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General Section

On the sum of squares of the middle-third Cantor set

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ABSTRACT

Let C be the middle-third Cantor set. In this paper, we show that for every $x \in [0,4]$, there exist $x_1, x_2, x_3, x_4 \in C$ such that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$, which was conjectured in Athreya et al. (2019) [1].

1. Introduction

The middle-third Cantor set

$$C = \left\{ \sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i} : \varepsilon_i \in \{0,2\} \right\}$$

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is a classical object in fractal geometry. The arithmetic on the middle-third Cantor set has been studied in [1,2,4,3,5,6,8]. The first classical result is that the set

$$C - C := \{x - y : x, y \in C\} \tag{1.1}$$

equals to the interval $[-1, 1]$. The proof of (1.1) was first given by H. Steinhaus in 1917. The result was rediscovered by J. F. Randolph in 1940 [7]. Using the symmetry of C, we can deduce that

$$C + C = C + (1 - C) = 1 + (C - C) = [0, 2],$$

where $C + C := \{x + y : x, y \in C\}$. The multiplication and division on middle-third Cantor set were discussed in [1]. Athreya, Reznick and Tyson [1] proved that

$$\mathcal{L}(C \cdot C) \geq \frac{17}{21} \text{ and } \frac{C}{C} = \bigcup_{n=-\infty}^{\infty} \left[\frac{2}{3} \cdot 3^n, \frac{3}{2} \cdot 3^n\right] \cup \{0\},$$

where $C \cdot C := \{xy : x, y \in C\}$, $\frac{C}{C} := \left\{\frac{x}{y} : x, y \in C, y \neq 0\right\}$ and $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$. Gu, Jiang, Xi and Zhao [3] gave the complete topological structure of $C \cdot C$. Moreover, they also proved that the Lebesgue measure of $C \cdot C$ is about 0.80955.

The main motivation of this paper is due to a conjecture posed by Athreya, Reznick and Tyson [1]. They conjectured $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 : \ x_i \in C\} = [0, 4]$ and claimed that there is strong numerical evidence supporting it. In this paper, we will prove this conjecture.

Fixing $\alpha > 1$, let C_α (the middle-$\frac{1}{\alpha}$ Cantor set) be generated by the iterated function system $\Phi = \{f_1(x) = rx, f_2(x) = rx + 1 - r\}$ with $r = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)$. Thus the classical middle-third Cantor set $C = C_3$. In the present paper we prove

Theorem 1.1. *Let C_α be the middle-$\frac{1}{\alpha}$ Cantor set for $\alpha > 1$. Then*

$$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 : \ x_i \in C_\alpha\} = [0, 4] \textit{ if and only if } \alpha \geq 3.$$

The proof of the above theorem is similar to the case $\alpha = 3$. We only give an outline of the proof of Theorem 1.1 for the middle-third Cantor set. Using the similarity of C, it suffices to prove that

$$(4/9, 4] \subseteq f(C^4)$$

where the function f is defined by (2.2). This is shown in Lemma 3.1 and 3.2. The basic idea to find intervals contained in $f(C^4)$ is due to [1, Lemma 3]. Notice that for the sum of four squares, the calculation is complicated. We divide the sum of four squares into two parts as

$$f(C^4) = g(C^3) + \{x^2 : x \in C\},$$

where the function g is defined by (2.1). We find some intervals contained in $g(C^3)$ and use the fourth number to translate these intervals so that they can cover the interval $(4/9, 4]$. With similar discussions as Lemma 2.3 and 2.4, in Corollary 2.5 we give a concrete condition to find the following intervals contained in $g(C^3)$, i.e.

$$[44/81, 67/81] \cup [8/9, 3] \subseteq g(C^3). \tag{1.2}$$

By the above inclusion, if we take $0, 1 \in C$, then

$$f(C^4) \supset g(C^3) \cup (g(C^3) + 1) \supset [8/9, 4].$$

It remains to prove that $f(C^4)$ can cover the points around $4/9$. We divide the interval $(4/9, 4/3]$ into the intervals of form

$$\left(\frac{4}{9} + \frac{8}{3^{2n+2}}, \frac{4}{9} + \frac{8}{3^{2n}}\right] \tag{1.3}$$

for every positive integer n. By the similarity of C and (1.2), we have that

$$\left[\frac{8}{3^{2n+2}}, \frac{27}{3^{2n+2}}\right] \cup \left[\frac{44}{3^{2n+2}}, \frac{67}{3^{2n+2}}\right] \subseteq g(C^3).$$

