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Hausdorff dimension of subsets of Moran fractals with prescribed group frequency of their codings

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Abstract
A well known result states that the set of numbers in base r in which the digits i occur with relative frequency p_i for $i = 0, \ldots, r-1$ is a set of Hausdorff dimension $-(1/\log r)\sum_{i=0}^{r-1} p_i \log p_i$. For instance, decimal numbers in which only the digits 1 and 6 occur, both with relative frequencies $\frac{1}{2}$, have Hausdorff dimension $\log 2/\log 10$. In this paper we generalize this result to the situation where one prescribes the relative frequencies of *groups* of digits in the expansion. For example, suppose we require that in the decimal expansion digits from $\{0, 1, 2\}$ occur with relative frequency $\frac{1}{2}$, and also that digits from $\{3, 4, \ldots, 9\}$ occur with this relative frequency. Our result shows that the Hausdorff dimension of this set is $(\log 2 + \frac{1}{2}\log 3 + \frac{1}{2}\log 7)/\log 10$. Actually, we take a much more general geometric viewpoint, considering subsets of Moran fractals specified by prescribing the relative frequencies of groups of symbols in their codings. We determine the Hausdorff dimension of such sets, and moreover give necessary and sufficient conditions for such a set to have positive Hausdorff measure in its dimension.

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1. Introduction

One of the first non-compact fractals sets to be studied is that of the numbers in the unit interval having an anomalous distribution of their digits in a fixed base r. According to Borel's theorem, almost all numbers will have their digits distributed according to the probability vector $(1/r, 1/r, \ldots, 1/r)$. So the set $M(p_0, \ldots, p_{r-1})$ of numbers for which the digits

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$0, 1, \ldots, r-1$ are distributed according to the probability vector $(p_0, \ldots, p_{r-1})$—not equal to uniform distribution—has measure zero, and it is interesting to obtain an idea of its size by the value of its Hausdorff dimension. Eggleston [2] proved that

$$\dim_H M(p_0, \ldots, p_{r-1}) = -\frac{1}{\log r} \sum_{i=0}^{r-1} p_i \log p_i.$$

In this paper we will generalize Eggleston's result by weakening the requirements on the relative frequency of the digits: partition the set of digits in k groups $\Gamma_1, \ldots, \Gamma_k$, i.e. the Γ_j are disjoint sets with union $\{0, \ldots, r-1\}$. Now prescribe the (overall) relative frequency c_j with which the digits from the jth group occur, and denote this set by $M(\Gamma_1, \ldots, \Gamma_k)$. As a special case of our result it will appear that

$$\dim_H M(\Gamma_1, \ldots, \Gamma_k) = \frac{1}{\log r} \sum_{j=1}^{k} c_j (\log \#\Gamma_j - \log c_j).$$

Here, we recall that the Hausdorff dimension of a set E in d-dimensional Euclidean space is obtained by considering the s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ of E defined by

$$\mathcal{H}^s(E) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(E),$$

where $\mathcal{H}^s_\delta(E)$ is obtained by covering E with sets D_i of diameter at most δ:

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} |D_i|^s : |D_i| \leqslant \delta \text{ and } \bigcup_{i=1}^{\infty} D_i \supseteq E \right\}.$$

The Hausdorff dimension of E is now the unique number in $[0, d]$, denoted by $\dim_H E$, such that

$$\mathcal{H}^s(E) = 0 \quad \text{if } s > \dim_H E \qquad \text{and} \qquad \mathcal{H}^s(E) = +\infty \quad \text{if } s < \dim_H E.$$

Our goal is to generalize Eggleston's result even further. The sets $M(p_0, \ldots, p_{r-1})$ can be considered as subsets of a one-dimensional self-similar set with constant scaling factor $1/r$ (the set [0, 1] (!)). We will consider so-called Moran fractals in d-dimensional space which admit different scalings $a_1, \ldots, a_r$. Various properties of these sets have been studied, e.g. in [1, 2, 4, 7, 8, 10, 12–14, 17]. Typically, the tool to describe and analyse Moran fractals is their coding in sequence space: each point in the set is coded via a finite to one coding map. For the examples above in the unit interval, e.g. the coding is just the expansion of a point in digits in base r. One can, therefore, specify subsets E of the Moran fractal by prescribing relative (group-)frequencies of their codings. Our main result (theorem 1) gives the Hausdorff dimension of the sets thus obtained. Our second result concerns the Hausdorff measure $\mathcal{H}^t(E)$ of E, when $t = \dim_H E$. This is called the Hausdorff measure of E in its dimension, and can be 0, finite and positive, or infinite. The value of $\mathcal{H}^t(E)$ is a well known way to assess the regularity of E. We shall give necessary and sufficient conditions for $\mathcal{H}^t(E)$ to be positive and finite (theorem 2). This result permits us (at the end of the paper), to solve a problem posed by Cawley and Mauldin in [1].

2. Notations and main results

A Moran fractal can be constructed as follows. Denote $\Omega = \{1, 2, \ldots, r\}$, where $r \geqslant 2$. The following notations will be used in this paper.

(i) $\Omega^* = \bigcup_{m=1}^{\infty} \Omega^m$ with $\Omega^m = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m)) : \sigma(j) \in \Omega\}$ for $m \in \boldsymbol{N}$; while $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \ldots) : \sigma(j) \in \Omega\}$.

(ii) $|\sigma|$ is used to denote the length of word $\sigma \in \Omega^*$. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \ldots, \sigma(|\sigma|), \tau(1), \ldots, \tau(|\tau|))$; and, for any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$ write $\tau * \sigma = (\tau(1), \ldots, \tau(|\tau|), \sigma(1), \sigma(2), \ldots)$.

(iii) $\sigma|m = (\sigma(1), \sigma(2), \ldots, \sigma(m))$ for $\sigma \in \Omega^\omega$ and $m \in \boldsymbol{N}$.

(iv) For $\sigma \in \Omega^m$, the cylinder set $C(\sigma)$ is defined as $C(\sigma) = \{\tau \in \Omega^\omega : \tau|m = \sigma\}$.

(v) When $f_i(x) : \boldsymbol{R}^d \to \boldsymbol{R}^d$, are maps for $1 \leqslant i \leqslant r$, we denote $f_\sigma(x) = f_{\sigma(1)} \circ \cdots \circ f_{\sigma(m)}(x)$ for $\sigma \in \Omega^m$ and $x \in \boldsymbol{R}^d$.

Fixing a nonempty compact set $J \subset \boldsymbol{R}^d$ with $\overline{\operatorname{int} J} = J$, a constant $0 < c < 1$ and positive real numbers $0 < a_i < 1$, $i = 1, 2, \ldots, r$, the related Moran fractal (or Moran set) is defined according to the following two steps.

Step 1. For each $\sigma \in \Omega^m$, $m \in N$, construct a compact set J_σ in $\boldsymbol{R}^d$ by induction:

- A family $\{J_j : j = 1, 2, \ldots, r\}$ of non-overlapping nonempty compact subsets of J is chosen for $m = 1$ such that $\overline{\operatorname{int} J_j} = J_j$, $|J_j| = a_j|J|$, where $|\cdot|$ denotes the diameter of a set, and J_j contains an open ball of diameter $c|J_j|$.
- Suppose that J_σ is given for some $\sigma \in \Omega^m$. Choose a family $\{J_{\sigma*i} : i = 1, 2, \ldots, r\}$ of non-overlapping nonempty compact subsets of J_σ such that $\overline{\operatorname{int} J_{\sigma*i}} = J_{\sigma*i}$, $|J_{\sigma*i}| = a_i|J_\sigma|$ and $J_{\sigma*i}$ contains an open ball of diameter $c|J_{\sigma*i}|$.

