Advances in Applied Mathematics 121 (2020) 102103

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Univoque bases of real numbers: Local dimension, Devil's staircase and isolated points

Derong Kong [a,*], Wenxia Li [b], Fan Lü [c], Zhiqiang Wang [b], Jiayi Xu [b]

[a] *College of Mathematics and Statistics, Chongqing University, Chongqing 401331, People's Republic of China*
[b] *Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200062, People's Republic of China*
[c] *Department of Mathematics, Sichuan Normal University, Chengdu 610068, People's Republic of China*

A R T I C L E I N F O

Article history:
Received 15 February 2020
Received in revised form 13 August 2020
Accepted 16 August 2020
Available online 25 August 2020

MSC:
primary 11A63
secondary 37B10, 26A30, 28A80, 68R15

Keywords:
Univoque bases
Hausdorff dimension
Devil's staircase
Critical values
Isolated points

A B S T R A C T

Given a positive integer M and a real number $x > 0$, let $\mathscr{U}(x)$ be the set of all bases $q \in (1, M+1]$ for which there exists a unique sequence $(d_i) = d_1 d_2 \dots$ with each digit $d_i \in \{0, 1, \dots, M\}$ satisfying

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i}.$$

The sequence (d_i) is called a *q-expansion* of x. In this paper we investigate the local dimension of $\mathscr{U}(x)$ and prove a 'variation principle' for unique non-integer base expansions. We also determine the critical values of $\mathscr{U}(x)$ such that when x passes the first critical value the set $\mathscr{U}(x)$ changes from a set with positive Hausdorff dimension to a countable set, and when x passes the second critical value the set $\mathscr{U}(x)$ changes from an infinite set to a singleton. Denote by $\mathbf{U}(x)$ the set of all unique q-expansions of x for $q \in \mathscr{U}(x)$. We give the Hausdorff dimension of $\mathbf{U}(x)$ and show that the dimensional function $x \mapsto \dim_H \mathbf{U}(x)$ is a non-increasing Devil's staircase. Finally, we investigate the topological structure of $\mathscr{U}(x)$. Although

* Corresponding author.
E-mail addresses: derongkong@126.com (D. Kong), wxli@math.ecnu.edu.cn (W. Li), lvfan1123@163.com (F. Lü), zhiqiangwzy@163.com (Z. Wang), dkxujy@163.com (J. Xu).

https://doi.org/10.1016/j.aam.2020.102103

the set $\mathscr{U}(1)$ has no isolated points, we prove that for typical $x > 0$ the set $\mathscr{U}(x)$ contains isolated points.

1. Introduction

Given a positive integer M and a real number $q \in (1, M+1]$, each point $x \in [0, M/(q-1)]$ can be written as

$$x = \pi_q((d_i)) := \sum_{i=1}^{\infty} \frac{d_i}{q^i}, \quad d_i \in \{0, 1, \ldots, M\} \ \forall i \geq 1. \tag{1.1}$$

The infinite sequence $(d_i) = d_1 d_2 \cdots$ is called a *q-expansion* of x with respect to the *alphabet* $\{0, 1, \cdots, M\}$.

Expansions in non-integer bases were pioneered by Rényi [35] and Parry [34]. Different from the integer base expansions Sidorov [36] (see also [10]) showed that for any $q \in (1, M+1)$ Lebesgue almost every $x \in [0, M/(q-1)]$ has a continuum of q-expansions. Furthermore, Erdős et al. [19,17,18] showed that for any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1, M+1)$ and $x \in [0, M/(q-1)]$ such that x has precisely k different q-expansions (see also cf. [38]). In particular, there is a great interest in unique q-expansions due to their close connections with open dynamical systems (cf. [13,22,25]). For more information on expansions in non-integer bases we refer to the surveys [24,37] and the survey chapter [15].

For $q > 1$ let $\mathcal{U}_q$ be the *univoque set* of $x \in I_q := [0, M/(q-1)]$ having a unique q-expansion, and let $\mathbf{U}_q := \pi_q^{-1}(\mathcal{U}_q)$ be the set of corresponding q-expansions. Dual to the univoque set $\mathcal{U}_q$ we consider in this paper the set of *univoque bases* of real numbers. For $x \geq 0$ let $\mathscr{U}(x)$ be the set of bases $q \in (1, M+1]$ such that x has a unique q-expansion, i.e.,

$$\mathscr{U}(x) = \{q \in (1, M+1] : x \in \mathcal{U}_q\}.$$

Clearly, for $x = 0$ the set $\mathscr{U}(0) = (1, M+1]$, because for each $q \in (1, M+1]$ the point 0 always has a unique q-expansion $0^\infty = 00\cdots$. So, it is interesting to investigate the set $\mathscr{U}(x)$ for $x > 0$.

When $x = 1$, the set $\mathscr{U} = \mathscr{U}(1)$ is well understood. Erdős et al. [19] showed that $\mathscr{U}$ is a Lebesgue null set of first category but it is uncountable. Later Daróczy and Kátai [12] showed that $\mathscr{U}$ has full Hausdorff dimension. Clearly, the largest element of $\mathscr{U}$ is $M+1$ since 1 has the unique expansion $M^\infty = MM\cdots$ in base $M+1$. Komornik and Loreti [26,27] found the smallest element $q_{KL} = q_{KL}(M)$ of $\mathscr{U}$, which was called the *Komornik-Loreti constant* by Glendinning and Sidorov [22]. Furthermore, they showed in [28] that its topological closure $\overline{\mathscr{U}}$ is a Cantor set: a non-empty compact set with neither isolated nor interior points. Hence,

$$(1, M+1] \setminus \overline{\mathscr{U}} = \bigcup (q_0, q_0^*), \tag{1.2}$$

where the left endpoints q_0 run through 1 and the set $\overline{\mathscr{U}} \setminus \mathscr{U}$, and the right endpoints q_0^* run through a subset $\mathscr{U}^*$ of $\mathscr{U}$ (cf. [13]). In particular, each left endpoint q_0 is algebraic, while each right endpoint q_0^*, called a *de Vries-Komornik number*, is transcendental (cf. [30]). Recently, Kalle et al. [23] showed that the set $\mathscr{U}$ has more weight close to $M+1$. For the detailed description of the local structure of $\mathscr{U}$ we refer to the recent paper [5].

However, for a general $x > 0$ we know very little about $\mathscr{U}(x)$. Lü, Tan and Wu [33] showed that for $M = 1$ and $x \in (0,1)$ the set $\mathscr{U}(x)$ is a Lebesgue null set but has full Hausdorff dimension. Recently, Dajani et al. [11] showed that the algebraic difference $\mathscr{U}(x) - \mathscr{U}(x)$ contains an interval for any $x \in (0,1]$. The smallest element of $\mathscr{U}(x)$ was investigated in [29,6]. In this paper we will investigate the set $\mathscr{U}(x)$ from the following perspectives. (i) We will determine the local dimension of $\mathscr{U}(x)$ and establish a so-called 'variation principle' in unique non-integer base expansions; (ii) We will determine the Hausdorff dimension of the symbolic set $\mathbf{U}(x)$ consisting of all expansions of x in base $q \in \mathscr{U}(x)$, and show that the function $x \mapsto \dim_H \mathbf{U}(x)$ is a non-increasing Devil's staircase (see Fig. 2); (iii) We will determine the critical values of $\mathscr{U}(x)$ such that when x passes the first critical value the set $\mathscr{U}(x)$ changes from positive Hausdorff dimension to a countable set, and when x passes the second critical value the set $\mathscr{U}(x)$ changes from an infinite set to a singleton; (iv) In contrast with $\mathscr{U} = \mathscr{U}(1)$ we will show that typically the set $\mathscr{U}(x)$ contains isolated points.

For $x > 0$ let

$$q_x := \min\left\{1+M, 1+\frac{M}{x}\right\}. \tag{1.3}$$

Then q_x is the largest base in $(1, M+1]$ such that the given x has an expansion with respect to the alphabet $\{0, 1, \ldots, M\}$.

Our first result focuses on the local dimension of $\mathscr{U}(x)$.

Theorem 1.1. *For any $x > 0$ and for any $q \in (1, q_x] \setminus \overline{\mathscr{U}}$ we have*

$$\lim_{\delta \to 0} \dim_H(\mathscr{U}(x) \cap (q-\delta, q+\delta)) = \lim_{\delta \to 0} \dim_H(\mathcal{U}_q \cap (x-\delta, x+\delta)).$$

Theorem 1.1 can be viewed as a 'variation principle' in unique non-integer base expansions. Recall from [14] the two dimensional univoque set

$$\mathbb{U} = \{(z,p) : z \text{ has a unique } p\text{-expansion}\}.$$

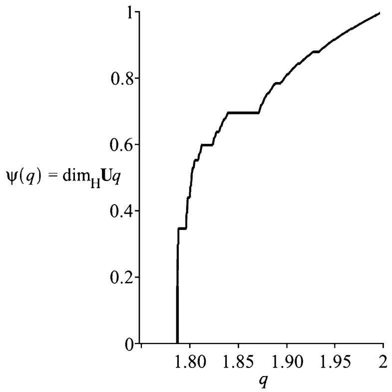

Fig. 1. The graph of $\psi : q \mapsto \dim_H \mathbf{U}_q$ with $M = 1$. $\psi(q)$ is positive if and only if $q > q_{KL} \approx 1.78723$, and $\psi(q) = 1$ if and only if $q = 2$.

Then the left hand side in Theorem 1.1 is the local dimension of the vertical slice $\mathbb{U} \cap \{z = x\} = \mathscr{U}(x)$ at the point (x, q), and the right hand side gives the local dimension of the horizontal slice $\mathbb{U} \cap \{p = q\} = \mathcal{U}_q$ at the same point (x, q). So Theorem 1.1 states that for any $x > 0$ and any $q \in (1, q_x] \setminus \overline{\mathscr{U}}$ the local dimension of $\mathbb{U}$ at the point (x, q) through the vertical slice is the same as that through the horizontal slice.

Let $\{0, 1, \ldots, M\}^{\mathbb{N}}$ be the set of all sequences $(d_i) = d_1 d_2 \ldots$ over the alphabet $\{0, 1, \ldots, M\}$. Equipped with the order topology on $\{0, 1, \ldots, M\}^{\mathbb{N}}$ induced by the metric

$$\rho((c_i), (d_i)) = (M + 1)^{-\inf\{i \geq 1 : c_i \neq d_i\}} \tag{1.4}$$

we can define the Hausdorff dimension of any subset of $\{0, 1, \ldots, M\}^{\mathbb{N}}$.

Note that $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q) \subset \{0, 1, \ldots, M\}^{\mathbb{N}}$ is the symbolic horizontal slice of the two-dimensional univoque set $\mathbb{U}$. The following result for the Hausdorff dimension of $\mathbf{U}_q$ was established in [25] and [4] (see Fig. 1).

Proposition 1.2 *([25,4]). The dimensional function* $\psi : q \mapsto \dim_H \mathbf{U}_q$ *is a non-decreasing Devil's staircase on* $(1, M + 1]$*. In particular,*

- ψ *is non-decreasing and continuous on* $(1, M + 1]$*;*
- ψ *is locally constant almost everywhere on* $(1, M + 1]$*;*
- $\psi(q) \in (0, 1]$ *if and only if* $q > q_{KL}$*. Furthermore,* $\psi(q) = 1$ *only when* $q = M + 1$*.*

The detailed study of the plateaus of ψ, i.e., the largest intervals for which ψ is constant, can be found in [1]. For the bifurcation set of ψ, which is the set of points where ψ is not locally constant, we refer to [3].

For $x > 0$ let $\Phi_x(q) = x_1(q) x_2(q) \ldots$ be the quasi-greedy q-expansion of x (see Section 2 for its definition). Now we define the symbolic set of univoque bases by

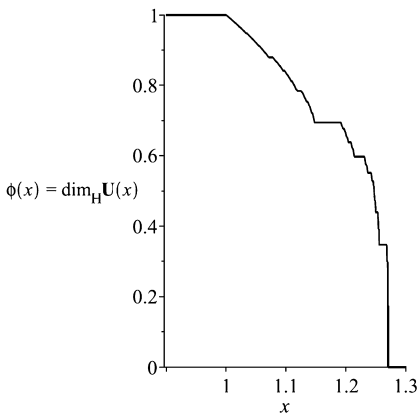

Fig. 2. The graph of $\phi : x \mapsto \dim_H \mathbf{U}(x)$ with $M = 1$. $\phi(x)$ is positive if and only if $x < x_{KL} \approx 1.27028$, and $\phi(x) = 1$ if and only if $x \leq 1$.

$$\mathbf{U}(x) := \{\Phi_x(q) : q \in \mathscr{U}(x)\}\,.$$

Observe that for each $q \in \mathscr{U}(x)$ the sequence $\Phi_x(q) \in \mathbf{U}(x)$ is the unique q-expansion of x. So, the map $q \mapsto \Phi_x(q)$ is bijective from $\mathscr{U}(x)$ to $\mathbf{U}(x)$. We will show in Proposition 3.1 that the map $q \mapsto \Phi_x(q)$ is locally bi-Hölder continuous on $\mathscr{U}(x)$.

