How smooth is a devil's staircase?

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Abstract

Let the Cantor set C in $\mathbf{R}$ be defined by $C = \bigcup_{j=0}^{r} h_j(C)$ with a disjoint union, where the h_j's are similitude mappings with ratios $0 < a_j < 1$. Let μ on C be the self-similar probability measure corresponding to the probability vector $(a_0^{\xi}, a_1^{\xi}, \cdots, a_r^{\xi})$, where $\xi = \dim_H C$ is the Hausdorff dimension of C. Let S be the set of points at which the probability distribution function $F(x)$ of μ has no derivative, finite or infinite. We prove that $\dim_H S = (\dim_H C)^2$ and $\dim_P S = \dim_B S = \dim_H C$.

Key Words: Hausdorff dimension; Packing dimension; Non-differentiability; Cantor function; Cantor measure.

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1 Introduction

Let $h_i(x) = ax + i(1-a), i = 0,1$ with $x \in [0,1]$ and $0 < a < \frac{1}{2}$. Then there exists a unique non-empty compact set $C \subset [0,1]$ such that

$$C = h_0(C) \bigcup h_1(C).$$

It is well-known that the Hausdorff dimension of C equals $\dim_H C = -\frac{\log 2}{\log a}$. Let μ be the uniform probability measure on C. Consider the distribution function which is often referred to as the Devil's staircase (for $a = \frac{1}{3}$):

$$F(x) = \mu([0,x]), \quad x \in [0,1].$$

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0,1] \setminus C$ and the upper derivative of $F(x)$ is infinite on C. Let S be the set of points at which $F(x)$ is not differentiable, i.e., the set of points in C at which the lower derivative of $F(x)$ is finite. S can be decomposed into

$$S = N^+ \bigcup N^- \bigcup \{t : t \text{ is an endpoint of } C\}, \tag{1}$$

where $N^+(N^-)$ is the set of non-end points of C at which the lower right (left) derivative of $F(x)$ is finite. Each $t \in C$ can be encoded by a unique $0-1$ sequence, denoted by $\tilde{t} = (t(1), t(2), \cdots)$, which satisfies $\{t\} = \bigcap_{n=1}^{\infty} h_{t(1)} \circ \cdots \circ h_{t(n)}([0,1])$. Now let $z(t,n)$ denote the position of the n-th zero in $\tilde{t}$. The set N^+ (symmetrically for N^-) is characterized by R.Darst [1] as follows:

[a] if $t \in N^+$, then $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \geq -\frac{\log a}{\log 2}$;

[b] if $\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} > -\frac{\log a}{\log 2}$, then $t \in N^+$.

By means of the above [a] and [b] R. Darst [1] proves that

$$\dim_H S = \dim_H N^+ = \left[\frac{\log 2}{\log a}\right]^2 = (\dim_H C)^2.$$

It is not difficult to show (see [1]) that $\dim_H S = \dim_H N^+ = \left[\frac{\log(r+1)}{\log a}\right]^2 = (\dim_H C)^2$ still holds for a little bit more general Cantor set C with $C = \bigcup_{j=0}^{r} h_j(C)$, where $h_j(x) = ax + (1-a)\frac{j}{r}$, $j = 0, 1, \cdots, r$ and $0 < a < \frac{1}{r+1}$.

The Cantor sets C described above are all homogeneous in the sense that all similitude mappings $h_j(x)$ have the same scaling factor a and the gaps between the images $h_j([0,1])$, $j = 0, 1, \cdots, r$, have the same length. In the following, let r be a positive integer and let the Cantor set C in $\mathbf{R}$ be defined by

$$C = \bigcup_{j=0}^{r} h_j(C),$$

where $h_j(x) = a_j x + b_j$, $j = 0, 1, \cdots, r$, with $0 < a_j < 1$. Without loss of generality we shall assume that $b_0 = 0$ and $a_r + b_r = 1$. We furthermore assume that the images $h_j([0,1])$, $j = 0, 1, \cdots, r$ are pairwise disjoint and are lined up from left to right. In this paper, we will determine the Hausdorff, box and packing dimensions of the set of non-differentiability points of the distribution function $F(x) = \mu([0,x])$ of the self-similar probability measure μ associated to the mappings $(h_j)_{j=0}^{r}$.