Step 2. The Moran fractal F associated with $\{0 < a_i < 1 : i = 1, 2, \ldots, r\}$ and $(J_\sigma)_{\sigma\in\Omega^*}$ is defined as the nonempty compact set

$$F = \bigcap_{m=1}^{\infty} \bigcup_{\sigma\in\Omega^m} J_\sigma. \tag{1}$$

The compact sets J_σ, $\sigma \in \Omega^*$ are generally referred to as component sets of F. In particular, J_σ is referred to as an mth level component set of F if $\sigma \in \Omega^m$. Define $\phi : \Omega^\omega \to \boldsymbol{R}^d$ by

$$\{\phi(\sigma)\} = \bigcap_{m=1}^{\infty} J_{\sigma|m}. \tag{2}$$

It is easy to see that $\phi(\Omega^\omega) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (1). But ϕ may not be injective. Let ρ be the metric on Ω^ω such that for any $\sigma, \tau \in \Omega^\omega$

$$\rho(\sigma, \tau) = 2^{-\min\{i:\sigma(i)\neq\tau(i)\}},$$

with the convention $\rho(\sigma, \sigma) = 0$. Let F be equipped with the Euclidean metric. Then, ϕ is continuous. Thus each $x \in F$ can be encoded via ϕ: a sequence $\sigma \in \Omega^\omega$ is called a location code of $x \in F$ if $\phi(\sigma) = x$. Therefore, ϕ is also called the coding map and Ω^ω is called the code space (or symbolic space). As a result, F is a projection of Ω^ω on $\boldsymbol{R}^d$ via ϕ.

Some comments about Moran fractals are listed below.

(C1) Moran fractals are regular fractals in the sense that it has been proven that $0 < \mathcal{H}^s(F) < \infty$, with $s = \dim_H F$, given by $\sum_{i=1}^r {a_i}^s = 1$ (see [9, 11, 13, 14]).

(C2) A Moran fractal is termed as map-specified if there exist similitude contractions f_i, $i = 1, 2, \ldots, r$, such that $J_\sigma = f_\sigma(J)$ for any $\sigma \in \Omega^*$. In this case F is actually the self-similar set determined by f_i, $1 \leqslant i \leqslant r$, which satisfy the open set condition with respect to the open set $O = \operatorname{int} J$ (i.e. $\bigcup_{i=1}^r f_i(O) \subseteq O$ with a disjoint union on the left) and the coding map ϕ in (2) can be changed into

$$\{\phi(\sigma)\} = \bigcap_{m=1}^{\infty} f_{\sigma|m}(\bar{O}) = \{\lim_{m\to\infty} f_{\sigma|m}(0)\}.$$

(C3) Let $0 < R < |J| \cdot \min_{1 \leqslant i \leqslant r} a_i$. A component set J_σ of F is termed as an R-size component set if

$$|J_\sigma| \leqslant R \qquad \text{and} \qquad |J_{\sigma|(|\sigma|-1)}| > R.$$

It is easy to see that for any $0 < R < |J| \cdot \min_{1 \leqslant i \leqslant r} a_i$, the set of all R-size component sets of F is a non-overlapping finite R-covering of F. Hence, by means of lemma 9.2 in [3], the requirement that J_σ contains an open ball of diameter $c|J_\sigma|$ implies an important fact: there exists a positive integer ϑ, independent of R and $x \in \boldsymbol{R}^d$, such that any ball $B_R(x)$ with radius R and centre at x intersects at most ϑ of the R-size component sets of F. Many analogues of this fact appear in this paper. A direct sequel leads to an important property of ϕ:

$$\sup_{x \in F} \#\{\phi^{-1}(x)\} < \vartheta. \tag{3}$$

Otherwise, suppose that for some $x \in F$ we have $\#\{\phi^{-1}(x)\} > \vartheta$. Take $\vartheta + 1$ different elements $\sigma_1, \sigma_2, \ldots, \sigma_{\vartheta+1}$ from this set. Let $m \in \boldsymbol{N}$ be such that $\sigma_1|m, \sigma_2|m, \ldots, \sigma_{\vartheta+1}|m$ differ from each other. Taking $R = \min_{1 \leqslant i \leqslant \vartheta+1} |J_{\sigma_i|m}|$, x lies in at least $\vartheta + 1$ R-size component sets of F, which implies that the ball $B_R(x)$ intersects at least $\vartheta + 1$ of the R-size component sets of F.

(C4) A more general Moran fractal structure was proposed by Wen (see [17] for details), where the code space $\Omega^\omega = \prod_{n=1}^\infty \Omega_n$ and corresponding to different Ω_n there are different scaling coefficients $\{0 < a_{n,j} < 1,\ j = 1, 2, \ldots, r_n\}$. Some dimension results of these generalized Moran sets can be found in [5–8, 11]. The class of generalized Moran sets clearly contains the class of Moran sets discussed here, and in fact is far larger, since a generalized Moran set often has different fractal dimensions, and has zero or infinite Hausdorff measure in its dimension.

For any $E \subseteq F$ there exists $\Lambda \subset \Omega^\omega$ such that $E = \phi(\Lambda)$. For certain Λ, it should be possible to determine the dimensions of the projections $E = \phi(\Lambda)$. Some solutions, which depend on the structure of Λ, can be found in [1, 2, 4, 7, 8, 10, 12–14], etc.

Let $2 \leqslant k \leqslant r$. Fix real numbers c_j, $j = 1, 2, \ldots, k$, such that $c_j \geqslant 0$ and $\sum_{j=1}^k c_j = 1$. Let Γ_j, $j = 1, \ldots, k$, be disjoint nonempty subsets of Ω with $\bigcup_{j=1}^k \Gamma_j = \Omega$. In this paper we will consider sets Λ specified by relative frequencies:

$$\Lambda = \hat{M}(\Gamma_1, \ldots, \Gamma_k) = \left\{ \sigma \in \Omega^\omega : \lim_{n \to \infty} \frac{\#\{1 \leqslant i \leqslant n : \sigma(i) \in \Gamma_j\}}{n} = c_j, 1 \leqslant j \leqslant k \right\}.$$

Let

$$M(\Gamma_1, \ldots, \Gamma_k) = \phi(\Lambda) = \phi(\hat{M}(\Gamma_1, \ldots, \Gamma_k)),$$

i.e. the subset of the Moran fractal F whose elements have their codings lying in Γ_j with a prescribed relative frequency c_j. We remark that $M(\Gamma_1, \ldots, \Gamma_k)$ is dense in F since $\sigma \in \hat{M}(\Gamma_1, \ldots, \Gamma_k)$ if and only if $i * \sigma \in \hat{M}(\Gamma_1, \ldots, \Gamma_k)$, $i \in \Omega$; and that $M(\Gamma_1, \ldots, \Gamma_k) = \bigcup_{i=1}^r f_i(M(\Gamma_1, \ldots, \Gamma_k))$ for the map-specified case (cf (C2)). Let

$$Z(t) = \sum_{j=1}^k c_j \log \sum_{i \in \Gamma_j} a_i^t - \sum_{j=1}^k c_j \log c_j,$$

where, here and throughout the whole paper, we adopt the convention $0 \cdot \log 0 = 0$. It is easy to see that the function $Z(t)$ has a unique zero in $[0, s]$ where s is defined by $\sum_{i=1}^r a_i^s = 1$, since $Z(t)$ is strictly decreasing with $Z(0) \geqslant 0$ and $Z(s) \leqslant 0$. In this paper, we obtain the following results.