It remains to choose special points in C as translations such that the translations of the above two intervals cover the intervals of form (1.3) for any $n \geq 1$. More precisely, we can choose the points $2/3, 2/3 + 3^{-2n} \in C$ for the interval $[44 \cdot 3^{-2n-2}, 67 \cdot 3^{-2n-2}]$, and the points $2/3, 2/3 + 3^{-2n}, 2/3 + 2 \cdot 3^{-2n} \in C$ for the interval $[8 \cdot 3^{-2n-2}, 27 \cdot 3^{-2n-2}]$. These points motivate the choices of some special points for the general case C_α.

This paper is organized as follows. In section 2, we discuss the set $\{x_1^2 + x_2^2 + x_3^2 : x_i \in C_\alpha\}$. The proof of Theorem 1.1 is arranged in the section 3.

2. Sum of three squares

As stated in the previous section, C_α is the unique nonempty compact set satisfying

$$C_\alpha = f_1(C_\alpha) \cup f_2(C_\alpha) = rC_\alpha \cup (rC_\alpha + 1 - r)$$

where $r = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)$. It follows that if $x \in C_\alpha$, then $rx \in C_\alpha$. We will use this simple observation in Lemma 3.1. For each positive integer n let

$$\mathcal{F}_n = \{f_\sigma([0,1]) : \sigma \in \{1,2\}^n\} \text{ and } F_n = \bigcup_{A \in \mathcal{F}_n} A,$$

where $f_\sigma(x) = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(x)$ for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \{1,2\}^n$. Then the sequence $F_n, n = 1, 2, \cdots$, of nonempty compact sets is decreasing and

$$C_\alpha = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \bigcup_{\sigma\in\{1,2\}^n} f_\sigma([0,1]).$$

It is easy to see that for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \{1,2\}^n$

$$f_\sigma(0) = \frac{1-r}{r}\sum_{k=1}^{n}(\sigma_k - 1)r^k$$

and so

$$f_\sigma([0,1]) = [f_\sigma(0), f_\sigma(1)] = \left[\frac{1-r}{r}\sum_{k=1}^{n}(\sigma_k-1)r^k, \frac{1-r}{r}\sum_{k=1}^{n}(\sigma_k-1)r^k + r^n\right].$$

Each element of $\mathcal{F}_n$, called an n-level basic interval, has length r^n. For an n-level basic interval $f_\sigma([0,1])$, it contains two $(n+1)$-level basic intervals $f_{\sigma 1}([0,1])$ and $f_{\sigma 2}([0,1])$. The interval $f_\sigma([0,1])$ shares the same left endpoint with $f_{\sigma 1}([0,1])$, and shares the same right endpoint with $f_{\sigma 2}([0,1])$. The length of the open interval $f_\sigma([0,1]) \setminus (f_{\sigma 1}([0,1]) \cup f_{\sigma 2}([0,1]))$ is $\frac{1}{\alpha}$ times that of $f_\sigma([0,1])$.

Denote by L_n the collection of left endpoints of all n-level basic intervals. For $u \in L_n$, we associate u with an n-level basic interval

$$I_u = [u,\ u + r^n]$$

and two $(n+1)$-level basic intervals denoted by

$$I_{u,0} = [u, u + r^{n+1}],\ I_{u,1} = [u + (1-r)r^n, u + r^n].$$

The key to discuss the sum of squares of Cantor set is the following lemma, which is an easy exercise in real analysis and also appears as Lemma 2 in [1].