In order to encode the elements of C, we introduce some notations. Let $\Omega = \{0, 1, \cdots, r\}$. We will write

(i) $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \cdots) : \sigma(j) \in \Omega\}$;

(ii) $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) : \sigma(j) \in \Omega\}$ for $k \in \mathbf{N}$ and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$;

(iii) $|\cdot|$ is used to denote the length of word. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \cdots, \sigma(|\sigma|), \tau(1), \cdots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \cdots, \tau(|\tau|), \sigma(1), \sigma(2), \cdots)$ for any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$;

(iv) $\sigma|k = (\sigma(1), \sigma(2), \cdots, \sigma(k))$ for $\sigma \in \Omega^\omega$ and $k \in \mathbf{N}$.

Denote $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \Omega^k$ and $x \in \mathbf{R}$. Then for $\sigma \in \Omega^k$, the intervals $h_{\sigma*0}([0,1]), h_{\sigma*1}([0,1]), \cdots, h_{\sigma*r}([0,1])$ are contained in $h_\sigma([0,1])$ in this order where the left endpoint of $h_{\sigma*0}([0,1])$ coincides with the left endpoint of $h_\sigma([0,1])$, and the right endpoint of $h_{\sigma*r}([0,1])$ coincides with the right endpoint of $h_\sigma([0,1])$. Moreover the length of interval $h_\sigma([0,1])$ equals $\lambda(h_\sigma([0,1])) = \prod_{j=1}^{k} a_{\sigma(j)} =: a_\sigma$ for $\sigma \in \Omega^k$. For $j = 1, 2, \cdots$, we define

$$C_j = \bigcup_{\sigma \in \Omega^j} h_\sigma([0,1]).$$

Then $C_j \downarrow C$ as $j \to \infty$ and $x \in C$ can be encoded by a unique $\sigma \in \Omega^\omega$ satisfying $\{x\} = \bigcap_{k=1}^\infty h_{\sigma|k}([0,1])$. We usually denote this unique code of x by $\tilde{x}$ and use $x(k)$ to denote the k-th component of $\tilde{x}$.

It is well-known that $\dim_H C = \dim_B C = \dim_P C = \xi$ where ξ is given by

$$\sum_{j=0}^{r} a_j^\xi = 1. \tag{2}$$

Let μ be the self-similar probability measure on C corresponding to the probability vector $(a_0^\xi, a_1^\xi, \cdots, a_r^\xi)$, i.e., the measure satisfying

$$\mu(h_\sigma([0,1])) = \prod_{j=1}^{k} a_{\sigma(j)}^\xi = a_\sigma^\xi, \quad \text{for any } \sigma \in \Omega^k,\ k \in \mathbf{N}.$$

Consider the distribution function of the probability measure μ, also called Cantor function:

$$F(x) = \mu([0,x]), \quad x \in [0,1].$$

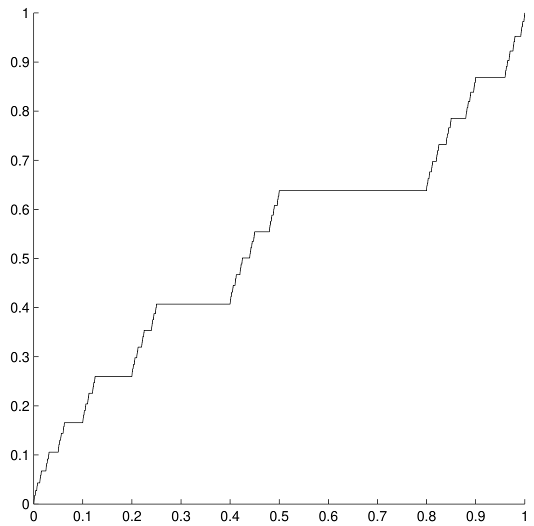

Figure 1: *The graph of $F(x)$ for the case $a_0 = 0.5$, $b_0 = 0$, $a_1 = 0.2$ and $b_1 = 0.8$.*

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0,1] \setminus C$. We will show that the upper derivative of $F(x)$ is infinite on C. Let S be the set of points at which $F(x)$ is not differentiable, i.e., the set of points in C at which the lower derivative of $F(x)$ is finite. The set S can be decomposed in the same way as in (1). The endpoints of

$h_\sigma([0,1])$ for a $\sigma \in \Omega^*$ will be called the endpoints of C. Obviously any endpoint e of C lies in C and except for a finite number of terms, its coding $\tilde{e}$ consists of either only the symbol 0 if e is the left endpoint of some $h_\sigma([0,1])$, or only the symbol r if e is the right endpoint of some $h_\sigma([0,1])$.