Theorem 1. $\dim_H M(\Gamma_1,\ldots,\Gamma_k) = t$ *where* t *is the unique non-negative real number such that*

$$\sum_{j=1}^{k} c_j \log c_j = \sum_{j=1}^{k} c_j \log \sum_{i\in\Gamma_j} a_i^t. \tag{4}$$

Theorem 2. *Suppose* $\dim_H M(\Gamma_1,\ldots,\Gamma_k) = t$. *Then, the following four statements are equivalent:*

(i) $0 < \mathcal{H}^t(M(\Gamma_1,\ldots,\Gamma_k)) < \infty$;
(ii) $\sum_{i=1}^{r} a_i^t = 1$, *i.e.* $t = \dim_H F$ (cf (C1));
(iii) $c_j = \sum_{i\in\Gamma_j} a_i^t$, $1 \leqslant j \leqslant k$;
(iv) $\dim_H M(\Gamma_1,\ldots,\Gamma_k) = \dim_H(F \setminus M(\Gamma_1,\ldots,\Gamma_k))$.

As an application of these results we rewrite, for convenience, $\Omega = \{0, 1, \ldots, r-1\}$. Take $J = [0, 1]$ and for each $\sigma \in \Omega^\omega$ let $J_\sigma = f_\sigma(J)$ (cf (C2)), where $f_i : [0,1] \to [0,1]$ is defined by

$$f_i(x) = \frac{1}{r}(x+i), \qquad i = 0, 1, \ldots, r-1.$$

Then by (1) we have that $F = [0, 1]$ is a map-specified Moran set. In this case, we have $a_i = 1/r$ for all i, and $\phi : \Omega^\omega = \{0, 1, \ldots, r-1\}^N \to [0, 1]$ is given by

$$\phi(\sigma) = \lim_{m\to\infty} f_{\sigma|m}(0) = \sum_{m=1}^{\infty} \frac{\sigma(m)}{r^m}.$$

Hence the r-ary expansion of $\phi(\sigma)$ is $\phi(\sigma) = (0.\sigma(1)\sigma(2)\ldots)_r$. Thus, $M(\Gamma_1,\ldots,\Gamma_k)$ consists of those $x \in F = [0, 1]$ for which the occurrence of digits of Γ_j in its base-r expansion has fixed relative frequency c_j. Theorems 1 and 2 yield that

$$\dim_H M(\Gamma_1,\ldots,\Gamma_k) = \frac{\sum_{j=1}^{k} c_j(\log \#\Gamma_j - \log c_j)}{\log r},$$

and $M(\Gamma_1,\ldots,\Gamma_k)$ has positive Hausdorff measure in its dimension if and only if $c_j = \#\Gamma_j/r$, implying $\dim_H M(\Gamma_1,\ldots,\Gamma_k) = 1$. In particular, taking $\Gamma_i = \{i-1\}$ for $1 \leqslant i \leqslant r$, $M(\Gamma_1,\ldots,\Gamma_r)$ just consists of those real numbers in $[0, 1]$ in whose r-ary expansion the digit i has density c_i. So we recover Eggleston's result ([12]), mentioned in the introduction.

3. Hausdorff dimension and measure property

In this section we will first determine the Hausdorff dimension of $M(\Gamma_1,\ldots,\Gamma_k)$. The usual way is to find an appropriate probability measure μ supported on the set in order to obtain a lower bound $\dim_H \mu$ for its Hausdorff dimension. The measure μ can be constructed as the image measure under ϕ of a probability measure $\hat{\mu}$ supported on $\hat{M}(\Gamma_1,\ldots,\Gamma_k)$. For $\hat{\mu}$ we take the infinite product probability measure on Ω^ω corresponding to probability vectors $(p_1, p_2, \ldots, p_r)$ satisfying $\sum_{i\in\Gamma_j} p_i = c_j$ for $j = 1,\ldots,k$. The key is to choose concrete p_is such that $\dim_H \mu$ reaches a maximum. The estimation of the upper bound of its Hausdorff dimension will be done by choosing an efficient sequence of coverings of the set.

Proof of theorem 1. We have to pay attention to the fact that some c_js may be zero. Without loss of generality we assume that $c_j > 0$ for $1 \leqslant j \leqslant k_1$ and $c_j = 0$ for $j > k_1$, where $k_1 \leqslant k$. Denote $\Omega_1 = \bigcup_{j=1}^{k_1} \Gamma_j$.

(A) $\dim_H M(\Gamma_1,\ldots,\Gamma_k) \geqslant t$. This bound is trivial when $t = 0$. Suppose $t > 0$. Let

$$b_i = \frac{a_i^t c_j}{\sum_{l\in\Gamma_j} a_l^t}, \qquad i \in \Gamma_j. \tag{5}$$

Thus, we have $\sum_{i\in\Omega_1} b_i = 1$ and $b_i = 0$ for $i \in \Omega\backslash\Omega_1$. Construct a probability measure $\hat{\mu}$ on Ω^ω by defining for any $\sigma \in \Omega^m$

$$\hat{\mu}(C(\sigma)) = \prod_{i=1}^{m} b_{\sigma(i)},$$

where $C(\sigma) = \{\tau \in \Omega^\omega : \tau|m = \sigma\}$ is the cylinder set determined by σ. Let μ on F be the image measure of $\hat{\mu}$ under ϕ. Let

$$\hat{M} = \left\{\sigma \in \Omega^\omega : \sigma \in \Omega_1^\omega \text{ and } \lim_{n\to\infty} \frac{\#\{1 \leqslant l \leqslant n : \sigma(l) = i\}}{n} = b_i, 1 \leqslant i \leqslant r\right\}. \tag{6}$$

Then $\hat{M} \subseteq \hat{M}(\Gamma_1, \ldots, \Gamma_k)$ and $M = \phi(\hat{M}) \subseteq M(\Gamma_1, \ldots, \Gamma_k)$. By Birkhoff's Ergodic theorem ([16]) we have for $\hat{\mu}$-a.e. σ

$$\lim_{n\to\infty} \frac{\#\{1 \leqslant l \leqslant n : \sigma(l) = i\}}{n} = b_i, \qquad 1 \leqslant i \leqslant r.$$