Lemma 2.1. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be continuous. If $\{K_j\}_{j\in\mathbb{N}}$ is a decreasing sequence of nonempty compact subsets of $\mathbb{R}^d$, then*

$$\varphi\left(\bigcap_{j=1}^{\infty} K_j\right) = \bigcap_{j=1}^{\infty} \varphi(K_j).$$

Proof. Since $\bigcap_{j=1}^{\infty} K_j \subseteq K_n$ for every $n \in \mathbb{N}$, we have

$$\varphi\left(\bigcap_{j=1}^{\infty} K_j\right) \subseteq \bigcap_{j=1}^{\infty} \varphi(K_j).$$

Conversely, assume that $y \in \bigcap_{j=1}^{\infty} \varphi(K_j)$. For every j, we can find $x_j \in K_j$ such that $\varphi(x_j) = y$. Since K_1 is compact, by Bolzano–Weierstrass Theorem, there is a convergent subsequence $x_{n_j} \to x$. Since φ is continuous, we have $\varphi(x) = y$. Note that the sequence $\{x_{n_j}\}_{j \geq m}$ is in K_m for every $m \in \mathbb{N}$. It follows from compactness that $x \in K_m$ for every $m \in \mathbb{N}$. Therefore, $y = \varphi(x) \in \varphi\left(\bigcap_{j=1}^{\infty} K_j\right)$, which completes the proof. $\square$

Define functions $g : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^4 \to \mathbb{R}$ by letting

$$g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \tag{2.1}$$

and

$$f(x_1, x_2, x_3, x_4) = g(x_1, x_2, x_3) + x_4^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{2.2}$$

For a positive integer k and a nonempty set $A \subseteq \mathbb{R}$, denote

$$A^k = \{(x_1, \cdots, x_k) : x_i \in A\}.$$

In order to show $f(C_\alpha^4) = [0, 4]$, we need to discuss the set $g(C_\alpha^3)$ and find some intervals in $g(C_\alpha^3)$. Note that $C_\alpha^3 = \bigcap_{n=1}^{\infty} F_n^3$. Applying Lemma 2.1 for the continuous function g, we obtain the following corollary.

Corollary 2.2. $g(C_\alpha^3) = \bigcap_{n=1}^{\infty} g(F_n^3)$.

If an interval $I \subseteq g(F_n^3)$ for every $n \in \mathbb{N}$, then $I \subseteq g(C_\alpha^3)$. The following two lemmas give a sufficient condition to find intervals in $g(C_\alpha^3)$.

Lemma 2.3. *Let $\alpha \geq 3$. For any $u, v, w \in L_n$, if*

$$\max\{u, v, w\} > 0 \tag{2.3}$$

and

$$4(1 - r) \max\{u, v, w\} \leq 2(u + v + w) + (1 + 2r)r^n, \tag{2.4}$$

then

$$g(I_u \times I_v \times I_w) = g((I_{u,0} \cup I_{u,1}) \times (I_{v,0} \cup I_{v,1}) \times (I_{w,0} \cup I_{w,1})).$$

Proof. At first we have $r = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right) \in [1/3, 1/2)$ since $\alpha \geq 3$. Write $t = u^2 + v^2 + w^2$. Without loss of generality, we can assume that $u \geq v \geq w$. By (2.3) we have $u > 0$ and so $u \geq f_{1^{n-1}2}(0) = (1 - r)r^{n-1} > r^n$. In addition, (2.4) reduces to

$$2v + 2w + (1 + 2r)r^n \geq 2(1 - 2r)u. \tag{2.5}$$

It is routine to verify that

$$\begin{aligned} g(I_{u,1}\times I_{v,0}\times I_{w,0}) =&[t+2u(1-r)r^n+(1-r)^2r^{2n},\\ &\quad t+2(u+rv+rw)r^n+(1+2r^2)r^{2n}],\\ g(I_{u,1}\times I_{v,0}\times I_{w,1}) =&[t+2(u+w)(1-r)r^n+2(1-r)^2r^{2n},\\ &\quad t+2(u+rv+w)r^n+(2+r^2)r^{2n}],\\ g(I_{u,1}\times I_{v,1}\times I_{w,0}) =&[t+2(u+v)(1-r)r^n+2(1-r)^2r^{2n},\\ &\quad t+2(u+v+rw)r^n+(2+r^2)r^{2n}], \end{aligned}$$

and

$$\begin{aligned} g(I_{u,1}\times I_{v,1}\times I_{w,1}) =&[t+2(u+v+w)(1-r)r^n+3(1-r)^2r^{2n},\\ &\quad t+2(u+v+w)r^n+3r^{2n}]. \end{aligned}$$