Observe that $\mathbf{U}(x) = \Phi_x(\mathscr{U}(x))$ is the symbolic vertical slice of the two dimensional univoque set $\mathbb{U}$. Comparing with Proposition 1.2 our second main result gives the Hausdorff dimension of $\mathbf{U}(x)$, and shows that the dimensional function $x \mapsto \dim_H \mathbf{U}(x)$ is a non-increasing Devil's staircase (see Fig. 2).

Theorem 1.3. *For any $x > 0$ the Hausdorff dimension of $\mathbf{U}(x)$ is given by*

$$\dim_H \mathbf{U}(x) = \dim_H \mathbf{U}_{q_x},$$

where q_x is defined in (1.3). Consequently, the dimensional function $\phi : x \mapsto \dim_H \mathbf{U}(x)$ is a non-increasing Devil's staircase on $(0, \infty)$. In particular,

(i) *ϕ is non-increasing and continuous on $(0, \infty)$;*
(ii) *ϕ is locally constant almost everywhere;*
(iii) *$\phi(x) \in (0, 1]$ if and only if $x < \frac{M}{q_{KL}-1}$. Furthermore, $\phi(x) = 1$ if and only if $x \leq 1$.*

Recall from [9] that the *generalized golden ratio* is defined by

$$q_G = q_G(M) := \begin{cases} k+1 & \text{if} \quad M = 2k; \\ \frac{k+1+\sqrt{k^2+6k+5}}{2} & \text{if} \quad M = 2k+1. \end{cases} \tag{1.5}$$

Note that $q_{KL} = q_{KL}(M)$ is the smallest element of $\mathscr{U} = \mathscr{U}(1)$ and $1 < q_G < q_{KL} < M + 1$. The following result on the critical values of $\mathcal{U}_q = \pi_q(\mathbf{U}_q)$ was first proven by Glendinning and Sidorov [22] for $M = 1$ and then proven in [31] for all $M \geq 2$. Furthermore, the Hausdorff dimension of $\mathcal{U}_q$ was given in [25]. For a set A we denote by $|A|$ its cardinality.

Proposition 1.4 *([22,31,25]). For any $q \in (1, M+1]$ the Hausdorff dimension of $\mathcal{U}_q$ is given by*

$$\dim_H \mathcal{U}_q = \frac{\dim_H \mathbf{U}_q}{\log q}.$$

Furthermore, we have the following properties.

- *If $q \in (1, q_G]$, then $\mathcal{U}_q = \left\{0, \frac{M}{q-1}\right\}$;*
- *If $q \in (q_G, q_{KL})$, then $|\mathcal{U}_q| = \aleph_0$;*
- *If $q = q_{KL}$, then $|\mathcal{U}_q| = 2^{\aleph_0}$ and $\dim_H \mathcal{U}_q = 0$;*
- *If $q \in (q_{KL}, M+1]$, then $\dim_H \mathcal{U}_q \in (0, 1]$. Furthermore, $\dim_H \mathcal{U}_q = 1$ if and only if $q = M + 1$.*

Here in Proposition 1.4 and throughout the paper we keep using base $M+1$ logarithms. By Theorem 1.3 and Proposition 1.4 we are able to determine the critical values of $\mathscr{U}(x)$ for $x > 0$ and $M \geq 1$. Set

$$x_G := \frac{M}{q_G - 1} \quad \text{and} \quad x_{KL} := \frac{M}{q_{KL} - 1}.$$

Since $1 < q_G < q_{KL} < M + 1$, it follows that $1 < x_{KL} < x_G$. Furthermore, by (1.3) it follows that $q_{x_G} = q_G$ and $q_{x_{KL}} = q_{KL}$.

Theorem 1.5. *Let $M \geq 1$. The set $\mathscr{U}(x)$ has zero Lebesgue measure for any $x > 0$. Furthermore,*

(i) *if $x \in (0, 1]$, then $\dim_H \mathscr{U}(x) = 1$;*
(ii) *if $x \in (1, x_{KL})$, then $0 < \dim_H \mathscr{U}(x) < 1$;*
(iii) *if $x \in [x_{KL}, x_G)$, then $|\mathscr{U}(x)| = \aleph_0$;*
(iv) *if $x \geq x_G$, then $\mathscr{U}(x) = \{q_x\}$.*

Remark 1.6.

- Theorem 1.5 (i) was first established in [33] for $M = 1$.
- In Lemma 4.6 we present a stronger result than Theorem 1.5 (ii): for $x \in (1, x_{KL})$ we have

$$0 < \dim_H \mathcal{U}_{q_x} \leq \dim_H \mathscr{U}(x) \leq \max_{q \in \mathscr{U}(x)} \dim_H \mathcal{U}_q < 1.$$

- In contrast with Proposition 1.4 for the univoque set $\mathcal{U}_q$, Theorem 1.5 shows that there is no $x > 0$ such that the set $\mathscr{U}(x)$ is uncountable but has zero Hausdorff dimension.

Recall that $\mathscr{U} = \mathscr{U}(1)$ has no isolated points and its closure $\overline{\mathscr{U}}$ is a Cantor set. Then it is natural to ask whether this is true for $\mathscr{U}(x)$? Our forth main result shows that typically this is not the case. Let

$$X_{iso} := \{x \in (0, \infty) : \mathscr{U}(x) \text{ contains isolated points}\}.$$

We show that for $M = 1$ the set X_{iso} is dense in $(0, \infty)$.

Theorem 1.7. *Let $M \geq 1$. The set X_{iso} is dense in $[0, 1]$. If $M = 1$, the set X_{iso} is dense in $(0, \infty)$.*

Remark 1.8.

- For $M \geq 1$ we show in Lemma 5.2 a slightly stronger property: for any $x \in [0, 1]$ any neighborhood of x in X_{iso} contains an interval.
- For $M = 1$ we show in Proposition 5.3 that $X_{iso} \supset (1, \infty)$. This means for any $x > 1$ the set $\mathscr{U}(x)$ contains isolated points.

The rest of the paper is arranged in the following way. In the next section we introduce the greedy and quasi-greedy expansions, and present some useful properties of unique expansions. In Section 3 we investigate the local dimension of $\mathscr{U}(x)$ and prove Theorem 1.1. Based on this we are able to calculate in Section 4 the Hausdorff dimension of the symbolic set $\mathbf{U}(x)$ and prove the irregularity of the dimensional function $x \mapsto \dim_H \mathbf{U}(x)$ (see Theorem 1.3). Furthermore, we determine the critical values of $\mathscr{U}(x)$ such that when x crosses the first critical value the Hausdorff dimension of $\mathscr{U}(x)$ vanishes, and when x crosses the second critical value the set $\mathscr{U}(x)$ degenerates to a singleton (see Theorem 1.5). The proof of Theorem 1.7 is presented in Section 5. Although the set $\mathscr{U}(1)$ has no isolated points, we show that typically $\mathscr{U}(x)$ contains isolated points. In the final section we pose some remarks and questions on $\mathscr{U}(x)$.

2. Preliminaries

In this section we recall some well-known properties from unique non-integer base expansions. First we need some terminology from symbolic dynamics (cf. [32]). Let $\{0, 1, \ldots, M\}^{\mathbb{N}}$ be the set of infinite sequences with digits from the alphabet $\{0, 1, \ldots, M\}$. Denote by σ the left shift on $\{0, 1, \ldots, M\}^{\mathbb{N}}$ such that $\sigma((c_i)) = (c_{i+1})$. By a *word* $\mathbf{c} = c_1 \ldots c_n$ we mean a finite string of digits with each digit c_i from $\{0, 1, \ldots, M\}$. Let $\{0, 1, \ldots, M\}^*$ be the set of all words including the empty word ϵ. For two words

$\mathbf{c}, \mathbf{d} \in \{0,1,\dots,M\}^*$ we write $\mathbf{cd}$ as a new word which is the concatenation of them. We denote by $\mathbf{c}^\infty = \mathbf{cc}\dots \in \{0,1,\dots,M\}^{\mathbb{N}}$ the periodic sequence which is the infinite concatenation of $\mathbf{c}$ with itself. Throughout the paper we will use the lexicographical ordering "$\prec, \preccurlyeq, \succ$" or "$\succcurlyeq$" between sequences and words in the usual way. For example, for two sequences $(c_i), (d_i) \in \{0,1,\dots,M\}^{\mathbb{N}}$ we write $(c_i) \prec (d_i)$ if $c_1 < d_1$, or there exists $n > 1$ such that $c_1 \dots c_{n-1} = d_1 \dots d_{n-1}$ and $c_n < d_n$. Furthermore, for two words $\mathbf{c}, \mathbf{d}$ we say $\mathbf{c} \prec \mathbf{d}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$. For a sequence (c_i) we denote its *reflection* by $\overline{(c_i)} = (M-c_1)(M-c_2)\dots \in \{0,1,\dots,M\}^{\mathbb{N}}$. Similarly, for a word $\mathbf{c} = c_1 \dots c_n$ we denote its reflection by $\overline{\mathbf{c}} := (M-c_1)\dots(M-c_n)$. If $c_n < M$, we write $\mathbf{c}^+ := c_1 \dots c_{n-1}(c_n+1)$; and if $c_n > 0$, we write $\mathbf{c}^- := c_1 \dots c_{n-1}(c_n - 1)$. So, $\overline{\mathbf{c}}, \mathbf{c}^+$ and $\mathbf{c}^-$ are all words in $\{0,1,\dots,M\}^*$.

2.1. Quasi-greedy and greedy expansions

Let $M \geq 1$ and $x > 0$. Recall from (1.3) that $q_x = \min\{1+M, 1+M/x\} = \max \mathscr{U}(x)$. For $q \in (1, q_x]$ let

$$\Phi_x(q) = x_1(q)x_2(q)\dots \in \{0,1,\dots,M\}^{\mathbb{N}}$$

be the *quasi-greedy* q-expansion of x, which is the lexicographically largest q-expansion of x not ending with 0^∞. In other words, $\Phi_x(q) = (x_i(q))$ is the q-expansion of x satisfying

$$\sum_{i=1}^{n} \frac{x_i(q)}{q^i} < x \quad \text{for all } n \geq 1.$$

In particular, for $x = 1$ and $q \in (1, q_1] = (1, M+1]$ we reserve the notation $\alpha(q) = (\alpha_i(q)) = \Phi_1(q)$ for the quasi-greedy q-expansion of 1.

Similarly, for $q \in (1, q_x]$ let

$$\Psi_x(q) = \tilde{x}_1(q)\tilde{x}_2(q)\dots \in \{0,1,\dots,M\}^{\mathbb{N}}$$

be the *greedy* q-expansion of x, which is the lexicographically largest q-expansion of x. Then $\Psi_x(q) = (\tilde{x}_i(q))$ is the q-expansion of x satisfying

$$\sum_{i=1}^{n} \frac{\tilde{x}_i(q)}{q^i} + \frac{1}{q^n} > x \quad \text{whenever} \quad \tilde{x}_n(q) < M.$$

If x has a unique q-expansion, i.e., $q \in \mathscr{U}(x)$, then $\Phi_x(q) = \Psi_x(q)$.

The following lemma for the quasi-greedy expansion $\Phi_x(q)$ and greedy expansion $\Psi_x(q)$ was essentially proven in [14].

Lemma 2.1.

(i) *Let $x > 0$. Then the map $q \mapsto \Phi_x(q)$ is left continuous and strictly increasing in $(1, q_x]$. Moreover, the sequence $\Phi_x(q) = (x_i(q))$ satisfies*

$$x_{n+1}(q)x_{n+2}(q)\cdots \preccurlyeq \alpha(q) \quad \text{whenever} \quad x_n(q) < M.$$

(ii) *For $x > 0$ the map $q \mapsto \Psi_x(q)$ is right continuous and strictly increasing in $(1, q_x]$. Moreover, the sequence $\Psi_x(q) = (\tilde{x}_i(q))$ satisfies*

$$\tilde{x}_{n+1}(q)\tilde{x}_{n+2}(q)\cdots \prec \alpha(q) \quad \text{whenever} \quad \tilde{x}_n(q) < M.$$

Proof. The monotonicity statements in (i) and (ii) are obvious by the definitions of Φ_x and Ψ_x respectively. The continuity statements follow from [14, Lemmas 2.3 and 2.5]. Finally, the lexicographical characterizations of $\Phi_x(q)$ and $\Psi_x(q)$ can be found in [8].