Therefore, $\hat{\mu}(\hat{M}) = \mu(M) = 1$. Now for $\sigma \in \Omega^*$ write $a(\sigma) = \prod_{i=1}^{|\sigma|} a_{\sigma(i)}$ and $b(\sigma) = \prod_{i=1}^{|\sigma|} b_{\sigma(i)}$. Then, for any $\sigma \in \hat{M}$ it is easy to verify that

$$\lim_{l\to\infty} \frac{\log b(\sigma|l)}{\log a(\sigma|l)} = \lim_{l\to\infty} \frac{\log \prod_{i=1}^{l} b_{\sigma(i)}}{\log \prod_{i=1}^{l} a_{\sigma(i)}} = \frac{\sum_{i\in\Omega_1} b_i \log b_i}{\sum_{i\in\Omega_1} b_i \log a_i}, \tag{7}$$

using ([16]). Note that

$$\begin{aligned}
\sum_{i\in\Omega_1} b_i \log b_i &= \sum_{j=1}^{k_1} \sum_{i\in\Gamma_j} b_i \log b_i = \sum_{j=1}^{k_1} \sum_{i\in\Gamma_j} b_i \left(t \log a_i + \log c_j - \log \sum_{l\in\Gamma_j} a_l^t\right) \\
&= t \sum_{j=1}^{k_1} \sum_{i\in\Gamma_j} b_i \log a_i + \sum_{j=1}^{k_1} \left[\left(\sum_{i\in\Gamma_j} b_i\right)\left(\log c_j - \log \sum_{l\in\Gamma_j} a_l^t\right)\right] \\
&= t \sum_{j=1}^{k_1} \sum_{i\in\Gamma_j} b_i \log a_i + \sum_{j=1}^{k_1} c_j \left(\log c_j - \log \sum_{l\in\Gamma_j} a_l^t\right) = t \sum_{i\in\Omega_1} b_i \log a_i,
\end{aligned} \tag{8}$$

by (5) and (4). From (7) and (8) it follows that for any $\sigma \in \hat{M}$

$$\lim_{l\to\infty} \frac{\log b(\sigma|l)}{\log a(\sigma|l)} = t.$$

Fix $0 < \epsilon < t$. Let $M^{(m)} = \phi(\hat{M}^{(m)})$ where

$$\hat{M}^{(m)} = \left\{\sigma \in \hat{M} : \frac{\log b(\sigma|l)}{\log a(\sigma|l)} > t - \epsilon \text{ for all } l \geqslant m\right\}. \tag{9}$$

Then,

$$1 = \hat{\mu}(\hat{M}) = \lim_{m\to\infty} \hat{\mu}(\hat{M}^{(m)}), \qquad 1 = \mu(M) = \lim_{m\to\infty} \mu(M^{(m)}).$$

Now fix m such that $\hat{\mu}(\hat{M}^{(m)}) > 0$. Let $\hat{\mu}_m$ be the restriction of $\hat{\mu}$ to $\hat{M}^{(m)}$ and let μ_m be the induced measure on $M^{(m)}$ of $\hat{\mu}_m$ by ϕ, i.e. for any Borel set $A \subseteq M^{(m)}$

$$\mu_m(A) = \hat{\mu}_m(\phi^{-1}(A)) = \hat{\mu}(\phi^{-1}(A) \cap \hat{M}^{(m)}).$$

By $B_R(x)$ we denote the closed ball with centre at x and radius R. Let $0 < R < (\min_{1\leqslant i\leqslant r} a_i)^m$. For each $\sigma \in \hat{M}^{(m)}$ there exists a positive integer $h(\sigma, R)$ such that

$$R \cdot \min_{1\leqslant i\leqslant r} a_i \leqslant a(\sigma|h(\sigma, R)) \leqslant R. \tag{10}$$

Note that $h(\sigma, R) > m$ and write $W = \{\sigma | h(\sigma, R) : \sigma \in \hat{M}^{(m)}\}$. For any fixed $x \in M^{(m)}$ let $W^* = \{\tau \in W : J_\tau \bigcap B_R(x) \bigcap M^{(m)} \neq \emptyset\}$. Then, there exists a finite positive constant ξ independent of R and x such that $\#W^* \leqslant \xi$ by lemma 9.2 in [3]. So

$$\mu_m(B_R(x)) \leqslant \hat{\mu}_m \left(\bigcup_{\tau \in W^*} C(\tau) \right) \leqslant \sum_{\tau \in W^*} \hat{\mu}(C(\tau)) = \sum_{\tau \in W^*} b(\tau) \leqslant \xi \cdot R^{t-\epsilon},$$

by (9) and (10). So we get

$$\liminf_{R \to 0} \frac{\log \mu_m(B_R(x))}{\log R} \geqslant t - \epsilon.$$

Theorem 1 in [15] tells us

$$\dim_H M(\Gamma_1, \ldots, \Gamma_k) \geqslant \dim_H M \geqslant \dim_H M^{(m)} \geqslant t - \epsilon,$$

which implies, letting $\epsilon \to 0$, $\dim_H M(\Gamma_1, \ldots, \Gamma_k) \geqslant t$.

(B) $\dim_H M(\Gamma_1, \ldots, \Gamma_k) \leqslant t$. This part of the proof is similar to that in [2]. For $\epsilon > 0$ and $m \in \boldsymbol{N}$, let

$$\Omega(m, \epsilon) = \{\sigma \in \Omega^m : (c_j - \epsilon)m \leqslant \#\{1 \leqslant i \leqslant m : \sigma(i) \in \Gamma_j\} \leqslant (c_j + \epsilon)m,\ 1 \leqslant j \leqslant k\}.$$

Note that for any given $\epsilon > 0$ and any given $\sigma \in \hat{M}(\Gamma_1, \ldots, \Gamma_k)$ there exists an $m_0 \in \boldsymbol{N}$ such that

$$\left| \frac{\#\{1 \leqslant i \leqslant m : \sigma(i) \in \Gamma_j\}}{m} - c_j \right| < \epsilon,$$

for all $m \geqslant m_0$, i.e. $\sigma \in \Omega(m, \epsilon)$ for all $m \geqslant m_0$. Then, for any $\epsilon > 0$, we have

$$\bigcup_{m_0=1}^{\infty} \bigcap_{m=m_0}^{\infty} \bigcup_{\sigma \in \Omega(m,\epsilon)} J_\sigma \supset M(\Gamma_1, \ldots, \Gamma_k).$$

Thus, in order to prove that $\dim_H M(\Gamma_1, \ldots, \Gamma_k) \leqslant t$, it is sufficient to show that, for every integer m_0 and every $\eta > 0$, there is an $\epsilon > 0$ (which is a function of η only) such that $\dim_H(\bigcap_{m=m_0}^{\infty} \bigcup_{\sigma \in \Omega(m,\epsilon)} J_\sigma) \leqslant t + \eta$.

