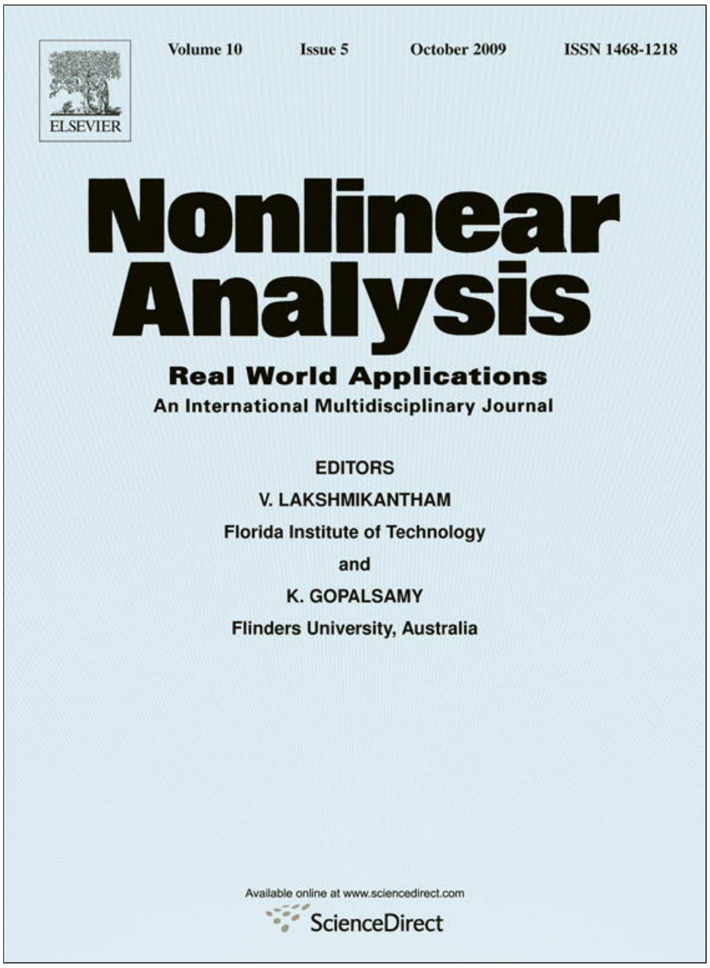
Volume 10
Issue 5
October 2009
ISSN 1468-1218
ELSEVIER
Nonlinear Analysis
Real World Applications
An International Multidisciplinary Journal
EDITORS
V. LAKSHMIKANTHAM
Florida Institute of Technology
and
K. GOPALSAMY
Flinders University, Australia
Available online at www.sciencedirect.com
ScienceDirect

Nonlinear Analysis: Real World Applications 10 (2009) 3240–3252

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Dimensions of subsets of Moran fractals related to frequencies of their codings

Wenxia Li *, Yuanyuan Yao

Department of Mathematics, East China Normal University, Shanghai 200062, PR China

ARTICLE INFO

Article history:
Received 19 February 2008
Accepted 8 October 2008

Keywords:
Moran fractals
Mixed group frequency
J-type packing
Hausdorff and packing dimensions

ABSTRACT

The Moran fractal considered in this paper is an extension of the self-similar sets satisfying the open set condition. We consider those subsets of the Moran fractal that are the union of an uncountable number of sets each of which consists of the points with their location codes having prescribed mixed group frequencies. It is proved that the Hausdorff and packing dimensions of each of these subsets coincide and are equal to the supremum of the Hausdorff (or packing) dimensions of the sets in the union. An approach is given to calculate their Hausdorff and packing dimensions. The main advantage of our approach is that we treat these subsets in a unified manner. Another advantage of this approach is that the values of the Hausdorff and packing dimensions do not need to be guessed a priori.

1. Introduction

Denote $\Omega = \{1, 2, \dots, r\}$, where $r \geq 2$. Let $\Omega^* = \bigcup_{m=0}^{\infty} \Omega^m$ with $\Omega^m = \{\sigma = (\sigma(1), \sigma(2), \dots, \sigma(m)) : \sigma(j) \in \Omega\}$ for $m \in \mathbb{N} \cup \{0\}$ (Ω^0 consists of the empty word $\emptyset$) and $\Omega^{\mathbb{N}} = \{\sigma = (\sigma(1), \sigma(2), \dots) : \sigma(j) \in \Omega\}$. We denote by $|\sigma|$ the length of word $\sigma \in \Omega^*$. For $\sigma \in \Omega^*$ and $\tau \in \Omega^* \cup \Omega^{\mathbb{N}}$, $\sigma * \tau$ denotes their concatenation. In particular, $\emptyset * \tau = \tau$. For $\sigma \in \Omega^m$, let $C(\sigma) = \{\tau \in \Omega^{\mathbb{N}} : \tau|m = \sigma\}$, called the cylinder set with base σ, where $\tau|m = (\tau(1), \tau(2), \dots, \tau(m))$.