Note that

$$\begin{aligned} &t+2(u+rv+rw)r^n+(1+2r^2)r^{2n}\\ &\quad-(t+2(u+w)(1-r)r^n+2(1-r)^2r^{2n})\\ =&2(ru+rv+2rw-w)r^n+(4r-1)r^{2n}\\ \geq&2(4r-1)wr^n+(4r-1)r^{2n}>0, \end{aligned}$$

and

$$\begin{aligned} &t+2(u+rv+w)r^n+(2+r^2)r^{2n}\\ &\quad-(t+2(u+v)(1-r)r^n+2(1-r)^2r^{2n})\\ =&2(ru+2rv-v+w)r^n+(4-r)r^{2n+1}\\ \geq&2(3r-1)vr^n+(4-r)r^{2n+1}>0, \end{aligned}$$

and

$$\begin{aligned} &t+2(u+v+rw)r^n+(2+r^2)r^{2n}\\ &\quad-(t+2(u+v+w)(1-r)r^n+3(1-r)^2r^{2n})\\ =&2(ru+rv+2rw-w)r^n+(6r-2r^2-1)r^{2n}\\ \geq&2(4r-1)wr^n+(6r-2r^2-1)r^{2n}>0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &g(I_{u,1}\times(I_{v,0}\cup I_{v,1})\times(I_{w,0}\cup I_{w,1}))\\ =&\left[t+2u(1-r)r^n+(1-r)^2r^{2n},\ t+2(u+v+w)r^n+3r^{2n}\right]. \end{aligned} \tag{2.6}$$

It is also routine to verify that

$$\begin{aligned} g(I_{u,0}\times I_{v,0}\times I_{w,0}) =&[t,\ t+2(u+v+w)r^{n+1}+3r^{2n+2}],\\ g(I_{u,0}\times I_{v,0}\times I_{w,1}) =&[t+2w(1-r)r^n+(1-r)^2r^{2n},\\ &\qquad t+2(ru+rv+w)r^n+(1+2r^2)r^{2n}],\\ g(I_{u,0}\times I_{v,1}\times I_{w,0}) =&[t+2v(1-r)r^n+(1-r)^2r^{2n},\\ &\qquad t+2(ru+v+rw)r^n+(1+2r^2)r^{2n}], \end{aligned}$$

and

$$\begin{aligned} g(I_{u,0}\times I_{v,1}\times I_{w,1}) =&[t+2(v+w)(1-r)r^n+2(1-r)^2r^{2n},\\ &\qquad t+2(ru+v+w)r^n+(2+r^2)r^{2n}]. \end{aligned}$$

Since $u>r^n$, we have

$$\begin{aligned} &t+2(u+v+w)r^{n+1}+3r^{2n+2}-(t+2w(1-r)r^n+(1-r)^2r^{2n})\\ =&2(ru+rv+2rw-w)r^n+(2r^2+2r-1)r^{2n}\\ \geq&2(3r-1)wr^n+2ur^{n+1}+(2r-1)r^{2n}\\ >&2(3r-1)wr^n+(4r-1)r^{2n}>0, \end{aligned}$$

and

$$\begin{aligned} &t+2(ru+rv+w)r^n+(1+2r^2)r^{2n}-(t+2v(1-r)r^n+(1-r)^2r^{2n})\\ =&2(ru+2rv-v+w)r^n+(r+2)r^{2n+1}\\ \geq&2(3r-1)vr^n+(r+2)r^{2n+1}>0, \end{aligned}$$

and

$$\begin{aligned} &t+2(ru+v+rw)r^n+(1+2r^2)r^{2n}\\ &\quad-(t+2(v+w)(1-r)r^n+2(1-r)^2r^{2n})\\ =&2(ru+rv+2rw-w)r^n+(4r-1)r^{2n}\\ \geq&2(4r-1)wr^n+(4r-1)r^{2n}>0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &g(I_{u,0}\times(I_{v,0}\cup I_{v,1})\times(I_{w,0}\cup I_{w,1}))\\ =&\left[t,\ t+2(ru+v+w)r^n+(2+r^2)r^{2n}\right]. \end{aligned} \tag{2.7}$$

It follows from condition (2.5) that

$$t+2(ru+v+w)r^n+(2+r^2)r^{2n}\geq t+2u(1-r)r^n+(1-r)^2r^{2n}.$$

Thus, the intervals in (2.6) and (2.7) overlap and so

$$g((I_{u,0} \cup I_{u,1}) \times (I_{v,0} \cup I_{v,1}) \times (I_{w,0} \cup I_{w,1})) = \left[t,\ t + 2(u+v+w)r^n + 3r^{2n}\right].$$