Since $\bigcap_{m=m_0}^{\infty} \bigcup_{\sigma \in \Omega(m,\epsilon)} J_\sigma$ is covered by $\{J_\sigma : \sigma \in \Omega(m, \epsilon)\}$, for $m \geqslant m_0$, it is sufficient to show that given $\eta > 0$, there is an $\epsilon > 0$ such that

$$\sum_{\sigma \in \Omega(m,\epsilon)} |J_\sigma|^{t+\eta} \to 0 \qquad m \to \infty.$$

Let

$$K = 4 \frac{\max_{1 \leqslant j \leqslant k_1} c_j}{\min_{1 \leqslant j \leqslant k_1} c_j} \max \left\{ \frac{\sum_{i \in \Gamma_{j_2}} a_i^{t+\eta}}{\sum_{i \in \Gamma_{j_1}} a_i^{t+\eta}},\ 1 \leqslant j_1, j_2 \leqslant k \right\}. \tag{11}$$

Let ϵ satisfy

$$0 < \epsilon < \frac{1}{r} \min_{1 \leqslant j \leqslant k_1} c_j \tag{12}$$

and

$$\prod_{j=1}^{k_1} \left(\frac{\sum_{i \in \Gamma_j} a_i^{t+\eta}}{\sum_{i \in \Gamma_j} a_i^t} \right)^{c_j} \cdot \prod_{j=k_1+1}^{k} \left(\frac{\sum_{i \in \Gamma_j} a_i^{t+\eta}}{\epsilon} \right)^{\epsilon} \cdot K^{2k\epsilon} < 1. \tag{13}$$

Now, we have

$$\sum_{\sigma \in \Omega(m,\epsilon)} |J_\sigma|^{t+\eta} = |J|^{t+\eta} \sum \frac{m!}{\prod_{j=1}^{k} d_j!} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j}, \tag{14}$$

where the summation is over sets of non-negative integers d_j, $1 \leqslant j \leqslant k$ such that

$$\sum_{j=1}^{k} d_j = m \quad \text{and} \quad m(c_j - \epsilon) \leqslant d_j \leqslant m(c_j + \epsilon), \qquad 1 \leqslant j \leqslant k. \tag{15}$$

Hence, for $i \in \Omega$ and $j \in \Omega_1$, if $m\epsilon > 1$

$$\frac{d_i + 1}{d_j} \leqslant \frac{m(c_i + \epsilon) + 1}{m(c_j - \epsilon)} \leqslant 4 \frac{\max_{1 \leqslant j \leqslant k_1} c_j}{\min_{1 \leqslant j \leqslant k_1} c_j}, \tag{16}$$

by (12) and (15). Let

$$Q_1 = \frac{m!}{\prod_{j=1}^{k} d_j!} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} \tag{17}$$

and Q_2 be the term of the sum (14) for which in (17) the d_{j_1} and d_{j_2}, for some $j_1 \neq j_2$ with $j_1 \in \Omega$ and $j_2 \in \Omega_1$, are replaced by $d_{j_1} + 1$ and $d_{j_2} - 1$, respectively, and the other d_js are kept fixed. Thus, when $m\epsilon > 1$

$$\frac{Q_1}{Q_2} = \frac{d_{j_1} + 1}{d_{j_2}} \cdot \frac{\sum_{i \in \Gamma_{j_2}} a_i^{t+\eta}}{\sum_{i \in \Gamma_{j_1}} a_i^{t+\eta}} \leqslant 4 \frac{\max_{1 \leqslant j \leqslant k_1} c_j}{\min_{1 \leqslant j \leqslant k_1} c_j} \cdot \frac{\sum_{i \in \Gamma_{j_2}} a_i^{t+\eta}}{\sum_{i \in \Gamma_{j_1}} a_i^{t+\eta}} \leqslant K, \tag{18}$$

by (11) and (16). Let Q_0 be the term of (14) for which d_j is the integer part of mc_j for $1 \leqslant j \leqslant k_1 - 1$, the integer part of $m\epsilon$ for $k_1 + 1 \leqslant j \leqslant k$. Therefore, there are real numbers $0 \leqslant \delta_j < 1$ with $j \neq k_1$ such that $d_j = mc_j - \delta_j$ for $1 \leqslant j \leqslant k_1 - 1$, $d_j = m\epsilon - \delta_j$ for $k_1 + 1 \leqslant j \leqslant k$, and $d_{k_1} = m - \sum_{j \neq k_1} d_j = mc_{k_1} - (k - k_1)m\epsilon + \sum_{j \neq k_1} \delta_j = mc_{k_1} - \delta_{k_1}$ with $\delta_{k_1} = (k - k_1)m\epsilon - \sum_{j \neq k_1} \delta_j$. Then, by means of the Stirling formula $n! = \sqrt{2\pi n}(n/e)^n \mathrm{e}^{\theta/12n}$ $(0 < \theta < 1)$ and (4) we have

$$\begin{aligned}
Q_0 &= \frac{m!}{\prod_{j=1}^{k} d_j!} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} = \frac{\sqrt{2\pi} m^{m+(1/2)} \mathrm{e}^{-m} \mathrm{e}^{\theta/12m}}{\prod_{j=1}^{k} \sqrt{2\pi} d_j^{d_j+(1/2)} \mathrm{e}^{-d_j} \mathrm{e}^{\theta_j/12d_j}} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} \\
&\leqslant K_1 \frac{m^{m+(1/2)}}{\prod_{j=1}^{k} d_j^{d_j+(1/2)}} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} \\
&= K_1 \frac{m^{m+(1/2)}}{\prod_{j=1}^{k_1} (mc_j - \delta_j)^{mc_j - \delta_j + (1/2)} \prod_{j=k_1+1}^{k} (m\epsilon - \delta_j)^{m\epsilon - \delta_j + (1/2)}} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} \\
&= K_1 \frac{m^{-(1/2)(k-1)}}{\prod_{j=1}^{k_1} (c_j - \delta_j/m)^{mc_j - \delta_j + (1/2)} \prod_{j=k_1+1}^{k} (\epsilon - \delta_j/m)^{m\epsilon - \delta_j + (1/2)}} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} \\
&\leqslant K_2 \frac{m^{-(1/2)(k-1)}}{\prod_{j=1}^{k_1} c_j^{mc_j} \prod_{j=k_1+1}^{k} \epsilon^{m\epsilon}} \cdot \prod_{j=1}^{k} \left(\sum_{i \in \Gamma_j} a_i^{t+\eta} \right)^{d_j} \\
&\leqslant K_3 \cdot m^{-(1/2)(k-1)} \cdot \prod_{j=1}^{k_1} \left(\frac{\sum_{i \in \Gamma_j} a_i^{t+\eta}}{\sum_{i \in \Gamma_j} a_i^{t}} \right)^{mc_j} \cdot \prod_{j=k_1+1}^{k} \left(\frac{\sum_{i \in \Gamma_j} a_i^{t+\eta}}{\epsilon} \right)^{m\epsilon},
\end{aligned} \tag{19}$$

where K_1, K_2 and K_3 are appropriate positive constants independent of m. Since (18) implies that for any term Q of (14)

$$\frac{Q}{Q_0} < K^{2km\epsilon}, \tag{20}$$

it follows that

$$\sum_{\sigma\in\Omega(m,\epsilon)}|J_\sigma|^{t+\eta} = |J|^{t+\eta}\sum\frac{m!}{\prod_{j=1}^k d_j!}\cdot\prod_{j=1}^k\left(\sum_{i\in\Gamma_j}a_i^{t+\eta}\right)^{d_j} < Q_0\cdot K^{2km\epsilon}\cdot(2m\epsilon)^k\cdot|J|^{t+\eta}$$
$$< K_4\cdot m^{(1/2)(k+1)}\cdot\left[\prod_{j=1}^{k_1}\left(\frac{\sum_{i\in\Gamma_j}a_i^{t+\eta}}{\sum_{i\in\Gamma_j}a_i^{t}}\right)^{c_j}\cdot\prod_{j=k_1+1}^{k}\left(\frac{\sum_{i\in\Gamma_j}a_i^{t+\eta}}{\epsilon}\right)^{\epsilon}\cdot K^{2k\epsilon}\right]^m$$
$$\to 0$$

as $m\to\infty$, by (19), (20) and (13), where K_4 is an appropriate positive constant independent of m. This completes the proof. QED

In theorem 1 we show that $\dim_H M(\Gamma_1,\ldots,\Gamma_k)=t$ by proving that for any positive real numbers ϵ and η $\dim_H M(\Gamma_1,\ldots,\Gamma_k)\geqslant t-\epsilon$ and $\dim_H M(\Gamma_1,\ldots,\Gamma_k)\leqslant t+\eta$. So we do not get any information about the value of $\mathcal{H}^t(M(\Gamma_1,\ldots,\Gamma_k))$. In the following, we will get the sufficient and necessary conditions for $M(\Gamma_1,\ldots,\Gamma_k)$ to have positive Hausdorff measure in its dimension.