In this paper, we will prove that $\dim_H S = \dim_H N^+ = \dim_H N^- = (\dim_H C)^2 = \xi^2$ and $\dim_B S = \dim_P S = \dim_H C = \xi$.

2 Codes of non-differentiability points

In this section, we characterize the set S of non-differentiability points by means of the behavior of their codings. We focus on N^+. Results on N^- can be obtained symmetrically.

Proposition 2.1 *The upper derivative of $F(x)$ is infinite for all $x \in C$.*

Proof For any t in C, t not a right endpoint, let the code be $\tilde{t} = (t(1), t(2), \cdots)$. Then $\tilde{t}$ has infinitely many entries lying in $\Omega \setminus \{r\}$. Suppose $\tilde{t}$ has a entry from $\Omega \setminus \{r\}$ in position j. Then t lies in the interval $h_{\tilde{t}|(j-1)}([0,1])$ but is not equal to the right endpoint u of $h_{\tilde{t}|(j-1)}([0,1])$, where

$$\tilde{u} = (t(1), \cdots, t(j-1), r, r, \cdots).$$

Note that u is also the right endpoint of $h_{\tilde{u}|j}([0,1])$ and that $t \notin h_{\tilde{u}|j}([0,1])$. Thus we have that $t, u \in h_{\tilde{t}|(j-1)}([0,1])$ and $(t,u] \supseteq h_{\tilde{u}|j}([0,1])$. Consider the slope of the line segment from the point $P = (t, F(t))$ on the graph of F to the point $Q = (u, F(u))$. We have

$$\begin{aligned} \frac{F(u)-F(t)}{u-t} &= \frac{\mu((t,u])}{u-t} \geq \frac{\mu(h_{\tilde{u}|j}([0,1]))}{|h_{\tilde{t}|(j-1)}([0,1])|} = \frac{a^\xi_{\tilde{t}|(j-1)} a^\xi_r}{a_{\tilde{t}|(j-1)}} \\ &= a^\xi_r (a_{\tilde{t}|(j-1)})^{\xi-1} \to \infty \text{ as } j \to \infty. \end{aligned}$$

Symmetrically the upper left derivative of t at a non-left-end point of C is infinite. **QED**

Proposition 2.2 *Let $\Gamma = \{0, 1, \cdots, r-1\}$. Let $\underline{a} = \min_{j \in \Gamma} a_j$ and $\bar{a} = \max_{j\in\Gamma} a_j$. Let $t \in C$ be not an endpoint of C and let $z(t,n)$ denote the position of the n-th occurrence of elements of Γ in $\tilde{t}$, then*

(I) if $t \in N^+$, then $\limsup_{n\to\infty}\left[\frac{n}{z(t,n)}(1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r}-1\right)+\frac{z(t,n+1)}{z(t,n)}-\xi^{-1}\right]\geq 0$;

(II) if $\limsup_{n\to\infty}\left[\frac{n}{z(t,n)}(1-\xi^{-1})\left(\frac{\log \underline{a}}{\log a_r}-1\right)+\frac{z(t,n+1)}{z(t,n)}-\xi^{-1}\right]> 0$, then $t \in N^+$.

Proof We first prove the statement (I), i.e., the lower-right derivative of $F(x)$ is infinite at a non-endpoint t of C when $\limsup_{n\to\infty}\left[\frac{n}{z(t,n)}(1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r}-1\right)+\frac{z(t,n+1)}{z(t,n)}-\xi^{-1}\right]< 0$. Consider such a point t with $\tilde{t}=(t(1),t(2),\cdots)$. Let k be a positive integer such that

$$\frac{n}{z(t,n)}(1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r}-1\right)+\frac{z(t,n+1)}{z(t,n)}-\xi^{-1}<q, \tag{3}$$