Remark 2.2. Taking $x = 1$ in Lemma 2.1 (i) it follows that the map $q \mapsto \Phi_1(q) = \alpha(q)$ is left-continuous and strictly increasing in $(1, M+1]$. In particular, the quasi-greedy expansion $\alpha(q) = (\alpha_i(q))$ satisfies $\alpha_{n+1}(q)\alpha_{n+2}(q)\ldots \preccurlyeq \alpha(q)$ whenever $\alpha_n(q) < M$. Indeed, one can verify (see also [16, Proposition 2.3]) that the map $q \mapsto \alpha(q)$ is bijective from $(1, M+1]$ to the set of sequences $(a_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying

$$a_{n+1}a_{n+2}\ldots \preccurlyeq a_1a_2\ldots \quad \text{for all } n \geq 0.$$

2.2. Unique expansions

For $q \in (1, M+1]$ we recall the symbolic univoque set

$$\mathbf{U}_q = \left\{(d_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}} : \pi_q((d_i)) \in \mathcal{U}_q\right\},$$

where π_q is the projection map define in (1.1). Then each sequence $(d_i) \in \mathbf{U}_q$ is the unique q-expansion of $\pi_q((d_i))$. So π_q is a bijective map from $\mathbf{U}_q$ to $\mathcal{U}_q$. The following lexicographical characterization of $\mathbf{U}_q$ was given by Erdős et al. [19] (see also [13]).

Lemma 2.3. *Let $q \in (1, M+1]$. Then $\mathbf{U}_q$ consists of all sequences $(d_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ satisfying*

$$\begin{cases} d_{n+1}d_{n+2}\ldots \prec \alpha(q) & \text{if} \quad d_n < M, \\ \overline{d_{n+1}d_{n+2}\ldots} \prec \alpha(q) & \text{if} \quad d_n > 0. \end{cases}$$

Observe that for any $x > 0$ and $q \in (1, M+1]$ we have $q \in \mathscr{U}(x)$ if and only if $\Phi_x(q) \in \mathbf{U}_q$. Recall that $\mathscr{U} = \mathscr{U}(1)$ is the set of bases for which 1 has a unique expansion. Komornik and Loreti [28] showed that its topological closure $\overline{\mathscr{U}}$ is a Cantor set as described in (1.2). Motivated by the work of de Vries and Komornik [13] we introduce the bifurcation set $\mathscr{V}$ of the set-valued map $q \mapsto \mathbf{U}_q$ defined by

$$\mathscr{V} := \{q \in (1, M+1] : \mathbf{U}_r \neq \mathbf{U}_q \ \forall r > q\}. \tag{2.1}$$

They showed in [13] that $\overline{\mathscr{U}} \subset \mathscr{V}$, and $\mathscr{V} \setminus \overline{\mathscr{U}}$ is countably infinite.

The following intimate connection between $\mathbf{U}_q$, $\overline{\mathscr{U}}$ and $\mathscr{V}$ was established by de Vries and Komornik [13] (see also [16]).

Lemma 2.4. *Let $M \geq 1$. The following statements hold true.*

(i) *The set-valued map $q \mapsto \mathbf{U}_q$ is non-decreasing with respect to the set-inclusion. Furthermore, for any connected component (q_0, q_0^*) of $(1, M+1] \setminus \overline{\mathscr{U}}$ and for any $p, q \in (q_0, q_0^*)$ the difference between $\mathbf{U}_p$ and $\mathbf{U}_q$ is at most countable.*
(ii) *For each connected component (q_L, q_R) of $(1, M+1] \setminus \mathscr{V}$ the set-valued map $q \mapsto \mathbf{U}_q$ is constant in $(q_L, q_R]$.*

3. Local dimension of $\mathscr{U}(x)$

In this section we will investigate the local dimension of $\mathscr{U}(x)$ by showing that the map Φ_x is locally bi-Hölder continuous from $\mathscr{U}(x)$ onto $\mathbf{U}(x)$. This provides a good estimation for the local dimension of $\mathscr{U}(x)$ via its symbolic set $\mathbf{U}(x)$. Based on this estimation we are able to prove the 'variation principle' as described in Theorem 1.1.

Recall that the symbolic set $\mathbf{U}(x) = \{\Phi_x(q) : q \in \mathscr{U}(x)\}$ and the metric ρ is defined in (1.4). First we show that the map $\Phi_x : \mathscr{U}(x) \mapsto \mathbf{U}(x)$ is locally bi-Hölder continuous.

Proposition 3.1. *Let $x > 0$ and $1 < a < b < M+1$. Then for any $p_1, p_2 \in \mathscr{U}(x) \cap (a, b)$,*

$$C_1 |p_1 - p_2|^{\frac{1}{\log a}} \leq \rho(\Phi_x(p_1), \Phi_x(p_2)) \leq C_2 |p_1 - p_2|^{\frac{1}{\log b}}, \tag{3.1}$$

where $C_1, C_2 > 0$ are constants independent of p_1 and p_2.

Proof. Take $p_1, p_2 \in \mathscr{U}(x) \cap (a, b)$ with $p_1 < p_2$. By Lemma 2.1 (i) we have $(x_i(p_1)) = \Phi_x(p_1) \prec \Phi_x(p_2) = (x_i(p_2))$. Then there exists an integer $n \geq 1$ such that

$$x_1(p_1) \cdots x_{n-1}(p_1) = x_1(p_2) \cdots x_{n-1}(p_2) \quad \text{and} \quad x_n(p_1) < x_n(p_2). \tag{3.2}$$

Note that

$$\sum_{i=1}^{n-1}\frac{x_i(p_1)}{p_1^i}<\sum_{i=1}^{\infty}\frac{x_i(p_1)}{p_1^i}=x=\sum_{i=1}^{\infty}\frac{x_i(p_2)}{p_2^i}\le\sum_{i=1}^{n-1}\frac{x_i(p_2)}{p_2^i}+\frac{M}{p_2^{n-1}(p_2-1)}.$$

Then by (3.2) it follows that

$$x(p_2-p_1)<\sum_{i=1}^{n-1}\frac{x_i(p_2)}{p_2^{i-1}}-\sum_{i=1}^{n-1}\frac{x_i(p_1)}{p_1^{i-1}}+\frac{M}{p_2^{n-2}(p_2-1)}\le\frac{M}{p_2^{n-2}(p_2-1)},$$

which implies

$$p_2-p_1<\frac{M}{p_2^{n-2}(p_2-1)x}. \tag{3.3}$$

Therefore, by (3.2) and (3.3) it follows that

$$\begin{aligned}\rho(\Phi_x(p_1),\Phi_x(p_2))^{\log a}=(M+1)^{-n\log a}=\frac{1}{a^n}&>\frac{1}{p_2^n}\\ &>\frac{(p_2-1)x}{Mp_2^2}|p_2-p_1|\\ &\ge\frac{(a-1)x}{Mb^2}|p_2-p_1|.\end{aligned}$$

This proves the first inequality of (3.1) by taking $C_1:=(\frac{(a-1)x}{Mb^2})^{1/\log a}$.

For the second inequality of (3.1) we note that $b<M+1$. So, $\alpha(b)\prec\alpha(M+1)=M^\infty$ by Lemma 2.1 (i). Then there exists $i_0\ge1$ such that

$$\alpha_1(b)\cdots\alpha_{i_0}(b)\prec M^{i_0}. \tag{3.4}$$

Since $p_2\in\mathscr{U}(x)$, we have $\Phi_x(p_2)\in\mathbf{U}_{p_2}$. Then by (3.2), (3.4) and Lemma 2.3 it follows that

$$\sum_{i=1}^{n}\frac{x_i(p_2)}{p_1^i}\ge\sum_{i=1}^{\infty}\frac{x_i(p_1)}{p_1^i}=x=\sum_{i=1}^{\infty}\frac{x_i(p_2)}{p_2^i}>\sum_{i=1}^{n}\frac{x_i(p_2)}{p_2^i}+\frac{1}{p_2^{n+i_0}}.$$

This implies

$$\frac{1}{p_2^{n+i_0}}<\sum_{i=1}^{n}\left(\frac{x_i(p_2)}{p_1^i}-\frac{x_i(p_2)}{p_2^i}\right)\le\sum_{i=1}^{\infty}\left(\frac{M}{p_1^i}-\frac{M}{p_2^i}\right)=\frac{M|p_2-p_1|}{(p_1-1)(p_2-1)}. \tag{3.5}$$

Hence, by (3.2) and (3.5) it follows that

$$\begin{aligned}\rho(\Phi_x(p_1),\Phi_x(p_2))^{\log b}=(M+1)^{-n\log b}=\frac{1}{b^n}&<\frac{1}{p_2^n}\\&<\frac{Mp_2^{i_0}}{(p_1-1)(p_2-1)}|p_2-p_1|\\&\le\frac{Mb^{i_0}}{(a-1)^2}|p_1-p_2|.\end{aligned}$$

This establishes the second inequality in (3.1) by taking $C_2:=(\frac{Mb^{i_0}}{(a-1)^2})^{1/\log b}$. □

The following lemma for the Hausdorff dimension under Hölder continuous maps is well-known (cf. [20]).

Lemma 3.2. *Let (X,d_1) and (Y,d_2) be two metric spaces, and let $f:X\to Y$. If there exist positive constants δ, C and λ such that*

$$d_2(f(x),f(y))\le Cd_1(x,y)^\lambda$$

for any $x,y\in X$ with $d_1(x,y)\le\delta$, then $\dim_H f(X)\le\frac{1}{\lambda}\dim_H X$.

By Proposition 3.1 and Lemma 3.2 we have the following estimation for the local dimension of $\mathscr{U}(x)$, which states that the local dimension of $\mathscr{U}(x)$ at any point $q\in(1,M+1)$ can be roughly estimated by the local dimension of the symbolic set $\mathbf{U}(x)$ at $\Phi_x(q)$.

Proposition 3.3. *Let $x>0$ and $1<a<b<M+1$. Then*

$$\frac{\dim_H\Phi_x\big(\mathscr{U}(x)\cap(a,b)\big)}{\log b}\le\dim_H\big(\mathscr{U}(x)\cap(a,b)\big)\le\frac{\dim_H\Phi_x\big(\mathscr{U}(x)\cap(a,b)\big)}{\log a}.$$

Proof. Excluding the trivial case we assume that $\mathscr{U}(x)\cap(a,b)$ contains infinitely many elements. Note that the map

$$\Phi_x:\mathscr{U}(x)\cap(a,b)\longrightarrow\Phi_x(\mathscr{U}(x)\cap(a,b));\qquad p\mapsto\Phi_x(p)$$

is bijective. Then its inverse map Φ_x^{-1} is well-defined. Hence, the proposition follows by Proposition 3.1 and Lemma 3.2. □

To prove Theorem 1.1 we still need the following lemma.

Lemma 3.4. *Fix $q\in(1,M+1)$. There exist constants $C_1,C_2>0$ such that for any $\mathbf{c}=(c_i),\mathbf{d}=(d_i)\in\mathbf{U}_q$ we have*

$$C_1\cdot\rho(\mathbf{c},\mathbf{d})^{\log q}\le|\pi_q(\mathbf{c})-\pi_q(\mathbf{d})|\le C_2\cdot\rho(\mathbf{c},\mathbf{d})^{\log q}. \tag{3.6}$$

Proof. Define a metric ρ_q on $\mathbf{U}_q$ by

$$\rho_q(\mathbf{c},\mathbf{d}) = q^{-\inf\{i\geq 1: c_i\neq d_i\}}$$

for any $\mathbf{c},\mathbf{d}\in\mathbf{U}_q$. Then the map $\pi_q : (\mathbf{U}_q,\rho_q)\to(\mathcal{U}_q,|\cdot|)$ is bi-Lipschitz (cf. [2, Lemma 2.2]). Since $\rho_q(\mathbf{c},\mathbf{d}) = \rho(\mathbf{c},\mathbf{d})^{\log q}$, this proves (3.6). □

Proof of Theorem 1.1. Let $x > 0$ and $q \in (1, q_x] \setminus \overline{\mathscr{U}}$. Note by (2.1) that $\overline{\mathscr{U}} \subset \mathscr{V}$ and the difference $\mathscr{V} \setminus \mathscr{U}$ is countable. Then there exists a $\delta > 0$ such that $q - \delta > 1$ and $(q-\delta, q)\cap\mathscr{V} = \emptyset$. So, by Lemma 2.1 (i) and Lemma 2.4 it follows that each $p \in \mathscr{U}(x)\cap(q-\delta,q)$ determines a unique $y = \pi_q(\Phi_x(p)) \in \mathcal{U}_q\cap(x-\eta,x)$ for some $\eta > 0$ depending on δ. This defines a bijection

$$\phi:\ \mathscr{U}(x)\cap(q-\delta,q)\to\mathcal{U}_q\cap(x-\eta,x);\quad p\mapsto\pi_q(\Phi_x(p)).$$

If the set $\mathscr{U}(x)\cap(q-\delta,q)$ is empty, then so is $\mathcal{U}_q\cap(x-\eta,x)$. In this case, it is trivial that

$$\lim_{\delta\to 0}\dim_H(\mathscr{U}(x)\cap(q-\delta,q)) = \lim_{\eta\to 0}\dim_H(\mathcal{U}_q\cap(x-\eta,x)), \tag{3.7}$$

and the limit is equal to zero. In the following we assume $\mathscr{U}(x)\cap(q-\delta,q)\neq\emptyset$. Then by (3.1) and (3.6) it follows that there exist constants $D_1, D_2 > 0$ such that

$$D_1|p_1-p_2|^{\frac{\log q}{\log(q-\delta)}} \leq |\phi(p_1)-\phi(p_2)| \leq D_2|p_1-p_2|$$

for all $p_1, p_2 \in \mathscr{U}(x)\cap(q-\delta,q)$. In other words, ϕ is 'nearly bi-Lipschitz' on $\mathscr{U}(x)\cap(q-\delta,q)$. By Lemma 3.2 this implies

$$\dim_H(\mathcal{U}_q\cap(x-\eta,x)) \leq \dim_H(\mathscr{U}(x)\cap(q-\delta,q)) \leq \frac{\log q}{\log(q-\delta)}\dim_H(\mathcal{U}_q\cap(x-\eta,x)).$$

Letting $\delta\to 0$, which implies $\eta\to 0$, we then establish (3.7) for any $x > 0$ and $q \in (1, q_x]\setminus\overline{\mathscr{U}}$.