Fix a constant $0 < c < 1$ and positive real numbers $0 < a_i < 1$, $i = 1, 2, \dots, r$. Assume that a collection $(J_\sigma)_{\sigma \in \Omega^*}$ of compact sets in $\mathbb{R}^d$ has the following features:

[A1] nested property: for $\sigma \in \Omega^*$ and $i \in \Omega$, $J_{\sigma * i} \subseteq J_\sigma$;

[A2] non-overlapping property: all J_σs with $\sigma \in \Omega^m$ are pairwise non-overlapping in the sense that $J_\sigma \cap J_\tau$ is of zero d-dimensional Lebesgue measure for any distinct $\sigma, \tau \in \Omega^m$;

[A3] regular sizes for J_σs: for $\sigma \in \Omega^*$ and $i \in \Omega$, $|J_{\sigma * i}| = a_i |J_\sigma| > 0$, where, if no confusion occurs, $|J_\sigma|$ denotes the diameter of J_σ;

[A4] regular sizes for the interior of J_σs: each J_σ, $\sigma \in \Omega^*$ contains an open ball with diameter $c|J_\sigma|$.

The *Moran fractal* associated with $(J_\sigma)_{\sigma \in \Omega^*}$ is defined as the nonempty compact set

$$F = \bigcap_{m=0}^{\infty} \bigcup_{\sigma \in \Omega^m} J_\sigma. \tag{1}$$

The definition of Moran fractals here is close to that in [6,16] with a bit variation, but simpler than that in [19] where a more general structure is discussed. (Readers can refer to [19] and more references therein for related results on this kind of structure.) Obviously, self-similar sets satisfying the open set condition are Moran fractals. The latter lose the property of similitude, but keep some typical properties exhibited by the former, e.g., they can be encoded by elements from $\Omega^{\mathbb{N}}$ and

$$\dim_H F = \overline{\dim}_B F = s, \quad 0 < \mathcal{H}^s(F) \leq \mathcal{P}^s(F) < \infty,$$

* Corresponding address: Department of Mathematics, East China Normal University, 500 Dong Chuan Road, Shanghai 200241, PR China. Tel.: +86 21 52682621; fax: +86 21 52682621.

E-mail address: wxli@math.ecnu.edu.cn (W. Li).

1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2008.10.021

where s is determined by $\sum_{j=1}^{r} a_j^s = 1$. The compact sets $J_\sigma, \sigma \in \Omega^*$ are generally referred to as component sets of F. In particular, J_σ is referred to as an mth level component set of F if $\sigma \in \Omega^m$. Define ϕ: $\Omega^{\mathbb{N}} \longrightarrow \mathbb{R}^d$ by

$$\{\phi(\sigma)\} \stackrel{\text{def}}{=} \bigcap_{m=0}^{\infty} J_{\sigma|m}. \tag{2}$$

It is easy to see that $\phi(\Omega^{\mathbb{N}}) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (2). But ϕ may not be injective. Let ρ be the metric on $\Omega^{\mathbb{N}}$ such that for any $\sigma, \tau \in \Omega^{\mathbb{N}}$

$$\rho(\sigma, \tau) \stackrel{\text{def}}{=} 2^{-\min\{i:\sigma(i)\neq\tau(i)\}},$$

with the convention $\rho(\sigma, \sigma) = 0$. Let F be equipped with the Euclidean metric. Then ϕ is continuous. Thus each $x \in F$ can be encoded via ϕ: a sequence $\sigma \in \Omega^{\mathbb{N}}$ is called a *location code* of $x \in F$ if $\phi(\sigma) = x$. Therefore, ϕ is also called the coding map and $\Omega^{\mathbb{N}}$ is called the code space (or symbolic space). As a result, F is a projection of $\Omega^{\mathbb{N}}$ on $\mathbb{R}^d$ via ϕ.

Let $0 < R < a_{\min}|J_\emptyset|$ where $a_{\min} \stackrel{\text{def}}{=} \min_{1\le i\le r} a_i$. A component set J_σ of F is termed as an R-size component set if

$$|J_\sigma| \le R \quad \text{and} \quad |J_{\sigma|(|\sigma|-1)}| > R.$$

It is easy to see that for any $0 < R < a_{\min}|J_\emptyset|$, the collection of all R-size component sets of F is a non-overlapping finite R-covering of F by [A1-3]. Hence, by means of Lemma 9.2 in [9], the requirement (see [A4]) that J_σ contains an open ball of diameter $c|J_\sigma|$ implies an important fact: there exists a positive integer ϑ, independent of R and $x \in \mathbb{R}^d$, such that any ball $B_R(x)$ with radius R and center at x intersects at most ϑ of the R-size component sets of F. Many analogues of this fact appear in this paper. A direct sequel leads to an important property of ϕ:

$$\sup_{x\in F} \#\{\phi^{-1}(x)\} < \vartheta,$$

where and throughout this paper, by $\#A$ we denote the cardinality of a finite set A. Otherwise, suppose that for some $x \in F$ we have $\#\{\phi^{-1}(x)\} > \vartheta$. Take $\vartheta + 1$ different elements $\sigma_1, \sigma_2, \ldots, \sigma_{\vartheta+1}$ from this set. Let $m \in \mathbb{N}$ be such that $\sigma_1|m, \sigma_2|m, \ldots, \sigma_{\vartheta+1}|m$ differ from each other. Taking $R = \min_{1\le i\le\vartheta+1} |J_{\sigma_i|m}|$, x lies in at least $\vartheta + 1$ R-size component sets of F, which implies that the ball $B_R(x)$ intersects at least $\vartheta + 1$ of the R-size component sets of F.