for some negative real number q whenever $n\geq k$. Let u be a positive number such that u is smaller than the distance between t and $[0,1]\setminus C_{\tilde{t}|l}$ with $l=z(t,k)$. Let x be a point in the segment $(t,t+u)$. Then $t,x\in h_{\tilde{t}|l}([0,1])$. We will see that $(F(x)-F(t))/(x-t)$ is large relative to k, so t is not in N^+. Let i denote the level at which $x\notin h_{\tilde{t}|i}([0,1])$ but $x\in h_{\tilde{t}|(i-1)}([0,1])$. Note that also $t\in h_{\tilde{t}|(i-1)}([0,1])$. Thus $x-t\leq |h_{\tilde{t}|(i-1)}([0,1])|=a_{\tilde{t}|(i-1)}$; also $i=z(t,n)$ for some $n>k$. Put $j=z(t,n+1)-1$, and by v we denote the right endpoint of $h_{\tilde{t}|j}([0,1])$, which implies that $\tilde{v}=(t(1),\cdots,t(j),r,r,\cdots)$ and $(t,v]\supseteq h_{\tilde{v}|(j+1)}([0,1])$. Then we have $t<v<x$ and $F(v)-F(t)=\mu((t,v])\geq \mu(h_{\tilde{v}|(j+1)}([0,1]))=(a_{\tilde{t}|j})^{\xi}a_r^{\xi}$. Therefore, defining

$$\widetilde{\Pi}_n=\prod_{i=1,t(i)\neq r}^{z(t,n)-1}a_{t(i)},$$

we have

$$\begin{aligned}
\frac{F(x)-F(t)}{x-t} &\geq \frac{(a_{\tilde{t}|j})^{\xi}a_r^{\xi}}{a_{\tilde{t}|(i-1)}}=\frac{a_r^{\xi}\left(\prod_{i=1}^{z(t,n+1)-1}a_{t(i)}\right)^{\xi}}{\prod_{i=1}^{z(t,n)-1}a_{t(i)}}=\frac{a_r^{\xi}\left[a_r^{z(t,n+1)-1-n}a_{t(z(t,n))}\widetilde{\Pi}_n\right]^{\xi}}{a_r^{z(t,n)-1-(n-1)}\widetilde{\Pi}_n}\\
&= a_{t(z(t,n))}^{\xi}\widetilde{\Pi}_n^{\xi-1}a_r^{(z(t,n+1)-n)\xi-z(t,n)+n}\\
&= a_{t(z(t,n))}^{\xi}\left[\widetilde{\Pi}_n^{\frac{\xi-1}{z(t,n)}}a_r^{\left(\frac{z(t,n+1)}{z(t,n)}-\frac{n}{z(t,n)}\right)\xi-1+\frac{n}{z(t,n)}}\right]^{z(t,n)}\\
&\geq \underline{a}^{\xi}\left[\bar{a}^{\frac{(n-1)(\xi-1)}{z(t,n)}}a_r^{\left(\frac{z(t,n+1)}{z(t,n)}-\frac{n}{z(t,n)}\right)\xi-1+\frac{n}{z(t,n)}}\right]^{z(t,n)}\\
&= \underline{a}^{\xi}\bar{a}^{1-\xi}\left[\left(\frac{\bar{a}}{a_r}\right)^{\frac{n(\xi-1)}{z(t,n)}}a_r^{\frac{z(t,n+1)}{z(t,n)}\xi-1}\right]^{z(t,n)}.
\end{aligned} \tag{4}$$

Let

$$Q=\left(\frac{\bar{a}}{a_r}\right)^{\frac{n(\xi-1)}{z(t,n)}}a_r^{\frac{z(t,n+1)}{z(t,n)}\xi-1}.$$

Taking logs and by (3) we have

$$\begin{aligned} \log Q &= \xi \log a_r \left[\frac{n}{z(t,n)}(1-\xi^{-1}) \left(\frac{\log \bar{a}}{\log a_r} - 1 \right) + \frac{z(t,n+1)}{z(t,n)} - \xi^{-1} \right] \\ &\geq \xi q \log a_r > 0. \end{aligned} \tag{5}$$

Since t is a non-end point, $z(t,n) \to \infty$ and the lower-right derivative of $F(x)$ is infinite at t by (4) and (5).