Let $Q=\{J_\sigma:\sigma\in\Omega^*\}$ and $Q_\sigma=\{J_{\sigma*\tau}:\tau\in\Omega^*\}$ for $\sigma\in\Omega^*$. For any $0\leqslant\alpha<\infty$ and any subset E of Moran set F define

$$\mathcal{H}_Q^\alpha(E)=\liminf_{\delta\to0}\left\{\sum_{i=1}^\infty|V_i|^\alpha:\{V_i\}\text{ is a non-overlapping }\delta\text{-covering of }E,\ V_i\in Q\right\}. \tag{21}$$

Similarly, for $0\leqslant\alpha<\infty$ and $E\subseteq F\bigcap J_\sigma$ define

$$\mathcal{H}_{Q_\sigma}^\alpha(E)=\liminf_{\delta\to0}\left\{\sum_{i=1}^\infty|V_i|^\alpha:\{V_i\}\text{ is a non-overlapping }\delta\text{-covering of }E,\ V_i\in Q_\sigma\right\}.$$

Lemma 3. (I) *If $E\subseteq F\bigcap J_\sigma$, then $h\mathcal{H}_{Q_\sigma}^\alpha(E)\leqslant\mathcal{H}_Q^\alpha(E)\leqslant\mathcal{H}_{Q_\sigma}^\alpha(E)$, where the positive real number h is independent of σ and E.*

(II) *If $E\subseteq F$, then*

$$4^{-d}(c\min_{1\leqslant i\leqslant r}a_i)^d\mathcal{H}_Q^\alpha(E)\leqslant\mathcal{H}^\alpha(E)\leqslant\mathcal{H}_Q^\alpha(E), \tag{22}$$

where the positive constant c can be found in the definition of the Moran set F. In particular, the covering $\{V_i\}$ of E in (21) *can be taken finite when E is compact.*

Proof. (I) It is clear that $\mathcal{H}_Q^\alpha(E)\leqslant\mathcal{H}_{Q_\sigma}^\alpha(E)$. Let $\delta<a(\sigma)|J|$. For any non-overlapping δ-covering $\{V_i\}$ of E with $V_i\in Q$, write $\mathcal{B}_i=\{V_i\}$ if $V_i\subseteq J_\sigma$ and

$$\mathcal{B}_i=\left\{J_{\sigma*\tau}:\ |J_{\sigma*\tau}|\leqslant|V_i|<|J_{\sigma*\tau|(|\sigma*\tau|-1)}|\text{ and }J_{\sigma*\tau}\bigcap V_i\bigcap E\neq\emptyset\right\},$$

if $V_i\subseteq J_\sigma$ does not hold. Then, there exists a finite positive constant h^{-1} independent of σ such that $\#\mathcal{B}_i\leqslant h^{-1}$ by the lemma 9.2 in [3]. Hence, $\mathcal{B}=\bigcup_i\mathcal{B}_i\subseteq Q_\sigma$ is a δ-covering of E. So we have

$$\sum_{U\in\mathcal{B}}|U|^\alpha\leqslant h^{-1}\sum_i|V_i|^\alpha.$$

Since for any $a,b\in\Omega^*$ either J_a and J_b are non-overlapping or $J_a\subseteq J_b$ (or $J_b\subseteq J_a$), we can assume that $\mathcal{B}$ is a non-overlapping covering (we can delete some elements from $\mathcal{B}$ if necessary). Therefore, we get $\mathcal{H}_Q^\alpha(E)\geqslant h\mathcal{H}_{Q_\sigma}^\alpha(E)$.

(II) The right inequality is clear. We only need to prove the left inequality. Let $\{W_i : 1 \leqslant i < p \leqslant \infty\}$ be an open δ-covering of E (where $p < \infty$ if E is compact). For any $1 \leqslant i < p$ let

$$\mathcal{A}_i = \{J_\sigma : |J_\sigma| \leqslant |W_i| < |J_{\sigma|(|\sigma|-1)}| \text{ and } J_\sigma \cap W_i \cap E \neq \emptyset\}.$$

Hence $\bigcup_{1\leqslant i<p} \mathcal{A}_i$ is a δ-covering of E with each of its elements in Q. Similarly as in (I), we can assume the $\bigcup_{1\leqslant i<p} \mathcal{A}_i$ is a non-overlapping covering of E (we can delete some J_σ from some $\mathcal{A}_i$ if necessary).

Again $\#\mathcal{A}_i$ is bounded by a constant by lemma 9.2 in [3]. But, this time, we would like to give an argument. In fact, let B_i be an open ball with centre in W_i and radius $2|W_i|$ and recall that for each $\sigma \in \Omega^*$, J_σ contains an open ball with diameter $c|J_\sigma|$. Then,

$$\mathcal{H}^d(B_i) = (4|W_i|)^d \geqslant \mathcal{H}^d\left(\bigcup_{J_\sigma\in\mathcal{A}_i} J_\sigma\right) \geqslant \sum_{J_\sigma\in\mathcal{A}_i} (c|J_\sigma|)^d \geqslant \left(c|W_i| \min_{1\leqslant i\leqslant r} a_i\right)^d \cdot \#\mathcal{A}_i,$$

which implies

$$\#\mathcal{A}_i \leqslant 4^d (c \min_{1\leqslant i\leqslant r} a_i)^{-d}. \tag{23}$$

Consequently, for any open δ-covering $\{W_i : 1 \leqslant i < p \leqslant \infty\}$ of E we can get a δ-covering $\bigcup_{1\leqslant i<p} \mathcal{A}_i$ of E, with each of its elements in Q (this covering is finite if $p < \infty$), such that

$$\begin{aligned}\sum_{J_\sigma\in\bigcup_{1\leqslant i<p}\mathcal{A}_i} |J_\sigma|^\alpha &= \sum_{1\leqslant i<p}\sum_{J_\sigma\in\mathcal{A}_i} |J_\sigma|^\alpha \leqslant \sum_{1\leqslant i<p} (|W_i|^\alpha \cdot \#\mathcal{A}_i) \\ &\leqslant 4^d (c \min_{1\leqslant i\leqslant r} a_i)^{-d} \sum_{1\leqslant i<p} |W_i|^\alpha \end{aligned}\tag{24}$$

by (23). Thus our result (22) is obtained by (24) and (21), taking the limit as $\delta \to 0$. QED