The proof for the right local dimension similar to (3.7) is more involved. The main obstacle in this case is that for a base $r \in \mathscr{U}(x)\cap(q, q+\delta)$ the expansion $\Phi_x(r)$ may not belong to $\mathbf{U}_q$, and then we have problems to build a bijective map similar to ϕ. Fortunately, if $(q, q+\delta)\cap\overline{\mathscr{U}} = \emptyset$ then the set of all such bases is at most countable. So, in the proof for the right local dimension we can throw away all of these bases $r \in \mathscr{U}(x)\cap(q, q+\delta)$ satisfying $\Phi_x(r)\notin\mathbf{U}_q$.

Take $q \in (1, q_x]\setminus\overline{\mathscr{U}}$. If $q = q_x \notin \overline{\mathscr{U}}$, then $q_x = 1 + M/x < M+1$. So, $\Psi_x(q_x) = M^\infty$, and thus x is the largest element of $\mathcal{U}_{q_x}$. Note that $q_x = \max\mathscr{U}(x)$. Then it is clear that

$$\mathscr{U}(x)\cap(q_x, q_x+\delta) = \mathcal{U}_{q_x}\cap(x, x+\zeta) = \emptyset\quad\text{for any } \delta,\zeta > 0. \tag{3.8}$$

In the following we assume $q \in (1, q_x) \setminus \overline{\mathscr{U}}$. Choose $\delta > 0$ such that $q + \delta < q_x$ and $(q, q+\delta) \cap \overline{\mathscr{U}} = \emptyset$. Let

$$\Gamma_{x,q,\delta} := \{r \in \mathscr{U}(x) \cap (q, q+\delta) : \Psi_x(r) \in \mathbf{U}_q\}.$$

By Lemma 2.1 (ii) and Lemma 2.4 (i) it follows that the difference between $\Gamma_{x,q,\delta}$ and $\mathscr{U}(x) \cap (q, q+\delta)$ is at most countable. So they have the same Hausdorff dimension. Observe that each $r \in \Gamma_{x,q,\delta}$ determines a unique $z = \pi_q(\Psi_x(r)) \in \mathcal{U}_q \cap (x, x+\zeta)$ for some $\zeta > 0$ depending on δ. This defines a bijection

$$\psi : \Gamma_{x,q,\delta} \to \mathcal{U}_q \cap (x, x+\zeta); \quad r \mapsto \pi_q(\Psi_x(r)).$$

Hence, by (3.1) and (3.6) we can prove that ψ is nearly bi-Lipschitz, and then by the same argument as in the proof of (3.7) we conclude that

$$\lim_{\delta \to 0} \dim_H(\mathscr{U}(x) \cap (q, q+\delta)) = \lim_{\delta \to 0} \dim_H \Gamma_{x,q,\delta} = \lim_{\zeta \to 0} \dim_H(\mathcal{U}_q \cap (x, x+\zeta)).$$

This, together with (3.7) and (3.8), completes the proof. □

4. Hausdorff dimension and critical values of $\mathbf{U}(x)$

Given $x > 0$, recall that the symbolic set $\mathbf{U}(x) = \{\Phi_x(q) : q \in \mathscr{U}(x)\}$ consists of all unique expansions of x with bases in $\mathscr{U}(x)$. Clearly, Φ_x is a bijective map from $\mathscr{U}(x)$ to $\mathbf{U}(x)$. Instead of looking at the set $\mathscr{U}(x)$ directly we focus on the symbolic set $\mathbf{U}(x)$. In this section we will investigate the Hausdorff dimension of $\mathbf{U}(x)$ with respect to the metric ρ defined in (1.4), and prove Theorem 1.3. Furthermore, by using Theorem 1.3 and Proposition 1.4 we determine the critical values of $\mathscr{U}(x)$, and then prove Theorem 1.5.

4.1. Hausdorff dimension of $\mathbf{U}(x)$

Our first result states that the set-valued map $x \mapsto \mathbf{U}(x)$ is non-increasing on $(1, \infty)$ with respect to the set inclusion.

Lemma 4.1. *The set-valued map $x \mapsto \mathbf{U}(x)$ is non-increasing on $(1, \infty)$.*

Proof. Let $x \in (1, \infty)$ and $(d_i) \in \mathbf{U}(x)$. Then there exists a unique base $q \in \mathscr{U}(x) \subseteq (1, M+1)$ such that

$$(d_i) = \Phi_x(q) \in \mathbf{U}_q. \tag{4.1}$$

Take $y \in (1, x)$. Then the equation

$$y = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \tag{4.2}$$

determines a unique base $\beta \in (q, M+1)$, since $\pi_{M+1}((d_i)) \le 1$. Observe by Lemma 2.4 (i) that the set-valued map $q \mapsto \mathbf{U}_q$ is non-decreasing. Then by (4.1) it follows that $(d_i) \in \mathbf{U}_q \subset \mathbf{U}_\beta$. In view of (4.2) this implies that

$$\Phi_y(\beta) = (d_i) \in \mathbf{U}_\beta.$$

So, $(d_i) \in \mathbf{U}(y)$, and thus $\mathbf{U}(x) \subseteq \mathbf{U}(y)$. This completes the proof. □

Now we turn to the Hausdorff dimension of $\mathbf{U}(x)$. This is based on the following lemma.

Lemma 4.2. *Given $x \in (0,1)$, let $(\varepsilon_i) = \Phi_x(M+1)$ be the quasi-greedy expansion of x in base $M+1$. Then there exist a word $\mathbf{w}$, a positive integer N and a strictly increasing sequence $(N_j) \subset \mathbb{N}$ such that*

$$\mathbf{U}_{N_j}(x) \subset \mathbf{U}(x) \quad \textit{for all } j \ge 1,$$

where

$$\mathbf{U}_{N_j}(x) := \left\{ \varepsilon_1 \ldots \varepsilon_{N+N_j} \mathbf{w} d_1 d_2 \ldots : \; d_{n+1} \ldots d_{n+N_j} \notin \left\{ 0^{N_j}, M^{N_j} \right\} \; \forall n \ge 0 \right\}.$$

Proof. The proof of this lemma is similar to [33, Section 4]. Let $(\varepsilon_i) \in \{0,1,\ldots,M\}^{\mathbb{N}}$ be the quasi-greedy expansion of x in base $M+1$. We distinguish two cases.

(I). (ε_i) **ends with** M^∞**.** Then we can write

$$(\varepsilon_i) = \varepsilon_1 \ldots \varepsilon_m \, M^\infty \quad \text{for some } m \ge 1 \text{ with } \varepsilon_m < M. \tag{4.3}$$

Let $\mathbf{w} = \epsilon$ be the empty word, $N = m$ and $N_j = m + j$ for $j \ge 1$. Take a sequence $(y_i) \in \mathbf{U}_{N_j}(x)$. Then it can be written as

$$(y_i) = \varepsilon_1 \ldots \varepsilon_{N+N_j} d_1 d_2 \ldots = \varepsilon_1 \ldots \varepsilon_m M^{N_j} d_1 d_2 \ldots, \tag{4.4}$$

where $(d_i) \in \{0,1,\ldots,M\}^{\mathbb{N}}$ contains neither N_j consecutive 0's nor N_j consecutive M's. Let q_j be the unique root in $(1, M+1)$ of the equation

$$x = \sum_{i=1}^{\infty} \frac{y_i}{q_j^i}.$$

Here we emphasize that $q_j < M+1$ because $\sum_{i=1}^{\infty} y_i/(M+1)^i < x$. We claim that (y_i) is the unique q_j-expansion of x.

Observe that the tail sequence

$$y_{m+1}y_{m+2}\ldots = M^{N_j}d_1d_2\ldots =: (\delta_i)$$

satisfies $\sigma^n((\delta_i)) \preccurlyeq (\delta_i)$ for all $n \geq 0$. Then by Remark 2.2 it follows that (δ_i) is the quasi-greedy expansion of 1 for some base $q \in (1, M+1]$, i.e., $\alpha(q) = (\delta_i)$. Note that $N_j > N = m$ and $\varepsilon_m < M$. Then by (4.4) it follows that the initial word $y_1 \ldots y_{N+N_j-1} = \varepsilon_1 \ldots \varepsilon_m M^{N_j-1}$ contains neither N_j consecutive 0's nor N_j consecutive M's. So, by the definition of (d_i) in (4.4) it follows that

$$0^{N_j} \prec y_{i+1}\ldots y_{i+N_j} \preccurlyeq M^{N_j} \quad \forall i \geq 0,$$

and the equality $y_{i+1}\ldots y_{i+N_j} = M^{N_j}$ holds if and only if $i = m$. This implies that $(\delta_i) = M^{N_j}d_1d_2\ldots$ is the lexicographically largest tail sequence of (y_i). Hence, by Lemma 2.3 it suffices to prove that $\alpha(q_j) \succ \alpha(q) = (\delta_i)$. In other words, it suffices to prove

$$\sum_{i=1}^{\infty} \frac{\delta_i}{q_j^i} < 1. \tag{4.5}$$

This follows from the following calculation: By (4.3) and (4.4) we obtain

$$\sum_{i=1}^{m} \frac{\varepsilon_i}{(M+1)^i} + \frac{1}{(M+1)^m} = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{(M+1)^i} = x = \sum_{i=1}^{\infty} \frac{y_i}{q_j^i} = \sum_{i=1}^{m} \frac{\varepsilon_i}{q_j^i} + \frac{1}{q_j^m} \sum_{i=1}^{\infty} \frac{\delta_i}{q_j^i},$$

which gives

$$\sum_{i=1}^{\infty} \frac{\delta_i}{q_j^i} = q_j^m \left(\frac{1}{(M+1)^m} + \sum_{i=1}^{m} \left(\frac{\varepsilon_i}{(M+1)^i} - \frac{\varepsilon_i}{q_j^i} \right) \right) \leq \frac{q_j^m}{(M+1)^m} < 1,$$

where the inequalities follow since $q_j < M + 1$. This proves (4.5).

Therefore, by Lemma 2.3 it follows that (y_i) is the unique q_j-expansion of x, i.e., $(y_i) \in \mathbf{U}(x)$. Hence, $\mathbf{U}_{N_j}(x) \subset \mathbf{U}(x)$.

(II). (ε_i) **does not end with** M^∞**.** Since (ε_i) is the quasi-greedy expansion of x in base $M+1$, (ε_i) does not end with 0^∞. Then there exists an integer $N \geq 3$ such that $\varepsilon_{N-2} > 0$. Choose $N_1 > N$ such that $\varepsilon_{N+N_1+1} > 0$ and

$$|\{1 \leq i \leq N_1 : \varepsilon_i > 0\}| \geq N+1, \qquad |\{1 \leq i \leq N_1 : \varepsilon_i < M\}| \geq N+1. \tag{4.6}$$

In fact, we can choose a strictly increasing sequence (N_j) such that $\varepsilon_{N+N_j+1} > 0$ for any $j \geq 1$. Set $\mathbf{w} = 0M$. Fix $j \geq 1$, and take a sequence

$$(y_i) = \varepsilon_1 \ldots \varepsilon_{N+N_j} 0M d_1 d_2 \ldots \in \mathbf{U}_{N_j}(x), \tag{4.7}$$

where the tail sequence (d_i) contains neither N_j consecutive 0's nor N_j consecutive M's. It follows from (4.6) that the initial word $\varepsilon_1 \dots \varepsilon_{N+N_j}$ contains neither N_j consecutive 0's nor N_j consecutive M's. Hence, by (4.7) it gives that (y_i) contains neither (N_j+1) consecutive 0's nor (N_j+1) consecutive M's. Note that the equation

$$x = \pi_{q_j}((y_i)) = \sum_{i=1}^{\infty} \frac{y_i}{q_j^i}$$

determines a unique $q_j \in (1, M+1)$. Here we emphasize that $q_j < M+1$, since $\varepsilon_{N+N_j+1} > 0 = y_{N+N_j+1}$ which implies that $\sum_{i=1}^{\infty} y_i/(M+1)^i < x$. Then by Lemma 2.3, to show that (y_i) is the unique q_j-expansion of x it suffices to show that $\alpha(q_j) \succ M^{N_j+1}0^\infty$, or equivalently, to prove

$$\sum_{i=1}^{N_j+1} \frac{M}{q_j^i} < 1. \tag{4.8}$$

Observe by (4.7) that

$$\sum_{i=1}^{N+N_j} \frac{\varepsilon_i}{q_j^i} < \sum_{i=1}^{\infty} \frac{y_i}{q_j^i} = x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{(M+1)^i} < \sum_{i=1}^{N+N_j} \frac{\varepsilon_i}{(M+1)^i} + \frac{1}{(M+1)^{N+N_j}}.$$

This, combined with $\varepsilon_{N-2} > 0$ and $q_j < M+1$, implies that

$$\frac{M+1-q_j}{q_j(M+1)^{N-2}} \le \frac{1}{q_j^{N-2}} - \frac{1}{(M+1)^{N-2}} \le \sum_{i=1}^{N+N_j} \left(\frac{\varepsilon_i}{q_j^i} - \frac{\varepsilon_i}{(M+1)^i} \right) < \frac{1}{(M+1)^{N+N_j}}.$$

Rearranging the above inequality yields

$$M+1-q_j < \frac{q_j}{(M+1)^{N_j+2}} < \frac{M}{q_j^{N_j+1}},$$

which gives $M(1-q_j^{-N_j-1}) < q_j - 1$. Thus,

$$\sum_{i=1}^{N_j+1} \frac{M}{q_j^i} = \frac{M(1-q_j^{-N_j-1})}{q_j-1} < 1,$$

proving (4.8).