For $\Gamma \subseteq \Omega$, $\sigma \in \Omega^{\mathbb{N}}$ and $m \in \mathbb{N}$, we denote by $f(\sigma, \Gamma, m)$ the ratio of the digits from Γ occurring in the first m entries of σ, i.e.,

$$f(\sigma, \Gamma, m) \stackrel{\text{def}}{=} \frac{\#\{1 \le \ell \le m : \sigma(\ell) \in \Gamma\}}{m}.$$

Whenever there exists the limit

$$f(\sigma, \Gamma) \stackrel{\text{def}}{=} \lim_{m\to\infty} f(\sigma, \Gamma, m) = \lim_{m\to\infty} \frac{\#\{1 \le \ell \le m : \sigma(\ell) \in \Gamma\}}{m}, \tag{3}$$

it is called the *group frequency* of the σ passing through Γ. When we write the symbol $f(\sigma, \Gamma)$ we are already assuming the existence of the limit in (3). Now let

$$\Gamma_i \neq \emptyset, \quad i = 1, 2, \ldots, k \text{ with } \bigcup_{i=1}^{k} \Gamma_i = \Omega.$$

We remark that some of Γ_is may have nonempty intersections. For $A \subseteq [0, 1]^k$, let

$$M_A = \phi(\widehat{M}_A) \text{ where } \widehat{M}_A \stackrel{\text{def}}{=} \left\{\sigma \in \Omega^{\mathbb{N}} : f(\sigma, \Gamma_j) = c_j, 1 \le j \le k, (c_1, \ldots, c_k) \in A\right\}. \tag{4}$$

For $n \in \mathbb{N}$ let Ξ_n be the set of (n-dimensional) probability vectors, i.e.,

$$\Xi_n = \left\{(\lambda_1, \ldots, \lambda_n) \in [0, 1]^n : \sum_{\ell=1}^{n} \lambda_\ell = 1\right\}. \tag{5}$$

Throughout this paper, we denote by log the natural logarithm. In the present paper, we obtain

Theorem 1.1. *Let M_A be defined as in (4). Let Ξ_r be defined as in (5). Then $\dim_H M_A = \dim_P M_A = t$ where*

$$t = \sup\left\{\frac{\sum_{\ell=1}^{r} p_\ell \log p_\ell}{\sum_{\ell=1}^{r} p_\ell \log a_\ell} : (p_\ell)_{\ell=1}^{r} \in \Xi_r \text{ and } \left(\sum_{\ell\in\Gamma_1} p_\ell, \ldots, \sum_{\ell\in\Gamma_k} p_\ell\right) \in A\right\}, \tag{6}$$

with the convention $0 \log 0 = 0$.

When A is a singleton, say $A = \{\vec{c}\}$ with $\vec{c} = (c_1, \ldots, c_k)$, $M_{\{\vec{c}\}}$ is the set of points whose location codes have a prescribed *mixed* group frequency c_j passing through $\Gamma_j, j = 1, \ldots, k$ (here we use the word "mixed"since some of Γ_is is allowed to have nonempty intersections). By Theorem 1.1,

$$\dim_H M_{\{\vec{c}\}} = \dim_P M_{\{\vec{c}\}} = \sup \left\{ \frac{\sum_{\ell=1}^{r} p_\ell \log p_\ell}{\sum_{\ell=1}^{r} p_\ell \log a_\ell} : (p_\ell)_{\ell=1}^{r} \in \Xi_r \text{ with } \sum_{\ell \in \Gamma_i} p_\ell = c_\ell,\ 1 \le i \le k \right\}.$$

This was also obtained in [12] where it was dealt with under a more general setting. Obviously,

$$M_A = \bigcup_{\vec{c} \in A} M_{\{\vec{c}\}},$$

i.e., M_A is a union of an uncountable number of sets each of which consists of the points with their location codes having prescribed mixed group frequencies. Thus, Theorem 1.1 indicates

$$\dim_H M_A = \dim_P M_A = \sup_{\vec{c} \in A} \dim_H M_{\{\vec{c}\}},$$

i.e., the Hausdorff and packing dimensions of M_A coincide and are equal to the supremum of the Hausdorff (or packing) dimensions of the sets in the union. Note that $M_{\{\vec{c}\}}$ may be an empty set for some $\vec{c} \in A$. Without loss of generality, we always assume that A is properly chosen such that $\widehat{M}_A \neq \emptyset$.

We consider some special cases as follows.

Case I: $k = 1$.

In this case, $\Gamma_1 = \Omega, A \subseteq [0, 1]$ and $M_A = F$ when $1 \in A(\subseteq [0, 1])$, $= \emptyset$ when $1 \notin A$. Thus, by Theorem 1.1

$$\dim_H F = \dim_P F = \sup \left\{ \frac{\sum_{\ell=1}^{r} p_\ell \log p_\ell}{\sum_{\ell=1}^{r} p_\ell \log a_\ell} : (p_\ell)_{\ell=1}^{r} \in \Xi_r \right\} = s,$$

where $\sum_{j=1}^{r} a_j^s = 1$. □

Case II: $k = r = \#\Omega$ (recall $\Omega = \{1, 2, \ldots, r\}$).