Note that

$$g(I_u \times I_v \times I_w) = \left[t,\ t + 2(u+v+w)r^n + 3r^{2n}\right].$$

Therefore, we conclude that

$$g(I_u \times I_v \times I_w) = g((I_{u,0} \cup I_{u,1}) \times (I_{v,0} \cup I_{v,1}) \times (I_{w,0} \cup I_{w,1})),$$

as desired. □

Lemma 2.4. *Let $\alpha \geq 3$. For any $u, v, w \in L_n$, if*

$$2(1-r)\max\{u,v,w\} + (1-2r)r^n \leq u + v + w, \tag{2.8}$$

then

$$g(I_u \times I_v \times I_w) \subseteq g(C_\alpha^3).$$

Proof. Note that the condition (2.8) implies (2.3) and (2.4).

For $k \geq n$, we define

$$\mathcal{F}_{1,k} = \{I \in \mathcal{F}_k : I \subseteq I_u\},\ \mathcal{F}_{2,k} = \{I \in \mathcal{F}_k : I \subseteq I_v\},\ \mathcal{F}_{3,k} = \{I \in \mathcal{F}_k : I \subseteq I_w\},$$

and

$$F_{1,k} = \bigcup_{A \in \mathcal{F}_{1,k}} A,\ \ F_{2,k} = \bigcup_{A \in \mathcal{F}_{2,k}} A,\ \ F_{3,k} = \bigcup_{A \in \mathcal{F}_{3,k}} A.$$

By Corollary 2.2, it suffices to show that for $k \geq n$,

$$g(I_u \times I_v \times I_w) \subseteq g(F_{1,k} \times F_{1,k} \times F_{1,k}). \tag{2.9}$$

We now prove it by induction on k.

When $k = n$, we have $F_{1,n} = I_u, F_{2,n} = I_v, F_{3,n} = I_w$, and thus

$$g(I_u \times I_v \times I_w) \subseteq g(F_{1,n} \times F_{2,n} \times F_{3,n}).$$

Next, assume that (2.9) is true for some $m \geq n$, i.e.,

$$g(I_u \times I_v \times I_w) \subseteq g(F_{1,m} \times F_{2,m} \times F_{3,m}). \tag{2.10}$$

Then, taking $x \in g(I_u \times I_v \times I_w)$, it follows from (2.10) that there exist $u', v', w' \in L_m$ such that

$$I_{u'} \subseteq I_u,\ I_{v'} \subseteq I_v,\ I_{w'} \subseteq I_w \ \text{ and } \ x \in g(I_{u'} \times I_{v'} \times I_{w'}).$$

Now condition (2.8) implies that $\max\{u, v, w\} > 0$, and thus

$$\max\{u', v', w'\} > 0.$$

Moreover, it follows from (2.8) that

$$\begin{aligned}(1-2r)\max\{u', v', w'\} &\leq (1-2r)\max\{u, v, w\} + (1-2r)r^n \\ &\leq u + v + w - \max\{u, v, w\} \\ &\leq u' + v' + w' - \max\{u', v', w'\},\end{aligned}$$

where the last inequality holds because the function

$$\psi(x, y, z) = x + y + z - \max\{x, y, z\},$$

i.e. the sum of two smallest elements among x, y, z, is increasing in its components. Therefore,

$$2(1-r)\max\{u', v', w'\} \leq u' + v' + w'.$$

Thus, applying Lemma 2.3, there exist $i, j, \ell \in \{0, 1\}$ such that

$$x \in g(I_{u',i} \times I_{v',j} \times I_{w',\ell}).$$

Obviously, we have $I_{u',i} \in \mathcal{F}_{1,m+1},\ I_{v',j} \in \mathcal{F}_{1,m+1}$ and $I_{w',\ell} \in \mathcal{F}_{1,m+1}$. Therefore,

$$x \in g(F_{1,m+1} \times F_{2,m+1} \times F_{3,m+1}).$$

This shows that (2.9) is true for $k = m + 1$. $\square$

Corollary 2.5. *For $\alpha \geq 3$,*

$$[a, b] \cup [2(1-r)^2, 3] \subseteq g(C_\alpha^3),$$

where $a = 2r^4 - 4r^3 + 3r^2 - 2r + 1$ and $b = r^4 - 2r^3 + 5r^2 - 2r + 1$.