Therefore, (y_i) is the unique q_j-expansion of x, i.e., $(y_i) \in \mathbf{U}(x)$. Hence, $\mathbf{U}_{N_j}(x) \subset \mathbf{U}(x)$ for all $j \ge 1$, completing the proof. □

The following lemma can be deduced from [4, Theorem 3.1].

Lemma 4.3. *For any $q \in (1, M+1]$ and $n \in \mathbb{N}$ let*

$$\mathbf{U}_{q,n} := \left\{(d_i) : \overline{\alpha_1(q)\ldots\alpha_n(q)} \prec d_{i+1}\ldots d_{i+n} \prec \alpha_1(q)\ldots\alpha_n(q)\ \forall i \geq 0\right\}. \tag{4.9}$$

Then

$$\lim_{n\to\infty} \dim_H \mathbf{U}_{q,n} = \dim_H \mathbf{U}_q.$$

Proof. Take $q \in (1, M+1]$, and set $\widetilde{\mathbf{U}}_q := \left\{(d_i) : \overline{\alpha(q)} \prec d_{i+1}d_{i+2}\ldots \prec \alpha(q)\ \forall i \geq 0\right\}$. Then by Lemma 2.3 we have $\widetilde{\mathbf{U}}_q \subseteq \mathbf{U}_q$. Furthermore, by [25, Lemma 2.5] it follows that

$$\dim_H \widetilde{\mathbf{U}}_q = \dim_H \mathbf{U}_q. \tag{4.10}$$

Define a sequence of subsets

$$\mathbf{V}_{q,n} := \left\{(d_i) : \overline{\alpha_1(q)\ldots\alpha_n(q)} \preccurlyeq d_{i+1}\ldots d_{i+n} \preccurlyeq \alpha_1(q)\ldots\alpha_n(q)\ \forall i \geq 0\right\}, \quad n \geq 1.$$

Then $\mathbf{U}_{q,n} \subseteq \widetilde{\mathbf{U}}_q \subseteq \mathbf{V}_{q,n}$ for all $n \geq 1$. So by (4.10) it suffices to prove

$$\lim_{n\to\infty} \dim_H \mathbf{U}_{q,n} = \lim_{n\to\infty} \dim_H \mathbf{V}_{q,n}. \tag{4.11}$$

Note that for any $n \geq 1$ the sets $\mathbf{U}_{q,n}$ and $\mathbf{V}_{q,n}$ are both subshifts of finite type, and then by [21, Proposition 3.1] it follows that

$$\dim_H \mathbf{U}_{q,n} = \frac{h_{top}(\mathbf{U}_{q,n})}{\log(M+1)} \quad \text{and} \quad \dim_H \mathbf{V}_{q,n} = \frac{h_{top}(\mathbf{V}_{q,n})}{\log(M+1)},$$

where $h_{top}(X)$ denotes the topological entropy of a subset $X \subseteq \{0, 1, \ldots, M\}^{\mathbb{N}}$. So, (4.11) follows directly from [4, Theorem 3.1] that $\lim_{n\to\infty} h_{top}(\mathbf{U}_{q,n}) = \lim_{n\to\infty} h_{top}(\mathbf{V}_{q,n})$. This completes the proof. □

Proof of Theorem 1.3. Note by Proposition 1.2 that the function $q \mapsto \dim_H \mathbf{U}_q$ is a non-decreasing Devil's staircase on $(1, M+1]$. Then by the definition of q_x it suffices to prove

$$\dim_H \mathbf{U}(x) = \dim_H \mathbf{U}_{q_x} \quad \text{for all } x > 0. \tag{4.12}$$

First we consider $x \in (0, 1)$. Let $(\varepsilon_i) = \Phi_x(M+1)$ be the quasi-greedy expansion of x in base $M+1$. Then by Lemma 4.2 there exist a word $\mathbf{w}$, a positive integer N and a strictly increasing sequence $(N_j) \subset \mathbb{N}$ such that

$$\mathbf{U}_{N_j}(x) = \left\{\varepsilon_1\ldots\varepsilon_{N+N_j}\mathbf{w}d_1d_2\ldots : (d_i) \in \Lambda_j\right\} \subset \mathbf{U}(x) \quad \text{for all } j \geq 1, \tag{4.13}$$

where

$$\Lambda_j := \Big\{(d_i) \in \{0,1,\ldots,M\}^{\mathbb{N}} : d_{n+1}\ldots d_{n+N_j} \notin \{0^{N_j}, M^{N_j}\}\ \ \forall n \geq 0\Big\}.$$

By (4.13) it follows that for any $j \geq 1$,

$$\dim_H \mathbf{U}(x) \geq \dim_H \mathbf{U}_{N_j}(x) = \dim_H \Lambda_j = \dim_H \mathbf{U}_{p_j} \tag{4.14}$$

where $p_j \in (1, M+1]$ satisfies

$$1 = \sum_{i=1}^{N_j} \frac{M}{p_j^i}.$$

Note that the function $q \mapsto \dim_H \mathbf{U}_q$ is continuous. Letting $j \to \infty$ in (4.14), so that $N_j \to \infty$ and hence $p_j \to M+1$, it follows by Proposition 1.2 that

$$\dim_H \mathbf{U}(x) \geq \dim_H \mathbf{U}_{M+1} = 1.$$

Note that $q_x = M+1$ for all $x \in (0,1)$. Hence, $\dim_H \mathbf{U}(x) = 1 = \dim_H \mathbf{U}_{q_x}$ for all $x \in (0,1)$. This proves (4.12) for $x \in (0,1)$.

Now we prove (4.12) for $x \geq 1$. Then $q_x = 1+M/x$, and the quasi-greedy q_x-expansion of x is M^∞. We claim that for any $N \in \mathbb{N}$ there exists an integer $J = J(N) > 0$ such that

$$\Gamma_{N,J} := \left\{M^J d_1 d_2 \ldots : (d_i) \in \mathbf{U}_{q_x,N}\right\} \subset \mathbf{U}(x), \tag{4.15}$$

where $\mathbf{U}_{q_x,N}$ is defined as in (4.9).

This can be verified by the following observation. Take $N \in \mathbb{N}$. Since $\Phi_x(q_x) = M^\infty$, by Lemma 2.1 we can choose J sufficiently large such that

$$\alpha_1(q_{N,J})\ldots\alpha_N(q_{N,J}) = \alpha_1(q_x)\ldots\alpha_N(q_x), \tag{4.16}$$

where $q_{N,J}$ is the positive root of the equation $\sum_{i=1}^{J} Mq^{-i} = x$. Note that each sequence $(y_i) \in \Gamma_{N,J}$ determines a unique base $p \in (1, q_x)$ via the equation

$$\sum_{i=1}^{\infty} \frac{y_i}{p^i} = x.$$

Since $M^J 0^\infty \prec (y_i) \prec M^\infty$, we must have $q_{N,J} < p < q_x$. Then by Lemma 2.1 and (4.16) it follows that $\alpha_1(p)\ldots\alpha_N(p) = \alpha_1(q_x)\ldots\alpha_N(q_x)$. So by Lemma 2.3 we conclude that each $(y_i) \in \Gamma_{N,J}$ is the unique expansion of x in some base $p \in (q_{N,J}, q_x)$. In other words, $\Gamma_{N,J} \subset \mathbf{U}(x)$, proving (4.15).

By (4.15) it follows that

$$\dim_H \mathbf{U}(x) \geq \dim_H \Gamma_{N,J} = \dim_H \mathbf{U}_{q_x,N} \quad \forall\, N \in \mathbb{N}. \tag{4.17}$$

By Lemma 4.3 it follows that $\lim_{N\to\infty} \dim_H \mathbf{U}_{q_x,N} = \dim_H \mathbf{U}_{q_x}$. Letting $N \to \infty$ in (4.17) we conclude that

$$\dim_H \mathbf{U}(x) \geq \dim_H \mathbf{U}_{q_x}.$$

The reverse inequality is obvious since $\mathbf{U}(x) \subset \mathbf{U}_{q_x}$ by Lemma 2.4 (i). This proves (4.12) for all $x \geq 1$. □

4.2. Critical values of $\mathscr{U}(x)$

Observe by Proposition 3.1 that the map $\Phi_x : \mathscr{U}(x) \to \mathbf{U}(x)$ is bijective and locally bi-Hölder continuous. So, to determine the critical values of $\mathscr{U}(x)$ is equivalent to determine the critical values of $\mathbf{U}(x)$. We do this by using Theorem 1.3 and Proposition 1.4.

Recall from (1.5) that $q_G = q_G(M) \in (1, M+1)$ is the generalized golden ratio. Then $x_G = M/(q_G - 1) > 1$. First we show that $\mathscr{U}(x)$ is a singleton for any $x \geq x_G$.

Lemma 4.4. *If $x \geq x_G$, then $\mathscr{U}(x) = \{q_x\}$.*

Proof. Note by Proposition 1.4 (i) that for $q \leq q_G$ the symbolic univoque set $\mathbf{U}_q = \{0^\infty, M^\infty\}$. Since for $x \geq x_G$ we have by (1.3) that $q_x \leq q_G$, so

$$\mathbf{U}(x) \subseteq \{0^\infty, M^\infty\} \quad \forall\, x \geq x_G.$$

If $0^\infty \in \mathbf{U}(x)$, then $x = \pi_q(0^\infty) = 0$, leading to a contradiction with our assumption that $x \geq x_G > 0$. So $\mathbf{U}(x) = \{M^\infty\}$, which implies $\mathscr{U}(x) = \{q_x\}$ for any $x \geq x_G$. □

In the following lemma we show that x_G is indeed a critical value for $\mathscr{U}(x)$. Recall that $q_{KL} \in (q_G, M+1)$ is the Komornik-Loreti constant. Then $x_{KL} = M/(q_{KL}-1) \in (1, x_G)$.

Lemma 4.5. *For any $x < x_G$ the set $\mathscr{U}(x)$ contains infinitely many elements. In particular, for $x \in [x_{KL}, x_G)$ we have $|\mathscr{U}(x)| = \aleph_0$.*

Proof. Let $x < x_G$. Then $q_x > q_G$. Note by Theorem 1.3 that $\dim_H \mathbf{U}(x) = \dim_H \mathbf{U}_{M+1} = 1$ for $x \in (0, 1]$. So it suffices to prove that $\mathbf{U}(x)$ contains infinitely many elements for $x \in (1, x_G)$. Take $x \in (1, x_G)$. Then by (1.3) it follows that $q_x = 1 + M/x \in (q_G, M+1)$ and the quasi-greedy expansion $\Phi_x(q_x) = M^\infty$. By Lemma 2.1 (i) it follows that for $k \in \mathbb{N}$ sufficiently large the equation

$$\pi_{p_k}(M^k \alpha(q_G)) = x \tag{4.18}$$

determines a unique base $p_k \in (q_G, q_x)$, and $p_k \nearrow q_x$ as $k \to \infty$. So, there exists $K = K(x) \in \mathbb{N}$ such that $p_k \in (q_G, q_x)$ for any $k \geq K$. Take $k \geq K$. Then $\alpha(p_k) \succ \alpha(q_G)$. By Lemma 2.3 it follows that $M^k \alpha(q_G) \in \mathbf{U}_{p_k}$. Therefore, by (4.18) we conclude that

$$M^k \alpha(q_G) \in \mathbf{U}(x) \quad \forall k \geq K.$$

This implies that $\mathscr{U}(x)$ is an infinite set for any $x < x_G$.