Proof of theorem 2. (iii) $\Rightarrow$ (ii). This is clear. (ii) $\Rightarrow$ (iii). Let $M(\Gamma_j, \Omega\backslash\Gamma_j) = \phi(\hat{M}(\Gamma_j, \Omega\backslash\Gamma_j))$, where $\hat{M}(\Gamma_j, \Omega\backslash\Gamma_j)$ is defined corresponding to the relative group-frequencies c_j and $1 - c_j$. For any fixed $1 \leqslant j \leqslant k$, since $M(\Gamma_1, \ldots, \Gamma_k) \subseteq M(\Gamma_j, \Omega\backslash\Gamma_j) \subseteq F$, by the definition of $M(\Gamma_1, \ldots, \Gamma_k)$ and $M(\Gamma_j, \Omega\backslash\Gamma_j)$, we have $\dim_H M(\Gamma_j, \Omega\backslash\Gamma_j) = t$. Hence, t is also such that

$$c_j \log c_j + (1 - c_j)\log(1 - c_j) = c_j \log\left(\sum_{i\in\Gamma_j} a_i^t\right) + (1 - c_j)\log\left(1 - \sum_{i\in\Gamma_j} a_i^t\right) \tag{25}$$

by (4) in theorem 1 and (ii). Note that at this moment it must be true that $0 < c_j < 1$. It is easy to check that the function $g(x) = c_j(\log c_j - \log x) + (1 - c_j)(\log(1 - c_j) - \log(1 - x))$ has a unique zero in (0, 1). Consequently, we get $c_j = \sum_{i\in\Gamma_j} a_i^t$ from (25).

(ii) $\Rightarrow$ (i). Construct a probability measure $\hat{\mu}$ on Ω^ω by requiring that for any $\sigma \in \Omega^m$

$$\hat{\mu}(C(\sigma)) = \prod_{i=1}^m a_{\sigma(i)}^t,$$

where $C(\sigma)$ is the cylinder determined by σ. Let μ on F be the image measure of $\hat{\mu}$ under ϕ. Let

$$\hat{M} = \left\{\sigma \in \Omega^\omega : \lim_{n\to\infty} \frac{\#\{1 \leqslant l \leqslant n : \sigma(l) = i\}}{n} = a_i^t,\ 1 \leqslant i \leqslant r\right\}.$$

Note that by the above we know (iii) also holds. Then, $\hat{M} \subseteq \hat{M}(\Gamma_1, \ldots, \Gamma_k)$, $M = \phi(\hat{M}) \subseteq M(\Gamma_1, \ldots, \Gamma_k)$ and $\mu(M) = \hat{\mu}(\hat{M}) = 1$ by Birkhoff's Ergodic theorem. By $B_R(x)$ we denote the closed ball with centre at x and radius R. Let R small enough be given. Then, for each $\sigma \in \hat{M}$ there exists a positive integer $h(\sigma, R)$ such that

$$R \cdot \min_{1 \leqslant i \leqslant r} a_i \leqslant a(\sigma|h(\sigma, R)) \leqslant R. \tag{26}$$

Let $W = \{\sigma|h(\sigma, R) : \sigma \in \hat{M}\}$. For any fixed $x \in M$ let $W^* = \{\tau \in W : J_\tau \bigcap B_R(x) \bigcap M \neq \emptyset\}$. Then, there exists a finite positive constant ξ independent of R and x such that $\#W^* \leqslant \xi$ by lemma 9.2 in [3]. Hence,

$$\mu(B_R(x)) \leqslant \hat{\mu}\left(\bigcup_{\tau \in W^*} C(\tau)\right) \leqslant \sum_{\tau \in W^*} \hat{\mu}(C(\tau)) = \sum_{\tau \in W^*} (a(\tau))^t \leqslant \xi R^t$$

by (26). So we get

$$\mathcal{H}^t(M(\Gamma_1, \ldots, \Gamma_k)) \geqslant \mathcal{H}^t(M) \geqslant \mu(M) = 1$$

by proposition 4.9(a) in [3]. On the other hand, we have

$$\mathcal{H}^t(M(\Gamma_1, \ldots, \Gamma_k)) \leqslant \mathcal{H}^t(F) < \infty.$$

(i) $\Rightarrow$ (ii). It is easy to check, that for any $m \in \boldsymbol{N}$

$$\hat{M}(\Gamma_1, \ldots, \Gamma_k) = \bigcup_{\sigma \in \Omega^m} \sigma * \hat{M}(\Gamma_1, \ldots, \Gamma_k),$$

where $\sigma * \hat{M}(\Gamma_1, \ldots, \Gamma_k) = \{\sigma * \tau : \tau \in \hat{M}(\Gamma_1, \ldots, \Gamma_k)\}$. Denote $\hat{M}(\sigma) = \sigma * \hat{M}(\Gamma_1, \ldots, \Gamma_k)$ and $M(\sigma) = \phi(\hat{M}(\sigma))$. Then, $M(\Gamma_1, \ldots, \Gamma_k) = \bigcup_{\sigma \in \Omega^m} M(\sigma)$. Note that $\{J_{\sigma * \tau} : \tau \in \mathcal{A} \subseteq \Omega^*\}$ is a non-overlapping covering of $M(\sigma)$ if and only if $\{J_\tau : \tau \in \mathcal{A} \subseteq \Omega^*\}$ is a non-overlapping covering of $M(\Gamma_1, \ldots, \Gamma_k)$ and $|J_{\sigma * \tau}| = a(\sigma)|J_\tau|$. Thus,

$$\mathcal{H}^t_{Q_\sigma}(M(\sigma)) = (a(\sigma))^t \mathcal{H}^t_Q(M(\Gamma_1, \ldots, \Gamma_k)). \tag{27}$$

Then, we have

$$\begin{aligned}
\mathcal{H}^t_Q(M(\Gamma_1, \ldots, \Gamma_k)) &= \mathcal{H}^t_Q\left(\bigcup_{\sigma \in \Omega^m} M(\sigma)\right) \leqslant \sum_{\sigma \in \Omega^m} \mathcal{H}^t_Q(M(\sigma)) \\
&\leqslant \sum_{\sigma \in \Omega^m} \mathcal{H}^t_{Q_\sigma}(M(\sigma)) = \sum_{\sigma \in \Omega^m} (a(\sigma))^t \mathcal{H}^t_Q(M(\Gamma_1, \ldots, \Gamma_k)) \\
&= \left(\sum_{i=1}^r a_i^t\right)^m \mathcal{H}^t_Q(M(\Gamma_1, \ldots, \Gamma_k)),
\end{aligned} \tag{28}$$

by lemma 3(I) and (27). In addition, for any $x \in M(\Gamma_1, \ldots, \Gamma_k)$, x must lie in at least one, but in at most ϑ of the $M(\sigma)$, $\sigma \in \Omega^m$ by (3). We claim that

$$\sum_{\sigma \in \Omega^m} \mathcal{H}^t_Q(M(\sigma)) \leqslant \vartheta \mathcal{H}^t_Q(M(\Gamma_1, \ldots, \Gamma_k)). \tag{29}$$

