Pisot number for any* $m \in \mathbb{N}$.
- $q_2(2m-1)$ *is a Perron number for any* $m \in \mathbb{N}$. *Furthermore,* $q_2(2m-1)$ *is neither a Pisot nor a Salem number.*

Proof. When $M = 2m$, by Proposition 3.5 it follows that

$$q_2(2m) = \frac{m+1+\sqrt{m^2+2m+5}}{2}.$$

Therefore, $\lim_{m\to\infty}\big(q_2(2m) - m\big) = 1$. Moreover, it is easy to check that $q_2(2m)$ is a Pisot number.

Now we consider $M = 2m-1$. First we prove the asymptotic result. By Theorem 4.9 it follows that $q_2 = q_2(2m-1)$ satisfies the following equation

$$q_2 = m - 1 + \frac{2m}{q_2} + \frac{m}{q_2^2} + \frac{1}{q_2^3}. \tag{4.15}$$

Observe that $q_2 < 2m$. Then by (4.15) it follows that

$$q_2 > m - 1 + \frac{2m}{2m} + \frac{m}{4m^2} = m + \frac{1}{4m}. \tag{4.16}$$

On the other hand, note that $q_2 > p_1 > m$. Then by (4.15) we have

$$q_2 < m - 1 + \frac{2m}{m} + \frac{m}{m^2} + \frac{1}{m^3} \le m + 1 + \frac{2}{m}.$$

This together with (4.16) implies that

$$\lim_{m\to\infty} \frac{q_2(2m-1)}{2m-1} = \frac{1}{2}.$$

Hence, by (4.15) we obtain

$$\lim_{m\to\infty} \left(q_2(2m-1) - \frac{2m-1}{2}\right) = \frac{3}{2}.$$

In the following we will prove that $q_2(2m-1)$ is a Perron number. Observe that $q_2 = q_2(2m-1)$ is a zero in $(p_1, p_2]$ of the function

$$f(x) := x^4 - (m-1)x^3 - 2mx^2 - mx - 1.$$

Moreover, one could verify that f is strictly increasing in (p_1, ∞). This implies that q_2 is the largest positive zero of the function f.

On the other hand, observe that f is the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & m & 2m-1 & m-1 \end{pmatrix}.$$

Clearly, A is a nonnegative integral matrix. Moreover, the matrix A is irreducible. Note that q_2 is the largest positive zero of the characteristic polynomial f of A. By Perron–Frobenius Theorem it follows that all of its Galois conjugates having modulus strictly smaller than q_2 (cf. [17, Theorem 4.2.3]). This implies that q_2 is a Perron number.

Finally, we show that q_2 is neither a Pisot nor a Salem number. Observe that f is the minimal polynomial for q_2. Moreover, one could verify that $f(-1) = -1 < 0$ and $f(-2) = 2m + 7 > 0$. This implies that f has a zero in $(-2, -1)$, i.e., q_2 has a Galois conjugate strictly larger than one in modulus. Therefore, $q_2(2m-1)$ is neither a Pisot nor a Salem number. □

Proof of Theorem 1.1. The theorem immediately follows by Propositions 3.5, 4.9 and 4.10. □

5. Proof of Theorem 1.2 and final remarks

In this section we will prove Theorem 1.2, and show that for $M = 2$ the smallest base $q_2 = q_2(2)$ of $\mathcal{B}_2(2)$ is also the smallest base of $\mathcal{B}_k(2)$ for any $k \in \mathbb{N}$.