Proof. Note that

$$\begin{aligned}&g\left([0, r] \times [1-r, 1] \times [1-r, 1]\right) \cup g\left([1-r, 1] \times [1-r, 1] \times [1-r, 1]\right) \\ &= \left[2(1-r)^2, 2 + r^2\right] \cup \left[3(1-r)^2, 3\right] = [2(1-r)^2, 3]\end{aligned}$$

and

$$g\left(\left[r-r^2,r\right]\times\left[r-r^2,r\right]\times\left[1-r,1-r+r^2\right]\right)=[a,b].$$

We claim that the intervals

$$g\left([0,r]\times[1-r,1]\times[1-r,1]\right),g\left([1-r,1]\times[1-r,1]\times[1-r,1]\right),$$

and

$$g\left(\left[r-r^2,r\right]\times\left[r-r^2,r\right]\times\left[1-r,1-r+r^2\right]\right)$$

are all included in $g(C_\alpha^3)$. Note that the intervals $[0,r],[1-r,1]$ are 1-level basic intervals and $[r-r^2,r],[1-r,1-r+r^2]$ are 2-level basic intervals. By Lemma 2.4 these just are done by checking condition (2.8) respectively for $n=1$ and $n=2$. In fact, we have

$$\begin{aligned}2(1-r)\cdot(1-r)+(1-2r)r-2(1-r)&=-r<0,\\2(1-r)\cdot(1-r)+(1-2r)r-3(1-r)&=-1<0,\end{aligned}$$

and

$$\begin{aligned}&2(1-r)\cdot(1-r)+(1-2r)r^2-\left(2(r-r^2)+(1-r)\right)\\=&-2r^3+5r^2-5r+1=-r(2r-1)(r-2)-(3r-1)<0.\quad\square\end{aligned}$$

3. The proof of Theorem 1.1

For $E\subseteq\mathbb{R}$ and $t\in\mathbb{R}$, we define $t\cdot E=\{tx:x\in E\}$.

Lemma 3.1. *If $E\subseteq f(C_\alpha^4)$, then $r^2\cdot E\subseteq f(C_\alpha^4)$. Similarly, if $E\subseteq g(C_\alpha^3)$, then $r^2\cdot E\subseteq g(C_\alpha^3)$.*

Proof. Assume that $E\subseteq f(C_\alpha^4)$. For $x\in E$, there are $x_1,x_2,x_3,x_4\in C_\alpha$ such that $x=x_1^2+x_2^2+x_3^2+x_4^2$. Then $r^2x=(rx_1)^2+(rx_2)^2+(rx_3)^2+(rx_4)^2\in f(C_\alpha^4)$. It follows that $r^2\cdot E\subseteq f(C_\alpha^4)$.

Similarly, the result for $g(C_\alpha^3)$ can be proved. $\square$

Lemma 3.2. *$f(C_\alpha^4)=[0,4]$ if and only if $(4r^2,4]\subseteq f(C_\alpha^4)$.*

Proof. Note that

$$0\in f(C_\alpha^4)\text{ and }(0,4]=\bigcup_{n=0}^{\infty}r^{2n}\cdot\left(4r^2,4\right].$$

The sufficiency follows from Lemma 3.1. $\square$

Now we are ready to prove Theorem 1.1.

The proof of Theorem 1.1. For $1 < \alpha < 3$, we have $0 < r < \frac{1}{3}$, which implies $4r^2 < (1-r)^2$. Note that C_α is contained in $[0,r] \cup [1-r,r]$. Assume that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $x_j \in C_\alpha$. If all x_j are contained in the interval $[0,r]$, then $x \le 4r^2$; otherwise $x \ge (1-r)^2$. Thus,

$$(4r^2, (1-r)^2) \cap f(C_\alpha^4) = \emptyset.$$

Therefore, it suffices to show $f(C_\alpha^4) = [0,4]$ when $\alpha \ge 3$.