This is not obvious, and needs some demonstration. Let $\{V_i\}$ be a non-overlapping δ-covering of $M(\Gamma_1, \ldots, \Gamma_k)$, with $V_i \in Q$ and $\delta < \min_{\sigma \in \Omega^m} |J_\sigma|$. For each $\sigma \in \Omega^m$, choose the covering $\mathcal{B}_\sigma$ of $M(\sigma)$ from $\{V_i\}$ by taking $\mathcal{B}_\sigma = \{V_i : V_i \cap M(\sigma) \neq \emptyset\}$. In this way, each V_i is chosen at most ϑ times. Otherwise, without loss of generality, suppose $V_1 \in \mathcal{B}_{\sigma_i}$ for $i = 1, 2, \ldots, \vartheta + 1$. Then, there exist $\tau_i \in C(\sigma_i), i = 1, 2, \ldots, \vartheta + 1$ such that $\phi(\tau_i) \in V_1 \cap M(\sigma_i)$. Take $R = |V_1|$ and consider the R-size component sets of F. Consequently, corresponding to each $\sigma_i, i = 1, 2, \ldots, \vartheta + 1$, there exists an R-size component set J_{ω_i} such that $J_{\omega_i} \subseteq J_{\sigma_i}$ and

$\phi(\tau_i) \in J_{\omega_i}$. Therefore, taking $x \in V_1$, we have that the ball $B_R(x)$ intersects at least $\vartheta + 1$ of the R-size component sets of F, which contradicts (C3). Thus,

$$\sum_{\sigma \in \Omega^m} \sum_{V \in \mathcal{B}_\sigma} |V|^t \leqslant \vartheta \sum_i |V_i|^t,$$

which leads to (29). From lemma 3(I) it follows that

$$\sum_{\sigma \in \Omega^m} \mathcal{H}_Q^t(M(\sigma)) \geqslant h \sum_{\sigma \in \Omega^m} \mathcal{H}_{Q_\sigma}^t(M(\sigma)). \tag{30}$$

Thus, by (27)–(30) we have for any $m \in \boldsymbol{N}$

$$\mathcal{H}_Q^t(M(\Gamma_1, \ldots, \Gamma_k)) \leqslant \left(\sum_{i=1}^r a_i^t\right)^m \mathcal{H}_Q^t(M(\Gamma_1, \ldots, \Gamma_k)) \leqslant h^{-1} \vartheta \mathcal{H}_Q^t(M(\Gamma_1, \ldots, \Gamma_k)). \tag{31}$$

Note that $0 < \mathcal{H}^t(M(\Gamma_1, \ldots, \Gamma_k)) < \infty$ if and only if $0 < \mathcal{H}_Q^t(M(\Gamma_1, \ldots, \Gamma_k)) < \infty$ by lemma 3(II). Hence, we obtain $\sum_{i=1}^r a_i^t = 1$ by (31).

(iv) $\Rightarrow$ (ii). If $\dim_H M(\Gamma_1, \ldots, \Gamma_k) = \dim_H(F \backslash M(\Gamma_1, \ldots, \Gamma_k))$, then

$$\dim_H F = \max\{\dim_H M(\Gamma_1, \ldots, \Gamma_k), \ \dim_H(F \backslash M(\Gamma_1, \ldots, \Gamma_k))\} = t.$$

Thus (ii) holds.

(ii)$\Rightarrow$(iv). On the other hand, since there exists at least one j, $1 \leqslant j \leqslant k$, such that $c_j > 0$, without loss of generality we can assume $c_1 > 0$. Fix $0 < \epsilon < c_1$ and take non-negative real numbers e_i, $1 \leqslant i \leqslant k$, such that

$$e_1 = c_1 - \epsilon, \quad e_2 = c_2 + \epsilon, \quad \text{and} \quad e_j = c_j, \qquad 2 < j \leqslant k.$$

By $M^*(\Gamma_1, \ldots, \Gamma_k)$ we denote the corresponding set with e_j in place of c_j, $1 \leqslant j \leqslant k$. Write $t(\epsilon) = \dim_H M^*(\Gamma_1, \ldots, \Gamma_k)$ where $t(\epsilon)$ is such that

$$(c_1 - \epsilon) \cdot \log(c_1 - \epsilon) + (c_2 + \epsilon) \cdot \log(c_2 + \epsilon) + \sum_{j=3}^k c_j \log c_j$$

$$= (c_1 - \epsilon) \cdot \log \sum_{i \in \Gamma_1} a_i^{t(\epsilon)} + (c_2 + \epsilon) \cdot \log \sum_{i \in \Gamma_2} a_i^{t(\epsilon)} + \sum_{j=3}^k c_j \log \sum_{i \in \Gamma_j} a_i^{t(\epsilon)}, \tag{32}$$

by theorem 1. Since $M^*(\Gamma_1, \ldots, \Gamma_k) \subseteq F \backslash M(\Gamma_1, \ldots, \Gamma_k) \subseteq F$, we get

$$t(\epsilon) = \dim_H M^*(\Gamma_1, \ldots, \Gamma_k) \leqslant \dim_H(F \backslash M(\Gamma_1, \ldots, \Gamma_k)) \leqslant \dim_H F = t, \tag{33}$$

by (ii). Note that $t(0) = t$ and $t(\epsilon)$ depends on ϵ continuously by (32). Then, $\lim_{\epsilon \to 0} t(\epsilon) = t$. So, $\dim_H(F \backslash M(\Gamma_1, \ldots, \Gamma_k)) = \dim_H M(\Gamma_1, \ldots, \Gamma_k) = t$ by (33). QED

Finally, we discuss question 2.18 in [1] by Cawley and Mauldin. When $k = r$ the sets Γ_j, $1 \leqslant j \leqslant r$, are all singletons. In this case formula (4) reduces to

$$t = \frac{\sum_{i=1}^r c_i \cdot \log c_i}{\sum_{i=1}^r c_i \cdot \log a_i}. \tag{34}$$

Thus theorem 1 yields

$$\dim_H M(\Gamma_1, \ldots, \Gamma_r) = \frac{\sum_{i=1}^r c_i \cdot \log c_i}{\sum_{i=1}^r c_i \cdot \log a_i}. \tag{35}$$

Now take positive real numbers p_i, $1 \leqslant i \leqslant r$ such that $\sum_{i=1}^r p_i = 1$. For any $q \in \boldsymbol{R}$ let $\beta(q)$ be the unique real number such that

$$\sum_{i=1}^r p_i^q a_i^{\beta(q)} = 1.$$

Let $\alpha(q) = -\beta'(q) = [\sum_{i=1}^{r} p_i^q a_i^{\beta(q)} \log p_i]/[\sum_{i=1}^{r} p_i^q a_i^{\beta(q)} \log a_i]$ and $f(q) = q\alpha(q)+\beta(q)$. Taking $c_i = p_i^q a_i^{\beta(q)}$, $1 \leqslant i \leqslant r$, it is easy to check $t = f(q)$ by (34). Thus, denoting—with this choice of (c_j)—$M_q = M(\Gamma_1, \ldots, \Gamma_r)$, (35) shows $\dim_H M_q = f(q)$, which was given by Cawley and Mauldin ([1]) for the multifractal decomposition of Moran fractals. Assume that $\log p_1/\log a_1 = \cdots = \log p_r/\log a_r$ does not hold. In this case $f(q) = s$ with $\sum_{i=1}^{r} a_i^s = 1$ if and only if $q = 0$ ([1]). Question 2.18 in [1] is: 'If $q \neq 0$, is it true that $0 < \mathcal{H}^{f(q)}(M_q) < \infty$?'. Here, our theorem 2 shows that $0 < \mathcal{H}^{f(q)}(M_q) < \infty$ if and only if $q = 0$.

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