Recall from Section 2 that $I_{q_2,2} = [0, 2/(q_2-1)]$ and the fundamental intervals

$$f_k(I_{q_2,2}) = \left[\frac{k}{q_2}, \frac{k}{q_2} + \frac{2}{q_2(q_2-1)}\right], \quad k = 0, 1, 2.$$

Then the *switch region* S_{q_2} is defined by

$$\begin{aligned} S_{q_2} &:= \bigcup_{k=1}^{2} f_{k-1}(I_{q_2,2}) \cap f_k(I_{q_2,2}) \\ &= \left[\frac{1}{q_2}, \frac{2}{q_2(q_2-1)}\right] \cup \left[\frac{2}{q_2}, \frac{1}{q_2} + \frac{2}{q_2(q_2-1)}\right]. \end{aligned} \tag{5.1}$$

Proof of Theorem 1.2. Let $M = 2$. Note that $q_2 = q_2(2) = 1 + \sqrt{2} \in (2, 3)$. Then almost every $x \in I_{q_2,2}$ has a continuum of different q_2-expansions w.r.t. the alphabet $\{0, 1, 2\}$ (cf. [20,5]). This yields that $q_2 \in \mathcal{B}_{2^{\aleph_0}}(2)$.

Now we prove $q_2 \in \mathcal{B}_{\aleph_0}(2)$. By Theorem 1.1 it gives that q_2 satisfies $q_2^2 = 2q_2 + 1$. This implies that $\alpha(q_2) = (20)^\infty$. Then

$$\overline{\alpha(q_2)} \preccurlyeq \alpha_{i+1}(q_2)\alpha_{i+2}(q_2)\cdots \preccurlyeq \alpha(q_2) \quad \text{for all} \quad i \geq 0.$$

By [14, Theorem 2.6] it follows that $x = 1 \in I_{q_2,2}$ has countably many q_2-expansions:

$$(20)^\infty, \quad \text{and} \quad (20)^k 210^\infty,\ (20)^k 12^\infty \quad \text{for all} \quad k \geq 0.$$

This establishes $q_2 \in \mathcal{B}_{\aleph_0}(2)$.

Finally, we will prove $q_2 \in \mathcal{B}_k(2)$ for all $k \geq 1$. This can be verified inductively by showing that the number

$$x_k = (1(00)^{k-1}1^\infty)_{q_2}$$

has exactly k different q_2-expansions. If $k = 1$, then by Proposition 2.3 it follows that $x_1 = (1^\infty)_{q_2}$ has a unique q_2-expansion.

Now suppose that x_k has exactly k different q_2-expansions. Note that $q_2^2 = 2q_2 + 1$, i.e., $(10^\infty)_{q_2} = (0210^\infty)_{q_2}$. This implies

$$x_{k+1} = (1(00)^k 1^\infty)_{q_2} = (021(00)^{k-1}1^\infty)_{q_2}. \tag{5.2}$$

By Proposition 2.3 it follows that $(00)^k 1^\infty \in \mathcal{U}'_{q_2}$. Moreover, note that

$$(21(00)^{k-1}1^\infty)_{q_2} = \frac{2}{q_2} + \frac{1}{q_2^2} + \frac{1}{q_2^{2k}(q_2-1)} > \frac{2}{q_2} + \frac{1}{q_2^2} = \frac{1}{q_2} + \frac{2}{q_2(q_2-1)}.$$

Then by (5.1) this implies $(21(00)^{k-1}1^\infty)_{q_2} \notin S_{q_2}$. By induction it follows that $(21(00)^{k-1}1^\infty)_{q_2}$ has exactly k different q_2-expansions. Hence, by (5.2) and Lemma 2.5 it follows that x_{k+1} has exactly $k+1$ different q_2-expansions. This implies that $q_2 \in \mathcal{B}_k(2)$ for any $k \geq 1$. Therefore, $\inf \mathcal{B}_k(2) \leq q_2$ for any $k \geq 1$.

On the other hand, take $q \in \mathcal{B}_k(2)$ with $k \geq 2$. Then there exists $x \in I_{q,2}$ having exactly k different q-expansions. By Lemma 2.5 and an affine transformation of x it follows that there exists $y \in I_q$ having precisely two different q-expansions. Thus, $q \geq q_2$ for any $k \geq 2$. This implies $\inf_{k\geq 2} \mathcal{B}_k(2) \geq q_2$.

Hence, we conclude that $q_2 = \min \mathcal{B}_k(2)$ for any $k \geq 2$. □

We mention that the proof of Theorem 1.2 does not work for $M = 2m$ with $m > 1$, since for large M we have more subintervals in the switch region as in (5.1) which makes the problem more involved.