In this case, $\Gamma_j = \{j\}$ for $1 \le j \le r, A \subseteq [0, 1]^r$ and

$$\begin{aligned} \widehat{M}_A &= \left\{ \sigma \in \Omega^{\mathbb{N}} : f(\sigma, \{j\}) = c_j, (c_1, \ldots, c_r) \in A \right\} \\ &= \left\{ \sigma \in \Omega^{\mathbb{N}} : \lim_{m \to \infty} \frac{\#\{1 \le \ell \le m : \sigma(\ell) = j\}}{m} = c_j, (c_1, \ldots, c_r) \in A \cap \Xi_r \right\}, \end{aligned}$$

and by Theorem 1.1

$$\dim_H M_A = \dim_P M_A = \sup \left\{ \frac{\sum_{\ell=1}^{r} p_\ell \log p_\ell}{\sum_{\ell=1}^{r} p_\ell \log a_\ell} : (p_\ell)_{\ell=1}^{r} \in A \cap \Xi_r \right\}.$$

In particular, for $\vec{c} = (c_1, \ldots, c_r) \in \Xi_r$

$$\dim_H M_{\{\vec{c}\}} = \dim_P M_{\{\vec{c}\}} = \frac{\sum_{\ell=1}^{r} c_\ell \log c_\ell}{\sum_{\ell=1}^{r} c_\ell \log a_\ell}. \quad \square \tag{7}$$

Case III: Γ_j, $1 \le j \le k$ are pairwise disjoint, i.e., a partition of Ω.

This is a more general case including the above two cases. Without loss of generality, we assume $\emptyset \neq A \subseteq \Xi_k$ since $M_{\{\vec{c}\}} = \emptyset$ for $\vec{c} \in A \setminus \Xi_k$. In this case, the dimension value t in (6) has an alternative expression by means of the method of Lagrange multipliers (also refer to the proof of Theorem 1.1 in Section 3. We formulate this case as a corollary.

Corollary 1.2. *Let Γ_j, $1 \le j \le k$ be a partition of Ω and A a nonempty subset of Ξ_k. Let M_A be defined as in (4). Then* $\dim_H M_A = \dim_P M_A = t = \sup_{(c_1,\ldots,c_k) \in A} x(c_1, \ldots, c_k)$*, where $x(c_1, \ldots, c_k)$ is the unique non-negative real root of the following equation in x,*

$$\sum_{j=1}^{k} c_j \log \sum_{i \in \Gamma_j} a_i^x - \sum_{j=1}^{k} c_j \log c_j = 0.$$

In particular, when $a_i = a$, $1 \le i \le r$

$$x(c_1, \dots, c_k) = \frac{\sum_{j=1}^{k} c_j(\log c_j - \log \# \Gamma_j)}{\log a}.$$

In particular, when A is a singleton this was given by [11, Theorem 1]. □

For $A \subseteq [0, 1]^k$ let

$$\Xi_A = \left\{ (p_1, \dots, p_r) \in \Xi_r : \left(\sum_{\ell \in \Gamma_1} p_\ell, \dots, \sum_{\ell \in \Gamma_k} p_\ell \right) \in A \right\} \subseteq \Xi_r.$$

It is easy to see that

$$M_A \supseteq \bigcup_{\vec{p} \in \Xi_A} M_{\{\vec{p}\}}$$

with the right side being composed of (in general, an uncountable number of) nonempty pairwise disjoint sets. We emphasize that the inclusion is proper in general since $\widehat{M}_A$ contains points for which not each digit frequency $f(\sigma, \{j\}), j \in \Omega$ is well-defined (this fact substantially complicates the problem of computing $\dim_H M_A$ and $\dim_P M_A$). From (7) and Theorem 1.1 it follows

$$\dim_H M_A = \dim_P M_A = \sup_{\vec{p} \in \Xi_A} \dim_H M_{\{\vec{p}\}} = \sup_{\vec{p} \in \Xi_A} \dim_P M_{\{\vec{p}\}}.$$

When A is compact (and then Ξ_A compact), the supremum can be reached. In this case, the dimension of M_A is carried by a single subset for which all the digit frequencies are known.

One can use equations or inequalities to produce the set A, e.g., take $A = \{(c_1, \dots, c_k) \in [0, 1]^k : c_1 = 2c_2, c_2 + c_3 \le 0.5\}$, or more general $A = \{(c_1, \dots, c_k) \in [0, 1]^k : g_i(c_1, \dots, c_k) = d_i \text{ for } i = 1, \dots, \ell; s_j \le h_j(c_1, \dots, c_k) \le t_j \text{ for} j = 1, \dots, n\}$. The main advantage of our approach is that we treat M_A in a unified manner (which could seem of different nature for each choices of g_is and h_js). Another advantage of this approach is that the values of the Hausdorff and packing dimensions do not need to be guessed a priori. In some works this a priori guess is crucial in order to construct auxiliary measures sitting on the set. These measures are then used to establish, rigorously, the values of the Hausdorff and packing dimensions.