On the other hand, observe by Lemma 2.4 (i) that $\mathbf{U}(x) \subseteq \mathbf{U}_{q_x}$ for any $x > 0$. Furthermore, $q_x \in (q_G, q_{KL})$ if and only if $x \in (x_{KL}, x_G)$. By using Proposition 1.4 (ii) it follows that $\mathbf{U}(x)$ is at most countable for any $x \in (x_{KL}, x_G)$. If $x = x_{KL}$, then $q_x = q_{KL}$ and $\Phi_x(q_x) = M^\infty$. Observe that

$$\begin{aligned}
\mathbf{U}(x) &= \{M^\infty\} \cup \{\Phi_x(p) : p \in \mathscr{U}(x) \cap (1, q_{KL})\} \\
&= \{M^\infty\} \cup \bigcup_{n=1}^{\infty} \left\{ \Phi_x(p) : p \in \mathscr{U}(x) \cap (1, q_{KL} - \frac{1}{2^n}) \right\} \\
&\subseteq \{M^\infty\} \cup \bigcup_{n=1}^{\infty} \mathbf{U}_{q_{KL} - \frac{1}{2^n}}.
\end{aligned}$$

Then by Proposition 1.4 (ii) we can deduce from the above equation that $\mathbf{U}(x)$ is also a countable set for $x = x_{KL}$. Therefore, $|\mathscr{U}(x)| = \aleph_0$ for any $x \in [x_{KL}, x_G)$. □

In the next lemma we demonstrate that x_{KL} is also a critical value of $\mathscr{U}(x)$.

Lemma 4.6.

(i) *If* $x \in (0, 1]$, *then* $\dim_H \mathscr{U}(x) = 1$;
(ii) *If* $x \in (1, x_{KL})$, *then*

$$0 < \dim_H \mathcal{U}_{q_x} \leq \dim_H \mathscr{U}(x) \leq \max_{q \in \overline{\mathscr{U}(x)}} \dim_H \mathcal{U}_q < 1.$$

Proof. (i) was first proven by Lü, Tan and Wu [33] for $M = 1$. For $M > 1$ the proof was given by Xu [39] in his thesis. For completeness we prove this by using Theorem 1.3 and Proposition 3.3.

For $x > 0$ note that $\mathscr{U}(x) \subset (1, q_x]$. Then by using the countable stability of Hausdorff dimension (cf. [20]) and Proposition 3.3 it follows that

$$\begin{aligned}
\dim_H \mathscr{U}(x) &= \dim_H \Big(\bigcup_n \mathscr{U}(x) \cap (1 + n^{-1}, q_x - n^{-1}) \Big) \\
&= \sup_n \dim_H \left(\mathscr{U}(x) \cap (1 + n^{-1}, q_x - n^{-1}) \right) \\
&\geq \sup_n \frac{1}{\log(q_x - n^{-1})} \dim_H \Phi_x \left(\mathscr{U}(x) \cap (1 + n^{-1}, q_x - n^{-1}) \right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\log q_x} \sup_n \dim_H \Phi_x \left(\mathscr{U}(x) \cap (1+n^{-1}, q_x - n^{-1})\right) \\
&= \frac{1}{\log q_x} \dim_H \left(\bigcup_n \Phi_x\left(\mathscr{U}(x) \cap (1+n^{-1}, q_x - n^{-1})\right)\right) \\
&= \frac{\dim_H \mathbf{U}(x)}{\log q_x} = \frac{\dim_H \mathbf{U}_{q_x}}{\log q_x},
\end{aligned}
$$

where the last equality follows by Theorem 1.3. Therefore, by Proposition 1.4 we obtain that

$$
\dim_H \mathscr{U}(x) \geq \dim_H \mathcal{U}_{q_x} \quad \forall\, x > 0. \tag{4.19}
$$

Note by (1.3) that $q_x = M+1$ for all $x \in (0,1]$. Then by (4.19) and Proposition 1.4 we conclude that $\dim_H \mathscr{U}(x) = \dim_H \mathcal{U}_{M+1} = 1$ for all $x \in (0,1]$.

Now we prove (ii). Let $x \in (1, x_{KL})$. Then $q_x \in (q_{KL}, M+1)$. The first two inequalities of (ii) follow from (4.19). For the remaining inequalities in (ii) we set

$$
\xi := \max_{q \in \overline{\mathscr{U}(x)}} \dim_H \mathcal{U}_q.
$$

Since $\overline{\mathscr{U}(x)} \subset (1, q_x]$ and $q_x \in \mathscr{U}(x) \cap (q_{KL}, M+1)$, by Proposition 1.2 it follows that $0 < \xi < 1$. Take $\varepsilon > 0$. By Proposition 3.3, Lemma 2.4 (i) and Proposition 1.2 it follows that for each $q \in \overline{\mathscr{U}(x)}$ there exists $\delta > 0$ such that

$$
\begin{aligned}
\dim_H(\mathscr{U}(x) \cap (q-\delta, q+\delta)) &\leq \frac{\dim_H \Phi_x(\mathscr{U}(x) \cap (q-\delta, q+\delta))}{\log(q-\delta)} \\
&\leq \frac{\log(q+\delta)}{\log(q-\delta)} \cdot \frac{\dim_H \mathbf{U}_{q+\delta}}{\log(q+\delta)} \\
&= \frac{\log(q+\delta)}{\log(q-\delta)} \dim_H \mathcal{U}_{q+\delta} \\
&\leq \dim_H \mathcal{U}_q + \varepsilon \leq \xi + \varepsilon.
\end{aligned} \tag{4.20}
$$

For each $q \in \overline{\mathscr{U}(x)}$ we choose a $\delta_q \in (0, M+1-q_x)$ satisfying (4.20). Then the collection $\left\{(q-\delta_q, q+\delta_q) : q \in \overline{\mathscr{U}(x)}\right\}$ forms an open cover of $\overline{\mathscr{U}(x)}$. Since $\overline{\mathscr{U}(x)}$ is compact, there exists a finite cover $\{(q_i - \delta_i, q_i + \delta_i)\}_{i=1}^N$ of $\overline{\mathscr{U}(x)}$, where $\delta_i := \delta_{q_i}$. By (4.20) this implies

$$
\begin{aligned}
\dim_H \mathscr{U}(x) &= \dim_H \left(\mathscr{U}(x) \cap \bigcup_{i=1}^N (q_i - \delta_i, q_i + \delta_i)\right) \\
&= \max_{1 \leq i \leq N} \dim_H(\mathscr{U}(x) \cap (q_i - \delta_i, q_i + \delta_i)) \\
&\leq \xi + \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. □

Proof of Theorem 1.5. By Lemmas 4.4–4.6 it suffices to prove that $\mathscr{U}(x)$ has zero Lebesgue measure. This result was first proven in [33] for $M = 1$ by using the Lebesgue density theorem. Here we present an alternate proof by using Proposition 3.3. By the same argument as in the proof of Lemma 4.6 (ii) one can easily verify that for $x > 0$,

$$\dim_H \left(\mathscr{U}(x) \cap (1, M+1-\frac{1}{2^n}) \right) < 1 \quad \text{for any } n \geq 1.$$

This implies that $\mathscr{U}(x) \cap (1, M+1-\frac{1}{2^n})$ has zero Lebesgue measure for all $n \geq 1$. Then we conclude that $\mathscr{U}(x)$ is a Lebesgue null set by observing

$$\mathscr{U}(x) \subseteq \{M+1\} \cup \bigcup_{n=1}^{\infty} \left(\mathscr{U}(x) \cap (1, M+1-\frac{1}{2^n}) \right). \quad \square$$

5. Isolated points of $\mathscr{U}(x)$

In this section we will consider the topological structure of $\mathscr{U}(x)$ when x varies in $(0, \infty)$. In particular, we will investigate the isolated points of $\mathscr{U}(x)$, and prove Theorem 1.7. Recall from (1.2) that $(1, M+1] \setminus \overline{\mathscr{U}} = \bigcup (q_0, q_0^*)$, and recall $\mathscr{V}$ from (2.1). Then for each connected component (q_0, q_0^*) of $(1, M+1] \setminus \overline{\mathscr{U}}$ we can write the elements of $\mathscr{V} \cap (q_0, q_0^*) = \{q_n\}_{n=1}^{\infty}$ in an increasing order as

$$q_0 < q_1 < q_2 < \cdots < q_n < q_{n+1} < \cdots, \quad \text{and} \quad q_n \nearrow q_0^* \text{ as } n \to \infty.$$

By Lemma 2.4 (ii) it follows that $\mathbf{U}_p = \mathbf{U}_{q_{n+1}}$ for any $p \in (q_n, q_{n+1}]$. For $n \geq 1$ set

$$\mathbf{U}_{q_{n+1}}^* := \mathbf{U}_{q_{n+1}} \setminus \mathbf{U}_{q_n} = \left\{ (d_i) \in \mathbf{U}_{q_{n+1}} : (d_i) \text{ ends with } \alpha(q_n) \text{ or } \overline{\alpha(q_n)} \right\}.$$

It was shown in [13] that $\mathbf{U}_{q_{n+1}}^*$ is dense in $\mathbf{U}_{q_{n+1}}$ for any $n \geq 1$.

First we give a sufficient condition for the set $\mathscr{U}(x)$ to include isolated points.

Proposition 5.1. *Let (q_0, q_0^*) be a connected component of $(1, M+1] \setminus \overline{\mathscr{U}}$, and let $\{q_n\}_{n=1}^{\infty} = \mathscr{V} \cap (q_0, q_0^*)$. Then for any*

$$x \in \bigcup_{n=1}^{\infty} \bigcup_{p \in (q_n, q_{n+1})} \pi_p(\mathbf{U}_{q_{n+1}}^*)$$

the set $\mathscr{U}(x)$ contains at least one isolated point.

Proof. For $n \geq 1$ let $x \in \pi_p(\mathbf{U}_{q_{n+1}}^*)$ for some $p \in (q_n, q_{n+1})$. In the following we will show that p is an isolated point of $\mathscr{U}(x)$. Note by the definition of $\mathscr{V}$ that $\Phi_x(p) \in \mathbf{U}_{q_{n+1}}^* \subset \mathbf{U}_{q_{n+1}} = \mathbf{U}_p$. Then $\Phi_x(p) = (x_i(p)) \in \mathbf{U}_p$. Furthermore, by the definition of $\mathbf{U}_{q_{n+1}}^*$ it follows that $\Phi_x(p)$ ends with $\alpha(q_n) = (a_1 \dots a_m \overline{a_1 \dots a_m})^\infty$ for some $m \geq 1$. So there exists $N \in \mathbb{N}$ such that

$$\Phi_x(p) = x_1(p) \dots x_N(p)(a_1 \dots a_m \overline{a_1 \dots a_m})^\infty.$$

Now suppose $p \in (q_n, q_{n+1})$ is not an isolated point of $\mathscr{U}(x)$. Then by Lemma 2.1 (i) there exists a $p' \in \mathscr{U}(x) \cap (q_n, q_{n+1})$ such that $p' \neq p$ and $\Phi_x(p') = (x_i(p'))$ coincides with $\Phi_x(p)$ for the first $N + 2m$ digits, i.e.,

$$x_1(p') \dots x_{N+2m}(p') = x_1(p) \dots x_N(p) a_1 \dots a_m \overline{a_1 \dots a_m}. \tag{5.1}$$

Observe that $\Phi_x(p') \in \mathbf{U}_{p'} = \mathbf{U}_{q_{n+1}}$ and $\alpha(q_{n+1}) = (a_1 \dots a_m \overline{a_1 \dots a_m}^+ \overline{a_1 \dots a_m} a_1 \dots a_m^-)^\infty$. Then by (5.1) and Lemma 2.3 it follows that

$$\Phi_x(p') = x_1(p) \dots x_{N_1}(p)(a_1 \dots a_m \overline{a_1 \dots a_m})^\infty = \Phi_x(p).$$

This implies $p' = p$ by Lemma 2.1 (i), leading to a contradiction with our hypothesis. So, p is an isolated point of $\mathscr{U}(x)$. □

Recall from Section 1 that $X_{iso} = \{x > 0 : \mathscr{U}(x) \text{ contains isolated points}\}$. By Lemma 4.4 we see that $\mathscr{U}(x) = \{q_x\}$ is a singleton for any $x \geq x_G = M/(q_G - 1)$. This implies that $[x_G, \infty) \subset X_{iso}$. In the following result we show that the set X_{iso} is dense in $[0, 1]$.