Assume that $\alpha \ge 3$. Note that $\frac{1}{3} \le r < \frac{1}{2}$. Then we have $(1-r)^2 \le 4r^2$. By Lemma 3.2, it suffices to prove that

$$((1-r)^2, 4] \subseteq f(C_\alpha^4). \tag{3.1}$$

In Corollary 2.5, we have $[2(1-r)^2, 3] \subseteq g(C_\alpha^3)$. Thus

$$f(C_\alpha^4) \supseteq f(C_\alpha^3 \times \{0,1\}) = (g(C_\alpha^3) + 0^2) \cup (g(C_\alpha^3) + 1^2) \supseteq \left[2(1-r)^2, 4\right] \tag{3.2}$$

Applying Corollary 2.5 and Lemma 3.1, we have

$$g(C_\alpha^3) \supseteq [ar^{2n}, br^{2n}] \cup [2(1-r)^2 r^{2n}, 3 \cdot r^{2n}] \ \text{ for } \ n = 0, 1, 2, \cdots, \tag{3.3}$$

where a, b are given in Corollary 2.5. For each positive integer n, since $1-r, 1-r+r^{2n} \in C_\alpha$, it follows that

$$\begin{aligned} f(C_\alpha^4) &\supseteq f\left(C_\alpha^3 \times \left\{1-r, 1-r+r^{2n}\right\}\right) \\ &= \left(g(C_\alpha^3) + (1-r)^2\right) \cup \left(g(C_\alpha^3) + (1-r+r^{2n})^2\right). \end{aligned}$$

Using (3.3), we have that

$$\begin{aligned} f(C_\alpha^4) &\supseteq [ar^{2n-2} + (1-r)^2, br^{2n-2} + (1-r)^2] \\ &\quad \cup [ar^{2n-2} + (1-r+r^{2n})^2, br^{2n-2} + (1-r+r^{2n})^2] \\ &= [ar^{2n-2} + (1-r)^2, br^{2n-2} + (1-r+r^{2n})^2] \\ &\supseteq [ar^{2n-2} + (1-r)^2, (b + 2r^2 - 2r^3) r^{2n-2} + (1-r)^2] \end{aligned} \tag{3.4}$$

where the last equality and the last inclusion hold because

$$\begin{aligned} & br^{2n-2} + (1-r)^2 - (ar^{2n-2} + (1-r+r^{2n})^2) \\ =& (b-a) r^{2n-2} - 2(1-r) r^{2n} - r^{4n} \\ =& (4r - r^2 - r^{2n}) r^{2n} \ge (4r - 2r^2) r^{2n} > 0, \end{aligned}$$

and

$$br^{2n-2} + (1-r+r^{2n})^2 - ((b+2r^2-2r^3)r^{2n-2} + (1-r)^2) = r^{4n} > 0.$$

For each positive integer n, by virtue of the fact

$$1-r, 1-r+r^{2n}, 1-r+r^{2n-1}-r^{2n} \in C_\alpha,$$

it follows that

$$\begin{aligned} f(C_\alpha^4) &\supseteq f\left(C_\alpha^3 \times \left\{1-r, 1-r+r^{2n}, 1-r+r^{2n-1}-r^{2n}\right\}\right) \\ &= \left(g(C_\alpha^3) + (1-r)^2\right) \cup \left(g(C_\alpha^3) + (1-r+r^{2n})^2\right) \\ &\quad \cup \left(g(C_\alpha^3) + (1-r+r^{2n-1}-r^{2n})^2\right). \end{aligned}$$

In terms of (3.3), we have that

$$\begin{aligned} f(C_\alpha^4) \supseteq\, & [2(1-r)^2 r^{2n} + (1-r)^2, 3r^{2n} + (1-r)^2] \\ & \cup [2(1-r)^2 r^{2n} + (1-r+r^{2n})^2, 3r^{2n} + (1-r+r^{2n})^2] \\ & \cup [2(1-r)^2 r^{2n} + (1-r+r^{2n-1}-r^{2n})^2, \\ & \qquad 3r^{2n} + (1-r+r^{2n-1}-r^{2n})^2] \\ =\, & [2(1-r)^2 r^{2n} + (1-r)^2, 3r^{2n} + (1-r+r^{2n-1}-r^{2n})^2] \\ \supseteq\, & [2(1-r)^2 r^{2n} + (1-r)^2, (2-r+2r^2)r^{2n-1} + (1-r)^2] \end{aligned} \tag{3.5}$$