As an application of Theorem 1.1, we consider two examples below. The first deals with a case where some frequencies $f(\sigma, \Gamma_i)$ have a linear relations, the second with a nonlinear relations. To simplify the computation, we only consider the case where Γ_is are pairwise disjoint.

Example 1. Let Γ_i, $1 \le i \le 4$ be disjoint nonempty subsets of Ω, α_1, α_2 positive numbers. Let $G = \phi(\widehat{G})$ where

$$\widehat{G} = \left\{ \sigma \in \Omega^{\mathbb{N}} : f(\sigma, \Gamma_j) = \alpha_j f(\sigma, \Gamma_{j+2}), j = 1, 2 \right\}.$$

Let $\Gamma_5 = \Omega \setminus \cup_{i=1}^{4} \Gamma_i$. When $\Gamma_5 \neq \emptyset$, Γ_i, $1 \le i \le 5$ is a partition of Ω. Let

$$A = \{(c_1, \dots, c_5) \in \Xi_5 : c_1 = \alpha_1 c_3 \text{ and } c_2 = \alpha_2 c_4\}.$$

Then $\widehat{G} = \widehat{M}_A$ and so $G = M_A$. Hence, Theorem 1.1 shows that $\dim_H G = \dim_P G = t$ with t determined by (6). By means of the method of Lagrange multipliers, we have that t is uniquely determined by

$$\sum_{j=1}^{2} (1 + \alpha_j) \alpha_j^{-\frac{\alpha_j}{1+\alpha_j}} \left(\sum_{\ell \in \Gamma_{j+2}} a_\ell^t \right)^{\frac{1}{1+\alpha_j}} \left(\sum_{\ell \in \Gamma_j} a_\ell^t \right)^{\frac{\alpha_j}{1+\alpha_j}} + \sum_{\ell \in \Gamma_5} a_\ell^t = 1.$$

In addition, if $\Gamma_5 = \emptyset$, t is then determined by

$$\sum_{j=1}^{2} (1 + \alpha_j) \alpha_j^{-\frac{\alpha_j}{1+\alpha_j}} \left(\sum_{\ell \in \Gamma_{j+2}} a_\ell^t \right)^{\frac{1}{1+\alpha_j}} \left(\sum_{\ell \in \Gamma_j} a_\ell^t \right)^{\frac{\alpha_j}{1+\alpha_j}} = 1.$$

Remark. We also can consider an easier case. Let Γ_1 and Γ_2 be disjoint nonempty subsets of Ω with $\Gamma_3 = \Omega \setminus (\Gamma_1 \cup \Gamma_2) \neq \emptyset$. For a positive number α let $G = \phi(\widehat{G})$ where

$$\widehat{G} = \left\{ \sigma \in \Omega^{\mathbb{N}} : f(\sigma, \Gamma_1) = \alpha f(\sigma, \Gamma_2) \right\}.$$

Then $\dim_H G = \dim_P G = t$ with t determined by

$$(1+\alpha)\alpha^{-\frac{\alpha}{1+\alpha}}\left(\sum_{\ell\in\Gamma_2} a_\ell^t\right)^{\frac{1}{1+\alpha}}\left(\sum_{\ell\in\Gamma_1} a_\ell^t\right)^{\frac{\alpha}{1+\alpha}}+\sum_{\ell\in\Gamma_3} a_\ell^t = 1. \tag{8}$$

Also t is determined by

$$(1+\alpha)\alpha^{-\frac{\alpha}{1+\alpha}}\left(\sum_{\ell\in\Gamma_2} a_\ell^t\right)^{\frac{1}{1+\alpha}}\left(\sum_{\ell\in\Gamma_1} a_\ell^t\right)^{\frac{\alpha}{1+\alpha}} = 1,$$

if $\Gamma_3 = \emptyset$.

Example 2. Let Γ_1 and Γ_2 be disjoint nonempty subsets of Ω such that $\Omega \setminus (\Gamma_1 \cup \Gamma_2) \neq \emptyset$. Let all a_i be equal, i.e., $a_i = a$, $i = 1, 2, \ldots, r$. Denote $\beta = \frac{\#\Gamma_2}{\#\Gamma_1}$. For $b > 1+\beta$, we consider the set $G = \phi(\widehat{G})$ where

$$\widehat{G} = \left\{\sigma \in \Omega^{\mathbb{N}} : bf(\sigma, \Gamma_1) = \mathrm{e}^{\beta - bf(\sigma,\Gamma_2)}\right\}.$$

Then

$$\dim_H G = \dim_P G = -\frac{1+\beta}{b}\log_a \#\Gamma_1 - \log_a b + \frac{b-1-\beta}{b}\log_a \frac{b-1-\beta}{r-\#\Gamma_1-\#\Gamma_2}.$$