At the end of this section we consider some questions on multiple expansions with multiple digits. The following theorem summarizes some results for $\mathcal{B}_k(M)$ with $M = 1$ obtained in [21,4,3,22].

Theorem 5.1.

(a) *The smallest element of* $\mathcal{B}_2(1)$ *is* $q_2(1) \approx 1.71064$, *the largest positive root of* $x^4 = 2x^2 + x + 1$.
(b) *The smallest element of* $\mathcal{B}_k(1)$ *for* $k \geq 3$ *is* $q_k(1) \approx 1.75488$, *the largest positive root of* $x^3 = 2x^2 - x + 1$.
(c) *The second smallest element of* $\mathcal{B}_{\aleph_0}(1)$ *is* $q_{\aleph_0}(1) \approx 1.64541$, *the largest positive root of* $x^6 = x^4 + x^3 + 2x^2 + x + 1$.
(d) $q_2(1) \in \mathcal{B}_1(1) \cap \mathcal{B}_2(1) \cap \mathcal{B}_{2^{\aleph_0}}(1)$ *and* $q_2(1) \notin \mathcal{B}_{\aleph_0}(1) \cup \bigcup_{k=3}^{\infty} \mathcal{B}_k(1)$.

In terms of Theorem 5.1 (d) it seems that the result holds true for any $M = 2m - 1$. Moreover, by Theorem 1.2 one may expect that the result holds true for any $M = 2m$. However, our method does not work for larger $M = 2m$. Thus we list the following questions:

Q1. Does Theorem 5.1 (d) holds for all $M = 2m - 1$?
Q2. Does Theorem 1.2 holds for all $M = 2m$?

In terms of Theorem 5.1 (a)–(b) and Theorem 1.2 we have accurate formulae for the smallest bases $q_k(1), q_k(2)$ for $k = 2, 3, \ldots$. Moreover, by Theorem 1.1 we have an accurate formula for the smallest base $q_2(M)$ for all $M \geq 1$. For $k \geq 3$ we denote by

$$q_k(M) = \inf \mathcal{B}_k(M).$$

Q3. What is $q_k(M)$ for $k \geq 3$ and $M \geq 3$?
Q4. Is it true that $q_k(M) \in \mathcal{B}_k(M)$ for any $k \geq 3$ and $M \geq 3$?

By Lemma 2.2 we know that the smallest base of $\mathcal{B}_{\aleph_0}(M)$ is the generalized golden ratio $p_1(M)$. Moreover, for $M = 1$ the second smallest base of $\mathcal{B}_{\aleph_0}(1)$ was determined in Theorem 5.1 (c). Denote by

$$q_{\aleph_0}(M) = \inf(\mathcal{B}_{\aleph_0}(M) \setminus \{p_1(M)\}).$$

Q5. What is $q_{\aleph_0}(M)$ for any $M \geq 2$?
Q6. Does $q_{\aleph_0}(M)$ belong to $\mathcal{B}_{\aleph_0}(M)$ for any $M \geq 2$?

Let $\mathcal{B}_{\aleph_0,1}(M)$ be the set of bases $q \in (1, M+1]$ such that 1 has countably many q-expansions w.r.t. the alphabet $\{0, 1, \ldots, M\}$. Note from [10] that the generalized golden ratio $p_1(M)$ is the smallest base of $\mathcal{B}_{\aleph_0,1}(M)$. Recently, the third author and her coauthors [23] considered $M = 1$ and determined the second smallest base $q_{\aleph_0,1}(1) \approx 1.68042$ of $\mathcal{B}_{\aleph_0,1}(1)$. For $M \geq 2$ set

$$q_{\aleph_0,1}(M) = \inf(\mathcal{B}_{\aleph_0,1}(M) \setminus \{p_1(M)\}).$$

Q7. What is $q_{\aleph_0,1}(M)$ for $M \geq 2$?
Q8. Is it true that $q_{\aleph_0,1}(M) \in \mathcal{B}_{\aleph_0,1}(M)$ for all $M \geq 2$?

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