where the last equality and inclusion hold because

$$\begin{aligned} & 3r^{2n} + (1-r)^2 - (2(1-r)^2 r^{2n} + (1-r+r^{2n})^2) \\ =& (6r - 2r^2 - 1 - r^{2n})r^{2n} \\ \geq& (6r - 3r^2 - 1)r^{2n} = [3(1-r)r + 3r - 1]r^{2n} > 0, \end{aligned}$$

$$\begin{aligned} & 3r^{2n} + (1-r+r^{2n})^2 - (2(1-r)^2 r^{2n} + (1-r+r^{2n-1}-r^{2n})^2) \\ =& -2r^{2n-1} + 7r^{2n} - 2r^{2n+2} - r^{4n-2} + 2r^{4n-1} \\ =& 2(3r-1)r^{2n-1} + (1-2r^2)r^{2n} - r^{4n-2} + 2r^{4n-1} \\ >& 2(3r-1)r^{2n-1} + r^{2n+1} - r^{4n-2} + 2r^{4n-1} \\ \geq& 2(3r-1)r^{2n-1} - r^{4n-2} + 3r^{4n-1} \\ =& 2(3r-1)r^{2n-1} + (3r-1)r^{4n-2} \geq 0, \end{aligned}$$

and

$$3r^{2n} + (1-r+r^{2n-1}-r^{2n})^2 - ((2-r+2r^2)r^{2n-1} + (1-r)^2) = (r^{2n-1} - r^{2n})^2 > 0.$$

Note that

$$a-(2-r+2r^2)r=2r^4-6r^3+4r^2-4r+1$$
$$=(2r-1)r^3-(5r^2-4r+1)r-(3r-1)<0,$$

which implies that the intervals in (3.4) and (3.5) overlap. It follows that for each positive integer n,

$$f(C_\alpha^4)\supseteq[2(1-r)^2r^{2n}+(1-r)^2,(b+2r^2-2r^3)r^{2n-2}+(1-r)^2]$$
$$\supseteq[2(1-r)^2r^{2n}+(1-r)^2,2(1-r)^2r^{2n-2}+(1-r)^2],$$

where the last inclusion holds because

$$(b+2r^2-2r^3)-2(1-r)^2=r^4-4r^3+5r^2+2r-1$$
$$=r^4+2r^2(1-2r)+(3r-1)(r+1)>0.$$

Therefore,

$$f(C_\alpha^4)\supseteq\bigcup_{n=1}^{\infty}[(1-r)^2(1+2r^{2n}),(1-r)^2(1+2r^{2n-2})]$$
$$=((1-r)^2,3(1-r)^2]. \tag{3.6}$$

By (3.2) and (3.6), we have

$$f(C_\alpha^4)\supseteq((1-r)^2,3(1-r)^2]\cup[2(1-r)^2,4]=((1-r)^2,4],$$

obtaining (3.1). □

CRediT authorship contribution statement

Zhiqiang Wang: Original draft preparation, find a solution. Kan Jiang: Find a solution. Wenxia Li: Supervision. Bing Zhao: Software.

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References

[1] Jayadev S. Athreya, Bruce Reznick, Jeremy T. Tyson, Cantor set arithmetic, Am. Math. Mon. 126 (1) (2019) 4–17.

[2] Carlos A. Cabrelli, Kathryn E. Hare, Ursula M. Molter, Sums of Cantor sets, Ergod. Theory Dyn. Syst. 17 (6) (1997) 1299–1313.

[3] Jiangwen Gu, Kan Jiang, Lifeng Xi, Bing Zhao, Multiplication on uniform λ-Cantor sets, arXiv:1910.08303, 2019.

[4] Kan Jiang, Lifeng Xi, Interiors of continuous images of the middle-third Cantor set, arXiv:1809.01880, 2018.

[5] Pedro Mendes, Fernando Oliveira, On the topological structure of the arithmetic sum of two Cantor sets, Nonlinearity 7 (2) (1994) 329–343.

[6] Marta Pawłowicz, Linear combinations of the classic Cantor set, Tatra Mt. Math. Publ. 56 (2013) 47–60.

[7] J.F. Randolph, Distances between points of the Cantor set, Am. Math. Mon. 47 (1940) 549–551.

[8] W.R. Utz, The distance set for the Cantor discontinuum, Am. Math. Mon. 58 (1951) 407–408.