Proof. Let $\Gamma_3 = \Omega \setminus (\Gamma_1 \cup \Gamma_2)$. Then Γ_i, $1 \le i \le 3$ is a partition of Ω. Let

$$A = \{(c_1, c_2, c_3) \in \Xi_3 : bc_1 = \mathrm{e}^{\beta - bc_2}\}.$$

Then $\widehat{G} = \widehat{M}_A$ and so $G = M_A$. Define $h(x) = \mathrm{e}^{\beta - bx}/b$. It follows from Corollary 1.2 that

$$\dim_H G = \dim_P G = \sup_{(c_1,c_2,c_3)\in A} \frac{\sum_{j=1}^{3} c_j(\log c_j - \log \#\Gamma_j)}{\log a} = \sup_{c_2\in B} \frac{\sum_{j=1}^{3} c_j(\log c_j - \log \#\Gamma_j)}{\log a}$$

where $B = \{c_2 \in [0, 1] : c_2 + h(c_2) \le 1\}$, $c_1 = h(c_2)$ and $c_3 = 1 - h(c_2) - c_2$. Let

$$H(c_2) = h(c_2)(\log h(c_2) - \log \#\Gamma_1) + c_2(\log c_2 - \log \#\Gamma_2) + (1 - h(c_2) - c_2)(\log(1 - h(c_2) - c_2) - \log \#\Gamma_3).$$

It is thus enough to determine the infimum of the function $H(c_2)$ on B. Note that

$$H'(c_2) = h'(c_2)(\log h(c_2) - \log \#\Gamma_1) + \log c_2 - \log \#\Gamma_2 - (1 + h'(c_2))(\log(1 - h(c_2) - c_2) - \log \#\Gamma_3).$$

It is easy to check that $H'(c_2) = 0$ for $c_2 = \frac{\beta}{b} \in \mathrm{int}(B)$. Some elementary calculus shows that $H(\frac{\beta}{b})$ is indeed a global minimum of H. Therefore,

$$\dim_H G = \dim_P G = \frac{H\left(\frac{\beta}{b}\right)}{\log a} = -\frac{1+\beta}{b}\log_a \#\Gamma_1 - \log_a b + \frac{b-1-\beta}{b}\log_a \frac{b-1-\beta}{r-\#\Gamma_1-\#\Gamma_2},$$

as desired. □

Finally, we notice a series of works on this general topics, e.g., by Barreira et al. [2–5] and Olsen et al. [1,14,15] etc., which are established in the framework of dynamical systems by using the thermodynamic formalism. Some of them concern the similar structure to that we are considering in the present paper. For example, we can regard $[0, 1]$ as a self-similar set (so a Moran fractal) created by the IFS $\{r^{-1}(x+k), k = 0, 1, \ldots, r-1\}$ with $2 \le r \in \mathbb{N}$ (accordingly all $a_i = 1/r$ in the definition of Moran fractal). In this case, (7) then gives [2, Corollary 12], the classical result of Eggleston in [7]. In addition, by taking both Γ_1 and Γ_2 as singletons, the result, shown by (8), in the remark of the above example 1 then coincides with [2, Theorem 2]; the result in the above Example 2 coincides with [2, Corollary 16] (we also obtain the packing dimensions of the related sets). Also some other results, such as Corollaries 13–15 in [2], can be deduced from Theorem 1.1 in the same way. Since Moran fractals considered in the present paper, in general case, cannot be dynamically defined, one hardly studies these kind of sets by means of thermodynamic formalism. This forces us to employ the initial definitions of Hausdorff and packing dimensions to establish our results.

The rest of this paper is arranged as follows. In Section 2, we give two lemmas which appeared in the previous papers of the authors and will be used for the proof of Theorem 1.1. The last section is mainly devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section, we give some preliminary observations. We first recall an equivalent definition of packing dimension for subsets of Moran fractals which allows us to use the collection of component sets as packings instead of the collection of balls centered at the subsets considered.

Let E be a subset of the Moran fractal F defined by (1). A collection $\{J_\sigma : \sigma \in \mathcal{A}\}$ with $\mathcal{A} \subseteq \Omega^*$ is said a *J-type* δ*-packing* of E if $|J_\sigma| \le \delta$ for each $\sigma \in \mathcal{A}, J_\sigma \cap E \neq \emptyset$ and $\operatorname{int} J_\sigma \cap \operatorname{int} J_\tau = \emptyset$ for distinct $\sigma, \tau \in \mathcal{A}$. For $s \ge 0$ we define a packing-type premeasure $\widetilde{\mathcal{P}}_0^s$ on subsets of the Moran fractal F by letting $\widetilde{\mathcal{P}}_0^s(E) = \lim_{\delta\downarrow 0} \widetilde{\mathcal{P}}_\delta^s(E)$ for $E \subseteq F$, where for each $\delta > 0$

$$\widetilde{\mathcal{P}}_\delta^s(E) = \sup\left\{\sum_{\sigma\in\mathcal{A}} |J_\sigma|^s : \{J_\sigma\}_{\sigma\in\mathcal{A}} \text{ is a } J\text{-type } \delta\text{-packing of } E\right\}.$$

Then a (outer) measure on subsets of F can be derived from $\widetilde{\mathcal{P}}_0^s$ in a routine way:

$$\begin{aligned}\widetilde{\mathcal{P}}^s(E) &= \inf\left\{\sum_{i=1}^\infty \widetilde{\mathcal{P}}_0^s(E_i) : E \subseteq \bigcup_{i=1}^\infty E_i, E_i \subseteq F\right\}\\ &= \inf\left\{\sum_{i=1}^\infty \widetilde{\mathcal{P}}_0^s(E_i) : E = \bigcup_{i=1}^\infty E_i, E_i \subseteq F\right\}.\end{aligned}$$