Now we turn to the proof of statement (II). Let t be such that

$$\limsup_{n\to\infty} \left[\frac{n}{z(t,n)}(1-\xi^{-1}) \left(\frac{\log \underline{a}}{\log a_r} - 1 \right) + \frac{z(t,n+1)}{z(t,n)} - \xi^{-1} \right] > 0.$$

Then there exists a sequence $\{n_k\}$ of positive integers such that

$$\frac{n_k}{z(t,n_k)}(1-\xi^{-1}) \left(\frac{\log \underline{a}}{\log a_r} - 1 \right) + \frac{z(t,n_k+1)}{z(t,n_k)} - \xi^{-1} > c, \tag{6}$$

for some positive constant c. Let x_k be the left endpoint of $h_{(\tilde{t}|j_k)*(t(j_k+1)+1)}([0,1])$, where $j_k = z(t,n_k) - 1$. Thus we have $\tilde{x}_k = (t(1), \cdots, t(j_k), t(j_k+1)+1, 0, \cdots, 0, \cdots)$. Let u_k be the right endpoint of $h_{\tilde{t}|(j_k+1)}([0,1])$. Then $\tilde{u}_k = (t(1), \cdots, t(j_k), t(j_k+1), r, r, r, \cdots)$. Thus $[u_k, x_k]$ is the gap on the right side of $h_{\tilde{t}|(j_k+1)}([0,1])$ and $\lambda([u_k,x_k]) = x_k - u_k = a_{\tilde{t}|j_k}\beta_{t(j_k+1)}$ where by β_j, $j = 0,1,\cdots,r-1$, we denote length of the gap between images $h_j([0,1])$ and $h_{j+1}([0,1])$. Note that $[t,x_k] \supseteq [u_k,x_k]$ and $\mu((t,x_k]) = \mu((t,u_k]) + \mu((u_k,x_k]) = \mu((t,u_k]) \leq \mu(h_{\tilde{t}|(z(t,n_k+1)-1)}([0,1]))$ since $\tilde{t}|(z(t,n_k+1)-1) = \tilde{u_k}|(z(t,n_k+1)-1)$. Therefore we have

$$F(x_k) - F(t) = \mu((t,x_k]) \leq \mu(h_{\tilde{t}|(z(t,n_k+1)-1)}([0,1])) = a^{\xi}_{\tilde{t}|(z(t,n_k+1)-1)},$$

and

$$x_k - t \geq \lambda([u_k,x_k]) = a_{\tilde{t}|(z(t,n_k)-1)}\beta_{t(z(t,n_k))}.$$

Denote $\underline{\beta} = \min_{j\in\{0,1,\cdots,r-1\}} \beta_j$. Then we obtain with a similar reasoning which led to (4)

$$\begin{aligned} \frac{F(x_k)-F(t)}{x_k - t} &\leq \frac{a^{\xi}_{\tilde{t}|(z(t,n_k+1)-1)}}{a_{\tilde{t}|(z(t,n_k)-1)}\beta_{\tilde{t}(z(t,n_k))}} \leq \frac{\left(\prod_{i=1}^{z(t,n_k+1)-1} a_{t(i)}\right)^{\xi}}{\underline{\beta}\prod_{i=1}^{z(t,n_k)-1} a_{t(i)}} \\ &\leq \frac{\bar{a}^{\xi}\underline{a}^{1-\xi}}{a_r^{\xi}\underline{\beta}} \left[\left(\frac{\underline{a}}{a_r}\right)^{\frac{n_k(\xi-1)}{z(t,n_k)}} a_r^{\frac{z(t,n_k+1)}{z(t,n_k)}\xi - 1} \right]^{z(t,n_k)}. \end{aligned} \tag{7}$$

Let

$$Q = \left(\frac{a}{a_r}\right)^{\frac{n_k(\xi-1)}{z(t,n_k)}} a_r^{\frac{z(t,n_k+1)}{z(t,n_k)}\xi-1}.$$

Taking logs and using (6) we obtain

$$\begin{aligned} \log Q &= \xi \log a_r \left[\frac{n_k}{z(t,n_k)}(1-\xi^{-1})\left(\frac{\log \underline{a}}{\log a_r} - 1\right) + \frac{z(t,n_k+1)}{z(t,n_k)} - \xi^{-1}\right] \\ &\leq \xi c \log a_r < 0. \end{aligned} \tag{8}$$

From (7) and (8) it follows that the right lower derivative at t is finite by letting $k \longrightarrow \infty$. **QED**

3 Dimensions of the set of non-differentiability points

In this section, we determine the dimensions of S. The following lemma is a special case of a result in [4].