Lemma 5.2. *For any $x \in [0, 1]$ and any $\delta > 0$ the intersection $X_{iso} \cap (x - \delta, x + \delta)$ contains an interval.*

Proof. Take $x \in [0, 1]$ and $\delta > 0$. Then there exist $y \in (x - \frac{\delta}{3}, x + \frac{\delta}{3})$ and an integer $N_1 = N_1(x, \delta) > 0$ such that the quasi-greedy expansion $\Phi_y(M + 1) = y_1 y_2 \dots$ contains neither N_1 consecutive 0's nor N_1 consecutive M's. By Lemmas 2.3 and 2.4 (i) this implies

$$(y_i) \in \mathbf{U}_q \quad \forall q > p_{N_1}, \tag{5.2}$$

where p_n is the root of $\sum_{i=1}^n \frac{M}{p_n^i} = 1$ in $(1, M + 1)$. Clearly, $p_n \nearrow M + 1$ as $n \to \infty$. Note that the map

$$g : [p_{N_1}, M + 1] \to \mathbb{R}; \quad q \mapsto \pi_q((y_i))$$

is continuous, and $g(M + 1) = y$. So there exists an integer $N_2 > N_1$ such that

$$g(q) \in \left(y - \frac{\delta}{3}, y + \frac{\delta}{3}\right) \subseteq \left(x - \frac{2\delta}{3}, x + \frac{2\delta}{3}\right) \quad \forall q \in [p_{N_2}, M + 1]. \tag{5.3}$$

Let $(q_0, q_0^*) \subset [p_{N_2}, M+1]$ be a connected component of $(1, M+1] \setminus \overline{\mathscr{U}}$, and write $(q_0, q_0^*) \setminus \mathscr{V} = \bigcup_{n=0}^{\infty}(q_n, q_{n+1})$. Take $n \geq 1$. Recall from [13, Theorem 1.4] that the set $\mathbf{U}_{q_{n+1}}^*$ is dense in $\mathbf{U}_{q_{n+1}}$ with respect to the metric ρ defined in (1.4), and note by (5.2) that $(y_i) \in \mathbf{U}_{q_{n+1}}$. Then there exists a sequence $(z_i) \in \mathbf{U}_{q_{n+1}}^*$ such that

$$|\pi_q((z_i)) - g(q)| = |\pi_q((z_i)) - \pi_q((y_i))| < \frac{\delta}{3} \quad \forall q \in (q_n, q_{n+1}). \tag{5.4}$$

Since $(q_n, q_{n+1}) \subset [p_{N_2}, M+1]$, by (5.3) and (5.4) it follows that

$$z^q := \pi_q((z_i)) \in (x - \delta, x + \delta) \quad \forall q \in (q_n, q_{n+1}).$$

Furthermore, by using Proposition 5.1 we obtain that $\mathscr{U}(z^q)$ contains isolated points for any $q \in (q_n, q_{n+1})$. In other words, $X_{iso} \cap (x - \delta, x + \delta)$ contains the sub-interval $(z^{q_{n+1}}, z^{q_n})$. $\square$

In the following we consider isolated points of $\mathscr{U}(x)$ for $x > 1$. When $M = 1$ we show that $X_{iso} \supset (1, \infty)$.

Proposition 5.3. *Let $M = 1$. Then for any $x > 1$ the set $\mathscr{U}(x)$ contains isolated points.*

Note by Lemma 4.4 that $X_{iso} \supset [x_G, \infty)$. Thus it suffices to prove that X_{iso} covers $(1, x_G)$. In the following we fix $M = 1$, and we will prove Proposition 5.3 in several steps. Let $(q_0, q_0^*) = (1, q_{KL})$ be the first connected component of $(1, 2] \setminus \overline{\mathscr{U}}$. Then $\mathscr{V} \cap (q_0, q_0^*) = \{q_1, q_2, q_3, \ldots\}$ satisfying

$$1 = q_0 < q_1 < q_2 < q_3 < \cdots < q_0^* = q_{KL}, \quad \text{and} \quad q_n \nearrow q_{KL} \text{ as } n \to \infty.$$

Furthermore, for each $n \geq 1$ the base $q_n \in (1, q_{KL})$ admits the quasi-greedy expansion

$$\alpha(q_n) = (\tau_1 \ldots \tau_{2^n}^-)^\infty, \tag{5.5}$$

where $(\tau_i)_{i=0}^\infty = 01101001\ldots$ is the classical Thue-Morse sequence (cf. [7]).

The following properties of the sequence (τ_i) are well known (see, for example, [27]).

Lemma 5.4. *For any integer $n \geq 0$ we have*

(i) $\tau_{2^n+1} \ldots \tau_{2^{n+1}} = \overline{\tau_1 \ldots \tau_{2^n}}^+$.
(ii) $\overline{\tau_1 \ldots \tau_{2^n - i}} \prec \tau_{i+1} \ldots \tau_{2^n} \preccurlyeq \tau_1 \ldots \tau_{2^n - i} \quad \forall\, 0 \leq i < 2^n$.

Now we construct sequences in $\mathbf{U}_{q_{n+1}}^*$.

Lemma 5.5. *For $n \geq 1$ and $k \geq 1$ let*

$$\mathbf{c}_{n,k} := \tau_1 \dots \tau_{2^{n-1}} (\overline{\tau_1 \dots \tau_{2^{n-1}}}^+)^k (\overline{\tau_1 \dots \tau_{2^n}}^+)^\infty.$$

Then $\mathbf{c}_{n,k} \in \mathbf{U}^*_{q_{n+1}}$ for all $k \geq 1$.

Proof. Note by (5.5) that $\mathbf{c}_{n,k}$ ends with $(\overline{\tau_1 \dots \tau_{2^n}^-})^\infty = \overline{\alpha(q_n)}$. Then by Lemma 2.3 it suffices to prove

$$\overline{\alpha(q_{n+1})} \prec \sigma^j(\mathbf{c}_{n,k}) \prec \alpha(q_{n+1}) \quad \forall j \geq 1, \tag{5.6}$$

where σ is the left-shift map. Since $\alpha(q_{n+1})$ begins with $\tau_1 \dots \tau_{2^n}$, we prove (5.6) by considering the following three cases.

(I). $1 \leq j < 2^{n-1}$. Then (5.6) follows by Lemma 5.4 (ii), which implies that

$$\overline{\tau_1 \dots \tau_{2^{n-1}-j}} \prec \tau_{j+1} \dots \tau_{2^{n-1}} \preccurlyeq \tau_1 \dots \tau_{2^{n-1}-j} \quad \text{and} \quad \overline{\tau_1 \dots \tau_j} \prec \tau_{2^{n-1}-j+1} \dots \tau_{2^{n-1}}.$$

(II). $2^{n-1} \leq j < (k+1)2^{n-1}$. Note that $\sigma^{2^{n-1}}(\mathbf{c}_{n,k}) = (\overline{\tau_1 \dots \tau_{2^{n-1}}}^+)^k (\overline{\tau_1 \dots \tau_{2^n}}^+)^\infty$. Then (5.6) again follows by Lemma 5.4 (ii), which implies that

$$\overline{\tau_1 \dots \tau_{2^{n-1}-i}} \prec \overline{\tau_{i+1} \dots \tau_{2^{n-1}}}^+ \preccurlyeq \tau_1 \dots \tau_{2^{n-1}-i} \quad \text{and} \quad \overline{\tau_1 \dots \tau_i} \prec \tau_{2^{n-1}-i+1} \dots \tau_{2^{n-1}} \tag{5.7}$$

for any $0 \leq i < 2^{n-1}$.

(III). $j \geq (k+1)2^{n-1}$. Then (5.6) follows from (5.7) with $n-1$ replaced by n. $\square$

By the definition of $\mathbf{c}_{n,k}$ it is easy to see that

$$\mathbf{c}_{n,k} \nearrow \mathbf{c}_{n,\infty} := \tau_1 \dots \tau_{2^{n-1}} (\overline{\tau_1 \dots \tau_{2^{n-1}}}^+)^\infty \quad \text{as } k \to \infty.$$

In the following lemma we construct sequences in $\mathbf{U}^*_{q_{n+1}}$ that decrease to $\mathbf{c}_{n,\infty}$.

Lemma 5.6. *For $n \geq 2$ and $k \geq 1$ let*

$$\mathbf{d}_{n,k} := \tau_1 \dots \tau_{2^{n-1}} (\overline{\tau_1 \dots \tau_{2^{n-1}}}^+)^k \, \overline{\tau_1 \dots \tau_{2^{n-2}}}^+ \, (\overline{\tau_1 \dots \tau_{2^n}}^+)^\infty.$$

Then $\mathbf{d}_{n,k} \in \mathbf{U}^*_{q_{n+1}}$ for all $k \geq 1$.

Proof. It is clear that $\mathbf{d}_{n,k}$ ends with $\overline{\alpha(q_n)}$. Then by Lemma 2.3 it suffices to prove

$$\overline{\alpha(q_{n+1})} \prec \sigma^j(\mathbf{d}_{n,k}) \prec \alpha(q_{n+1}) \quad \forall j \geq 1. \tag{5.8}$$

Since $\alpha(q_{n+1})$ begins with $\tau_1 \dots \tau_{2^n}$, by Cases (I) and (II) in the proof of Lemma 5.5 we only need to verify (5.8) for $j \geq k2^{n-1} + 2^{n-2}$. Observe by Lemma 5.4 (i) that

$$\begin{aligned}\sigma^{k2^{n-1}+2^{n-2}}(\mathbf{d}_{n,k}) &= \tau_1\ldots\tau_{2^{n-2}}\overline{\tau_1\ldots\tau_{2^{n-2}}}^+(\overline{\tau_1\ldots\tau_{2^{n-1}}}\tau_1\ldots\tau_{2^{n-1}})^\infty \\ &= (\tau_1\ldots\tau_{2^{n-1}}\overline{\tau_1\ldots\tau_{2^{n-1}}})^\infty = \alpha(q_n).\end{aligned}$$

Then by the same argument as in the proof of Case III in Lemma 5.5 it follows that (5.8) holds for all $j \geq k2^{n-1}+2^{n-2}$. This completes the proof. □

Clearly,

$$\mathbf{d}_{n,k} \searrow \mathbf{d}_{n,\infty} := \tau_1\ldots\tau_{2^{n-1}}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^\infty = \mathbf{c}_{n,\infty} \quad \text{as } k\to\infty.$$

Now we are ready to prove Proposition 5.3.

Proof of Proposition 5.3. Let $M = 1$. By Lemma 5.5 and Proposition 5.1 it follows that

$$X_{iso} \supset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcup_{p\in(q_n,q_{n+1})}\pi_p(\mathbf{c}_{n,k}) = \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}(\pi_{q_{n+1}}(\mathbf{c}_{n,k}),\pi_{q_n}(\mathbf{c}_{n,k})), \tag{5.9}$$

where the bases $q_n \in \mathscr{V}$ are defined as in (5.5). Similarly, by Lemma 5.6 and Proposition 5.1 it follows that

$$X_{iso} \supset \bigcup_{n=2}^{\infty}\bigcup_{k=1}^{\infty}\bigcup_{p\in(q_n,q_{n+1})}\pi_p(\mathbf{d}_{n,k}) = \bigcup_{n=2}^{\infty}\bigcup_{k=1}^{\infty}(\pi_{q_{n+1}}(\mathbf{d}_{n,k}),\pi_{q_n}(\mathbf{d}_{n,k})). \tag{5.10}$$

In the following we will show that the unions in (5.9) and (5.10) are sufficient to cover $(1, x_G)$.

First we prove that the union in (5.9) covers $(1, x_G)$ up to a countable set. By (5.5) and Lemma 5.4 (i) it follows that

$$\begin{aligned}\pi_{q_{n+1}}(\mathbf{c}_{n,k+1}) &= \pi_{q_{n+1}}(\tau_1\ldots\tau_{2^{n-1}}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^{k+1}\overline{\tau_1\ldots\tau_{2^{n-1}}}(\tau_1\ldots\tau_{2^{n-1}}\overline{\tau_1\ldots\tau_{2^{n-1}}})^\infty) \\ &< \pi_{q_{n+1}}(\tau_1\ldots\tau_{2^{n-1}}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^{k+2}0^\infty) \\ &= \pi_{q_{n+1}}(\tau_1\ldots\tau_{2^n}0^{2^{n-1}k}\,\overline{\tau_1\ldots\tau_{2^{n-1}}}^+0^\infty) + \pi_{q_{n+1}}(0^{2^n}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^k0^\infty).\end{aligned} \tag{5.11}$$

On the other hand, by (5.5) we obtain

$$\begin{aligned}\pi_{q_n}(\mathbf{c}_{n,k}) &= \pi_{q_n}(\tau_1\ldots\tau_{2^{n-1}}\overline{\tau_1\ldots\tau_{2^{n-1}}}^+0^\infty) + \pi_{q_n}(0^{2^n}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^k0^\infty) \\ &= 1 + \pi_{q_n}(0^{2^n}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^k0^\infty).\end{aligned} \tag{5.12}$$