By $\mathcal{P}^s$ we denote the s-dimensional packing measure (Readers can refer to [9,13] for its definition). It is easy to show that there exists a positive number b such that $\mathcal{P}^s(E) \le b\widetilde{\mathcal{P}}^s(E)$ for any $E \subseteq F$. But the (outer) measures $\widetilde{\mathcal{P}}^s$ and $\mathcal{P}^s$ may not be equivalent (the equivalence means there are positive constants c_1, c_2 such that $c_1\widetilde{\mathcal{P}}^s(E) \le \mathcal{P}^s(E) \le c_2\widetilde{\mathcal{P}}^s(E)$ for any $E \subseteq F$.) However, they do determine the same critical index.

Lemma 2.1 *([10, Theorem 1.1]). Let $E \subset F$. Then*

$$\overline{\dim}_B E = \inf\left\{s : \widetilde{\mathcal{P}}_0^s(E) = 0\right\} = \sup\left\{s : \widetilde{\mathcal{P}}_0^s(E) = +\infty\right\},$$

and

$$\dim_P E = \inf\left\{s : \widetilde{\mathcal{P}}^s(E) = 0\right\} = \sup\left\{s : \widetilde{\mathcal{P}}^s(E) = +\infty\right\}.$$

Let Λ_j, $1 \le j \le n$ be a partition of Ω, i.e., $\Omega = \cup_{j=1}^n \Lambda_j$ with disjoint union. For $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Xi_n$ (a probability vector), $m \in \mathbb{N}$ and $\epsilon > 0$ let

$$\begin{aligned}\Omega_{\vec{\lambda}}(m,\epsilon) &\overset{\text{def}}{=} \{\sigma \in \Omega^{\mathbb{N}} : \lambda_j - \epsilon \le f(\sigma, \Lambda_j, m) \le \lambda_j + \epsilon, 1 \le j \le n\}\\ &= \{\sigma \in \Omega^{\mathbb{N}} : (\lambda_j - \epsilon)m \le \#\{1 \le \ell \le m : \sigma(\ell) \in \Lambda_j\} \le (\lambda_j + \epsilon)m, 1 \le j \le n\}.\end{aligned} \tag{9}$$

Then $\{J_{\sigma|m} : \sigma \in \Omega_{\vec{\lambda}}(m,\epsilon)\}$ is a sub-collection of all mth level component sets of F. The following lemma essentially comes from [11, Theorem 1] which describes an important property of these sub-collections. For readers' convenience, we give its proof in detail.

Lemma 2.2. *Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Xi_n$ be a probability vector. Suppose that the positive number s satisfies*

$$\prod_{j=1}^n \lambda_j^{\lambda_j} > \prod_{j=1}^n \left(\sum_{i\in\Lambda_j} a_i^s\right)^{\lambda_j}, \tag{10}$$

where we adopt the convention $0^0 = 1$. Then there exists an $\epsilon_0 = \epsilon_0(s, \vec{\lambda})$ such that for any $0 < \epsilon \le \epsilon_0$

$$\sum_{m=m_0}^\infty \sum_{\sigma\in\Omega_{\vec{\lambda}}(m,\epsilon)} |J_{\sigma|m}|^s \to 0, \quad \text{as } m_0 \to +\infty,$$

where $\Omega_{\vec{\lambda}}(m,\epsilon)$ is defined by (9).

Proof. Since some of λ_js may be zeros, for simplicity we, without loss of generality, assume that $\lambda_j > 0$ for $1 \le j \le n_1$ and $\lambda_j = 0$ for $j > n_1$, where $n_1 \le n$. Let

$$K = \frac{r + (r+1)\max\limits_{1\le j\le n_1} \lambda_j}{(r-1)\min\limits_{1\le j\le n_1} \lambda_j} \cdot \max\left\{\frac{\sum_{i\in\Lambda_{j_2}} a_i^s}{\sum_{i\in\Lambda_{j_1}} a_i^s}, 1 \le j_1, j_2 \le n\right\}. \tag{11}$$

For $\epsilon > 0$ (more exactly, we require that ϵ satisfies the following (12), take

$$C_*(\epsilon) = \left(1 - \frac{(n-n_1)\epsilon}{\lambda_{n_1}}\right)^{-\lambda_{n_1}} (\lambda_{n_1} - (n-n_1)\epsilon)^{(n-n_1)\epsilon},$$

and

$$C_{**}(\epsilon) = \left(\sum_{i\in\Lambda_{n_1}} a_i^s\right)^{-(n-n_1)\epsilon}.$$