Lemma 3.1 *Let* $\Gamma = \{0, 1, \cdots, r-1\}$ *and* $z(t,n)$ *denote the position of the* n*-th occurrence of elements of* Γ *in* $\tilde{t}$. *For given* $0 < p \leq 1$, *let*

$$C(p) = \left\{ t \in C \setminus \{\text{right endpoints of } C\} : \limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \geq p^{-1} \right\}. \tag{9}$$

Let $\eta = \eta(p)$ *be such that*

$$p \log \sum_{j\in\Omega} a_j^\eta + (1-p)\log a_r^\eta = 0. \tag{10}$$

Then we have $\dim_H C(p) = \eta$ *and* $\dim_P C(p) = \dim_B C(p) = \dim_H C = \xi$ *where* ξ *is defined in* (2).

It is easy to verify that $\eta(p)$ is strictly increasing and continuous in p and that $\eta(0+) < \eta(p) \leq \eta(1) = \xi$. We also consider for $0 < p \leq 1$ and with the same Γ

$$C^*(p) = \left\{ t \in C \setminus \{\text{right endpoints of } C\} : \limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} > p^{-1} \right\}. \tag{11}$$

Directly from Lemma 3.1 it follows that $\dim_P C^*(p) = \dim_B C^*(p) = \dim_H C = \xi$. Moreover, $\dim_H C^*(p) = \eta(p)$. To see that this follows from Lemma 3.1, approximate $C^*(p)$ by a union of $C(p_k)$'s, where $p_k \uparrow p$.

Lemma 3.2 *Let* $\Gamma = \{0, 1, \cdots, r-1\}$, $\underline{a} = \min_{j\in\Gamma} a_j$, $\bar{a} = \max_{j\in\Gamma} a_j$, *and let*

$$p_1^{-1} = \max\left\{\xi^{-1}, (\xi^{-1}-1)\frac{\log \underline{a}}{\log a_r} + 1\right\};\;\; p_2^{-1} = \min\left\{\xi^{-1}, (\xi^{-1}-1)\frac{\log \bar{a}}{\log a_r} + 1\right\}.$$

Then $C^*(p_1) \subseteq N^+ \subseteq C(p_2)$ *and* $\eta(p_1) \leq \dim_H(N^+) \leq \eta(p_2)$.

Proof By Lemma 3.1 it suffices to prove $C^*(p_1) \subseteq N^+ \subseteq C(p_2)$. Let $t \in N^+$. By Proposition 2.2 we have

$$\limsup_{n\to\infty}\left[\frac{n}{z(t,n)}(1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r} - 1\right) + \frac{z(t,n+1)}{z(t,n)} - \xi^{-1}\right] \geq 0.$$

Now if $\frac{\log \bar{a}}{\log a_r} > 1$, then $\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} \geq \xi^{-1}$. If not, we use that $\frac{n}{z(t,n)} \leq 1$ and so $\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} \geq \xi^{-1} - (1-\xi^{-1})\left(\frac{\log \bar{a}}{\log a_r} - 1\right)$. So we find that

$$\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} \geq p_2^{-1},$$

i.e., $t \in C(p_2)$. On the other hand, $t \in C^*(p_1)$ implies in a similar way that $t \in N^+$. **QED**

Theorem 3.3 *Let* $C = \bigcup_{j=0}^{r} h_j(C)$ *be the Cantor set determined by* $\{h_j(x) = a_j x + b_j : 0 \leq j \leq r\}$, *and let* ξ *be its Hausdorff dimension. If* S *is the set of non-differentiability points of the Cantor distribution determined by* $a_0, \cdots, a_r$ *on* C, *then* $\dim_H S = \xi^2 = (\dim_H C)^2$ *and* $\dim_B S = \dim_P S = \dim_H C = \xi$.