Since $\pi_{q_{n+1}}(\tau_1\ldots\tau_{2^n}0^{2^{n-1}k}\overline{\tau_1\ldots\tau_{2^{n-1}}}^+0^\infty) < 1$ for any $n \geq 1, k \geq 1$, by (5.11) and (5.12) it follows that $\pi_{q_{n+1}}(\mathbf{c}_{n,k+1}) < \pi_{q_n}(\mathbf{c}_{n,k})$. Therefore, the intervals $J_k := (\pi_{q_{n+1}}(\mathbf{c}_{n,k}), \pi_{q_n}(\mathbf{c}_{n,k}))$ with $k \geq 1$ are pairwise overlapping. So,

$$\bigcup_{k=1}^{\infty}(\pi_{q_{n+1}}(\mathbf{c}_{n,k}),\pi_{q_n}(\mathbf{c}_{n,k}))=(\pi_{q_{n+1}}(\mathbf{c}_{n,1}),\pi_{q_n}(\mathbf{c}_{n,\infty})),\tag{5.13}$$

where we recall that $\mathbf{c}_{n,\infty}=\tau_1\ldots\tau_{2^{n-1}}(\overline{\tau_1\ldots\tau_{2^{n-1}}}^+)^\infty$. Note by Lemma 5.4 (i) that

$$\mathbf{c}_{n,1}=\tau_1\ldots\tau_{2^{n-1}}\overline{\tau_1\ldots\tau_{2^{n-1}}}^+(\overline{\tau_1\ldots\tau_{2^n}}^+)^\infty=\tau_1\ldots\tau_{2^n}(\overline{\tau_1\ldots\tau_{2^n}}^+)^\infty=\mathbf{c}_{n+1,\infty}.\tag{5.14}$$

Write $z_n:=\pi_{q_n}(\mathbf{c}_{n,\infty})$. Then by (5.9), (5.13) and (5.14) it follows that

$$X_{iso}\supset\bigcup_{n=1}^{\infty}(z_{n+1},z_n).$$

Observe that $z_1=\pi_{q_1}(\mathbf{c}_{1,\infty})=\pi_{q_1}(1^\infty)=x_G$. Furthermore, since $q_n\nearrow q_{KL}$ and $\mathbf{c}_{n,\infty}\searrow\tau_1\tau_2\ldots$ as $n\to\infty$, we have

$$z_n=\pi_{q_n}(\mathbf{c}_{n,\infty})\searrow\pi_{q_{KL}}(\tau_1\tau_2\ldots)=1\quad\text{as } n\to\infty.$$

Therefore,

$$X_{iso}\supset(1,x_G)\setminus\{z_n:n\ge 2\}.$$

To complete the proof it remains to prove $z_n\in X_{iso}$ for all $n\ge 2$. We will show that all of these points z_n belong to the union in (5.10). Recall that for $n\ge 2$ the sequence $\mathbf{d}_{n,k}$ decreases to $\mathbf{d}_{n,\infty}=\mathbf{c}_{n,\infty}$ as $k\to\infty$. Since $\mathbf{d}_{n,k}$ and $\mathbf{c}_{n,\infty}$ are both quasi-greedy q_n-expansions, by Lemma 2.1 (i) it follows that

$$\pi_{q_n}(\mathbf{d}_{n,k})>\pi_{q_n}(\mathbf{c}_{n,\infty})=z_n\quad\forall k\ge 1.\tag{5.15}$$

On the other hand, since

$$\lim_{k\to\infty}\pi_{q_{n+1}}(\mathbf{d}_{n,k})=\pi_{q_{n+1}}(\mathbf{c}_{n,\infty})<\pi_{q_n}(\mathbf{c}_{n,\infty})=z_n,$$

by (5.15) there exists $K\in\mathbb{N}$ such that for all $k\ge K$ we have

$$z_n\in(\pi_{q_{n+1}}(\mathbf{d}_{n,k}),\pi_{q_n}(\mathbf{d}_{n,k}))\subset X_{iso}\quad\forall n\ge 2.\tag{5.16}$$

This completes the proof. □

Remark 5.7. By Proposition 5.1 and (5.16) it follows that for any $z_n=\pi_{q_n}(\mathbf{c}_{n,\infty})$ with $n\ge 2$ and for any $k\ge K$ the equation

$$\pi_{p_k}(\mathbf{d}_{n,k})=z_n$$

determines a unique $p_k \in (q_n, q_{n+1})$. Note by Lemma 5.6 that $\mathbf{d}_{n,k} \in \mathbf{U}^*_{q_{n+1}}$. Then $z_n \in \pi_{p_k}(\mathbf{U}^*_{q_{n+1}})$. So, by the proof of Proposition 5.1 it follows that p_k is an isolated point of $\mathscr{U}(z_n)$. Observe that for any $k \neq j$ we have $\mathbf{d}_{n,k} \neq \mathbf{d}_{n,j}$, and thus $p_k \neq p_j$. This means that for each $n \geq 2$ the set $\mathscr{U}(z_n)$ contains infinitely many isolated points.

Proof of Theorem 1.7. The theorem follows by Lemma 5.2 and Proposition 5.3. □

6. Final remarks and questions

At the end of this paper we pose some questions. In view of Theorem 1.1 it is natural to ask the following question.

Question 1. Does Theorem 1.1 hold for any $x > 0$ and any $q \in \overline{\mathscr{U}}$?

By Theorem 1.3 and Theorem 1.2 it follows that the bifurcation set of $\phi : x \mapsto \dim_H \mathbf{U}(x)$ can be easily obtained from the bifurcation set of $\psi : q \mapsto \dim_H \mathbf{U}_q$. More precisely, x is a bifurcation point of ϕ if and only if q_x is a bifurcation point of ψ. The same correspondence holds for the plateaus of ϕ and ψ. Motivated by Lemma 4.1 and the works studied in [13,3] we ask the following question.

Question 2. Can we describe the bifurcation sets of the set-valued map $x \mapsto \mathbf{U}(x)$? In other words, can we describe the following sets

$$E_1 = \{x : \mathbf{U}(y) \neq \mathbf{U}(x)\ \forall y > x\} \quad \text{and} \quad E_2 = \{x : \dim_H(\mathbf{U}(x) \setminus \mathbf{U}(y)) > 0\ \forall y > x\}?$$

Although we can calculate the Hausdorff dimension of $\mathbf{U}(x)$ as in Theorem 1.5, we are not able to determine the Hausdorff dimension of $\mathscr{U}(x)$ for $x \in (1, x_{KL})$.

Question 3. What is the Hausdorff dimension of $\mathscr{U}(x)$ for $x \in (1, x_{KL})$?

Finally, for the isolated points of $\mathscr{U}(x)$ we have shown in Theorem 1.7 that X_{iso} is dense in $(0, \infty)$ for $M = 1$. Our proof does not work for $M \geq 2$ in the interval $(1, x_G)$.

Question 4. Is it true that X_{iso} is dense in $(0, \infty)$ for any $M \geq 2$? We conjecture that

$$\mathscr{U}(x) \text{ contains isolated points} \quad \Longleftrightarrow \quad x \in (0,1) \cup (1, \infty).$$

Up to now we know very little about the topological structure of $\mathscr{U}(x)$. Clearly, for $x \geq x_G$ the set $\mathscr{U}(x) = \{q_x\}$ is a singleton.

Question 5. When is $\mathscr{U}(x)$ a closed set for $x \in (0, x_G)$?

Acknowledgments

The authors thank the anonymous referee for many useful suggestions, especially for the simplification of the proof of Theorem 1.1. D. Kong thanks Pieter Allaart for providing the Maple codes of Fig. 1. He was supported by NSFC No. 11971079 and the Fundamental and Frontier Research Project of Chongqing No. cstc2019jcyj-msxmX0338 and No. cx2019067. W. Li was supported by NSFC No. 11671147, 11571144 and Science and Technology Commission of Shanghai Municipality (STCSM) No. 13dz2260400. F. Lü was supported by NSFC No. 11601358.

References

[1] R. Alcaraz Barrera, S. Baker, D. Kong, Entropy, topological transitivity, and dimensional properties of unique q-expansions, Trans. Am. Math. Soc. 371 (5) (2019) 3209–3258.
[2] P.C. Allaart, On univoque and strongly univoque sets, Adv. Math. 308 (2017) 575–598.
[3] P.C. Allaart, S. Baker, D. Kong, Bifurcation sets arising from non-integer base expansions, J. Fractal Geom. 6 (4) (2019) 301–341.
[4] P.C. Allaart, D. Kong, On the continuity of the Hausdorff dimension of the univoque set, Adv. Math. 354 (2019) 106729.
[5] P.C. Allaart, D. Kong, Relative bifurcation sets and the local dimension of univoque bases, Ergod. Theory Dyn. Syst. (2020), https://doi.org/10.1017/etds.2020.38.
[6] P.C. Allaart, D. Kong, On the smallest base in which a number has a unique expansion, arXiv: 2006.07927, 2020.
[7] J.-P. Allouche, J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in: Sequences and Their Applications, Singapore, 1998, in: Springer Ser. Discrete Math. Theor. Comput. Sci., Springer, London, 1999, pp. 1–16.
[8] C. Baiocchi, V. Komornik, Greedy and quasi-greedy expansions in non-integer bases, arXiv:0710.3001v1, 2007.
[9] S. Baker, Generalized golden ratios over integer alphabets, Integers 14 (A15) (2014) 28.
[10] K. Dajani, M. de Vries, Invariant densities for random β-expansions, J. Eur. Math. Soc. 9 (1) (2007) 157–176.
[11] K. Dajani, V. Komornik, D. Kong, W. Li, Algebraic sums and products of univoque bases, Indag. Math. 29 (4) (2018) 1087–1104.
[12] Z. Daróczy, I. Kátai, On the structure of univoque numbers, Publ. Math. (Debr.) 46 (3–4) (1995) 385–408.
[13] M. de Vries, V. Komornik, Unique expansions of real numbers, Adv. Math. 221 (2) (2009) 390–427.
[14] M. de Vries, V. Komornik, A two-dimensional univoque set, Fundam. Math. 212 (2) (2011) 175–189.
[15] M. de Vries, V. Komornik, Expansions in non-integer bases, in: Combinatorics, Words and Symbolic Dynamics, in: Encyclopedia Math. Appl., vol. 159, Cambridge Univ. Press, Cambridge, 2016, pp. 18–58.
[16] M. de Vries, V. Komornik, P. Loreti, Topology of the set of univoque bases, Topol. Appl. 205 (2016) 117–137.
[17] P. Erdős, M. Horváth, I. Joó, On the uniqueness of the expansions $1 = \sum q^{-n_i}$, Acta Math. Hung. 58 (3–4) (1991) 333–342.
[18] P. Erdős, I. Joó, On the number of expansions $1 = \sum q^{-n_i}$, Ann. Univ. Sci. Bp. Rolando Eötvös Nomin., Sect. Math. 35 (1992) 129–132.
[19] P. Erdős, I. Joó, V. Komornik, Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems, Bull. Soc. Math. Fr. 118 (1990) 377–390.
[20] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, third edition, John Wiley & Sons, Ltd., Chichester, 2014.
[21] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, Math. Syst. Theory 1 (1) (1967) 1–49.
[22] P. Glendinning, N. Sidorov, Unique representations of real numbers in non-integer bases, Math. Res. Lett. 8 (2001) 535–543.

[23] C. Kalle, D. Kong, W. Li, F. Lü, On the bifurcation set of unique expansions, Acta Arith. 188 (4) (2019) 367–399.
[24] V. Komornik, Expansions in noninteger bases, Integers 11B (A9) (2011) 30.
[25] V. Komornik, D. Kong, W. Li, Hausdorff dimension of univoque sets and devil's staircase, Adv. Math. 305 (2017) 165–196.
[26] V. Komornik, P. Loreti, Unique developments in non-integer bases, Am. Math. Mon. 105 (7) (1998) 636–639.
[27] V. Komornik, P. Loreti, Subexpansions, superexpansions and uniqueness properties in non-integer bases, Period. Math. Hung. 44 (2) (2002) 197–218.
[28] V. Komornik, P. Loreti, On the topological structure of univoque sets, J. Number Theory 122 (1) (2007) 157–183.
[29] D. Kong, On small univoque bases of real numbers, Acta Math. Hung. 150 (1) (2016) 194–208.
[30] D. Kong, W. Li, Hausdorff dimension of unique beta expansions, Nonlinearity 28 (1) (2015) 187–209.
[31] D. Kong, W. Li, F.M. Dekking, Intersections of homogeneous Cantor sets and beta-expansions, Nonlinearity 23 (11) (2010) 2815–2834.
[32] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, Cambridge, 1995.
[33] F. Lü, B. Tan, J. Wu, Univoque sets for real numbers, Fundam. Math. 227 (1) (2014) 69–83.
[34] W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hung. 11 (1960) 401–416.
[35] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung. 8 (1957) 477–493.
[36] N. Sidorov, Almost every number has a continuum of β-expansions, Am. Math. Mon. 110 (9) (2003) 838–842.
[37] N. Sidorov, Arithmetic dynamics, in: Topics in Dynamics and Ergodic Theory, in: London Math. Soc. Lecture Note Ser., vol. 310, Cambridge Univ. Press, Cambridge, 2003, pp. 145–189.
[38] N. Sidorov, Expansions in non-integer bases: lower, middle and top orders, J. Number Theory 129 (4) (2009) 741–754.
[39] J. Xu, Lebesgue measure and Hausdorff dimension of univoque sets for $\beta > 1$, Thesis (M.Sc.), East China Normal University, Shanghai, 2019.