Proof Since $S = N^+ \bigcup N^- \bigcup \{\text{the endpoints of } C\}$ and because of the symmetry between N^+ and N^-, it suffices to determine the dimensions of N^+. By Lemma 3.2 and Lemma 3.1, $\dim_B N^+ = \dim_P N^+ = \dim_H C = \xi$ is trivial. We define $a = \min_{j\in\Omega} a_j$. Let $0 < \delta < a^2$, and let

$$\Omega_\delta = \{\sigma \in \Omega^* : a_\sigma \leq \delta \text{ and } a_{\sigma|(|\sigma|-1)} > \delta\}.$$

Note that for each $\sigma \in \Omega_\delta$ we have $a\delta < a_\sigma \leq \delta$. Thus for any $\sigma, \tau \in \Omega_\delta$

$$\frac{\log \delta}{\log a + \log \delta} \leq \frac{\log a_\sigma}{\log a_\tau} \leq \frac{\log a + \log \delta}{\log \delta}. \tag{12}$$

Here we would like to relate Ω_δ to ξ by showing that

$$\lim_{\delta\to 0} \frac{\log \#\Omega_\delta}{-\log \delta} = \xi, \tag{13}$$

where ξ is defined in (2) and $\#\Omega_\delta$ denotes the number of elements of Ω_δ. By $N_\delta(C)$ we denote the smallest number of sets of diameter at most δ that cover C. By Lemma 9.2 in [2] there exists a positive constant q independent of δ such that

$$q\#\Omega_\delta \le N_\delta(C) \le \#\Omega_\delta. \tag{14}$$

Hence (13) holds by (14) and the fact $\lim_{\delta\to 0} \frac{\log N_\delta(C)}{-\log\delta} = \xi$. Now let $\mathcal{H}_\delta = \{h_\sigma : \sigma \in \Omega_\delta\}$. Note that h_σ is a similitude mapping with ratio $0 < a_\sigma < 1$ for each $\sigma \in \mathcal{H}_\delta$, that the family $\mathcal{H}_\delta$ of similitude mappings still satisfies the open set condition, and that the unique self-similar set determined by $\mathcal{H}_\delta$ equals C:

$$C = \bigcup_{\sigma\in\mathcal{H}_\delta} h_\sigma(C).$$

We also have that ξ defined in (2) satisfies $\sum_{\sigma\in\Omega_\delta} a_\sigma^\xi = 1$. If we denote μ_δ the self-similar probability measure on C corresponding to the probability vector $(a_\sigma^\xi : \sigma \in \Omega_\delta)$, then $\mu_\delta = \mu$ since $\mu_\delta(h_\tau([0,1])) = \mu(h_\tau([0,1]))$ for any $\tau \in \Omega_\delta^k$ and $k \in \mathbf{N}$. Hence for the corresponding non-differentiability points N_δ^+ we have $N_\delta^+ = N^+$ for all $\delta > 0$.

There exists a unique $\sigma \in \Omega_\delta$ with $\sigma(j) = r$ for $j = 1, 2, \cdots, |\sigma|$. We denote this special element by σ_δ. Note that σ_δ plays the same role in Ω_δ as r in Ω, in the sense that $h_{\sigma_\delta}([0,1])$ is the right most interval in $[0,1]$ of the intervals $(h_\sigma([0,1]))_{\sigma\in\Omega_\delta}$. Let

$$\Gamma_\delta := \Omega_\delta \setminus \{\sigma_\delta\} \ \text{ and } \ a_\delta := a_{\sigma_\delta} = a_r^{|\sigma_\delta|}.$$

We will use the notations $\underline{a}_\delta = \min_{\sigma\in\Gamma_\delta} a_\sigma$, $\bar{a}_\delta = \max_{\sigma\in\Gamma_\delta} a_\sigma$,

$$p_1(\delta) = \left(\max\left\{\xi^{-1}, (\xi^{-1} - 1)\frac{\log \underline{a}_\delta}{\log a_\delta} + 1\right\}\right)^{-1} \tag{15}$$

and

$$p_2(\delta) = \left(\min\left\{\xi^{-1}, (\xi^{-1} - 1)\frac{\log \bar{a}_\delta}{\log a_\delta} + 1\right\}\right)^{-1}.$$

Then by Lemma 3.1 and Lemma 3.2 with Ω replaced by Ω_δ, Γ replaced by Γ_δ, a_r replaced by a_δ and p_1, p_2 by $p_1(\delta)$ respectively $p_2(\delta)$, we have

$$\eta(p_1(\delta)) \leq \dim_H N_\delta^+ \leq \eta(p_2(\delta)), \tag{16}$$

where $\eta(p_1(\delta))$ and $\eta(p_2(\delta))$ are defined by formula (10):

$$p_1(\delta) \log \sum_{\sigma \in \Omega_\delta} a_\sigma^{\eta(p_1(\delta))} + (1 - p_1(\delta)) \log a_\delta^{\eta(p_1(\delta))} = 0, \tag{17}$$

and

$$p_2(\delta) \log \sum_{\sigma \in \Omega_\delta} a_\sigma^{\eta(p_2(\delta))} + (1 - p_2(\delta)) \log a_\delta^{\eta(p_2(\delta))} = 0.$$

Since $N_\delta^+ = N^+$ for all $\delta > 0$ it follows from (16) that $\dim_H N^+ = \xi^2$ holds if for any $\varepsilon > 0$ there exists a $\delta^* > 0$ such that when $0 < \delta < \delta^*$ we have $|\eta(p_1(\delta)) - \xi^2| < \varepsilon$ and $|\eta(p_2(\delta)) - \xi^2| < \varepsilon$. Verification of this claim will be given in the following only for $\eta(p_1(\delta))$, since the same argument can be employed for $\eta(p_2(\delta))$. To alleviate the notation we will write $p_\delta := p_1(\delta)$ and $\eta_\delta := \eta(p_1(\delta))$. For $x \in (0, \xi]$ let

$$T_\delta(x) = p_\delta \frac{\log \sum_{\sigma \in \Omega_\delta} a_\sigma^x}{x \log a_\delta} + 1 - p_\delta$$

Note that

$$a^x \delta^x \# \Omega_\delta \leq \sum_{\sigma \in \Omega_\delta} a_\sigma^x \leq \delta^x \# \Omega_\delta,$$

which implies

$$\frac{x \log \delta + \log \# \Omega_\delta}{x \log a_\delta} \leq \frac{\log \sum_{\sigma \in \Omega_\delta} a_\delta^x}{x \log a_\delta} \leq \frac{x \log a + x \log \delta + \log \# \Omega_\delta}{x \log a_\delta}. \tag{18}$$

Therefore by (18) we have

$$0 \leq T_\delta(x) - \left[1 + p_\delta \left(\frac{\log \delta}{\log a_\delta} - 1\right) + \frac{1}{x} \frac{p_\delta \log \# \Omega_\delta}{\log a_\delta}\right] \leq \frac{p_\delta \log a}{\log a_\delta}. \tag{19}$$

Note that by (12) and (15) we have

$$\lim_{\delta \downarrow 0} p_\delta = \xi. \tag{20}$$

Since $a\delta < a_\delta \leq \delta$, and by (13) and (20) we have

$$\lim_{\delta \downarrow 0} p_\delta \left(\frac{\log \delta}{\log a_\delta} - 1\right) = \lim_{\delta \downarrow 0} \frac{p_\delta \log a}{\log a_\delta} = 0 \text{ and } \lim_{\delta \downarrow 0} \frac{p_\delta \log \# \Omega_\delta}{\log a_\delta} = -\xi^2.$$

Thus by (19) we have for all $x \in (0, \xi]$

$$\lim_{\delta \downarrow 0} T_\delta(x) = 1 - \frac{\xi^2}{x} =: T_0(x). \tag{21}$$

Now for any given $\varepsilon > 0$ satisfying $0 < \xi^2 - \varepsilon < \xi^2 + \varepsilon < \xi < 1$, we see that

$$T_0(\xi^2 - \varepsilon) = \frac{-\varepsilon}{\xi^2 - \varepsilon} < -\varepsilon \text{ and } T_0(\xi^2 + \varepsilon) = \frac{\varepsilon}{\xi^2 + \varepsilon} > \varepsilon.$$

By (21) we can take $\delta^* > 0$ so that for $0 < \delta < \delta^*$

$$|T_\delta(\xi^2 - \varepsilon) - T_0(\xi^2 - \varepsilon)| < \frac{\varepsilon}{2} \text{ and } |T_\delta(\xi^2 + \varepsilon) - T_0(\xi^2 + \varepsilon)| < \frac{\varepsilon}{2}.$$

Then necessarily for these δ

$$T_\delta(\xi^2 - \varepsilon) < -\frac{\varepsilon}{2} \text{ and } T_\delta(\xi^2 + \varepsilon) > \frac{\varepsilon}{2}.$$

Then for $0 < \delta < \delta^*$

$$\xi^2 - \varepsilon < \eta_\delta < \xi^2 + \varepsilon,$$

since $T_\delta(x)$ is strictly increasing in x and $T_\delta(\eta_\delta) = 0$. **QED**

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