Clearly, $\lim_{\epsilon\downarrow 0} C_*(\epsilon) = \lim_{\epsilon\downarrow 0} C_{**}(\epsilon) = 1$. Let $\epsilon_0 = \epsilon_0(s, \vec{\lambda})$ be such that for $0 < \epsilon \le \epsilon_0$

$$0 < \epsilon < \frac{1}{r} \min_{1\le j\le n_1} \lambda_j \tag{12}$$

and

$$C_*(\epsilon)C_{**}(\epsilon) \prod_{j=1}^{n_1} \left(\frac{\sum_{i\in\Lambda_j} a_i^s}{\lambda_j}\right)^{\lambda_j} \cdot \prod_{j=n_1+1}^{n} \left(\frac{\sum_{i\in\Lambda_j} a_i^s}{\epsilon}\right)^{\epsilon} \cdot K^{2n\epsilon} < K_* < 1, \tag{13}$$

where K_* is a number satisfying (we remind the readers of (10).

$$\prod_{j=1}^{n_1} \left(\frac{\sum_{i\in\Lambda_j} a_i^s}{\lambda_j}\right)^{\lambda_j} < K_* < 1.$$

In the following we fix an ϵ with $0 < \epsilon \le \epsilon_0$. Now we have

$$\sum_{\sigma\in\Omega_{\vec{\lambda}}(m,\epsilon)} |J_{\sigma|m}|^s = |J_\emptyset|^s \sum \frac{m!}{\prod_{j=1}^{n} d_j!} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Gamma_j} a_i^s\right)^{d_j}, \tag{14}$$

where the summation is over sets of non-negative integers d_j, $1 \le j \le n$ such that

$$\sum_{j=1}^{n} d_j = m \quad \text{and} \quad m(\lambda_j - \epsilon) \le d_j \le m(\lambda_j + \epsilon), \quad 1 \le j \le n. \tag{15}$$

Let Q_0 be the term of the sum (14) for which d_j is the integer part of $m\lambda_j$ for $1 \le j \le n_1 - 1$, the integer part of $m\epsilon$ for $n_1 + 1 \le j \le n$. Therefore there are real numbers $0 \le \delta_j < 1$ with $j \ne n_1$ such that $d_j = m\lambda_j - \delta_j$ for $1 \le j \le n_1 - 1$, $d_j = m\epsilon - \delta_j$ for $n_1 + 1 \le j \le n$ (In fact, all δ_js are equal, so are all d_js, for $n_1 + 1 \le j \le n$), and $d_{n_1} = m - \sum_{j\ne n_1} d_j = m\lambda_{n_1} - (n-n_1)m\epsilon + \sum_{j\ne n_1} \delta_j = m\lambda_{n_1} - \delta_{n_1}$ with $\delta_{n_1} = (n-n_1)m\epsilon - \sum_{j\ne n_1} \delta_j$. Note that when m is big enough, we have

$$\frac{\left(\lambda_j - \frac{\delta_j}{m}\right)^{m\lambda_j - \delta_j + 1/2}}{\lambda_j^{m\lambda_j}} = \left(1 - \frac{\delta_j}{m\lambda_j}\right)^{m\lambda_j} \left(\lambda_j - \frac{\delta_j}{m}\right)^{-\delta_j + 1/2} > \text{constant} > 0, \quad 1 \le j < n_1,$$

$$\frac{\left(\epsilon - \frac{\delta_j}{m}\right)^{m\epsilon - \delta_j + 1/2}}{\epsilon^{m\epsilon}} = \left(1 - \frac{\delta_j}{m\epsilon}\right)^{m\epsilon} \left(\epsilon - \frac{\delta_j}{m}\right)^{-\delta_j + 1/2} > \text{constant} > 0, \quad n_1 < j \le n,$$

and

$$\begin{aligned}\frac{\left(\lambda_{n_1} - \frac{\delta_{n_1}}{m}\right)^{m\lambda_{n_1} - \delta_{n_1} + 1/2}}{\lambda_{n_1}^{m\lambda_{n_1}}} &= \left(1 - \frac{\delta_{n_1}}{m\lambda_{n_1}}\right)^{m\lambda_{n_1}} \left(\lambda_{n_1} - \frac{\delta_{n_1}}{m}\right)^{-\delta_{n_1} + 1/2} \\ &= \left(1 - \frac{(n-n_1)m\epsilon - \sum_{j\ne n_1} \delta_j}{m\lambda_{n_1}}\right)^{m\lambda_{n_1}} \left(\lambda_{n_1} - \frac{(n-n_1)m\epsilon - \sum_{j\ne n_1} \delta_j}{m}\right)^{-(n-n_1)m\epsilon + \sum_{j\ne n_1} \delta_j + 1/2}\end{aligned}$$

$$\geq \text{ constant } \left[\left(1 - \frac{(n-n_1)\epsilon}{\lambda_{n_1}}\right)^{\lambda_{n_1}} (\lambda_{n_1} - (n-n_1)\epsilon)^{-(n-n_1)\epsilon} \right]^m$$
$$= \text{constant } C_*(\epsilon)^{-m} > 0.$$

Then by means of Stirling formula $n! = \sqrt{2\pi n}(n/e)^n e^{\frac{\theta}{12n}}$ $(0 < \theta < 1)$, we have that when m is big enough

$$\begin{aligned}
Q_0 &= \frac{m!}{\prod_{j=1}^{n} d_j!} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} = \frac{\sqrt{2\pi}\, m^{m+\frac{1}{2}} e^{-m} e^{\frac{\theta}{12m}}}{\prod_{j=1}^{n} \sqrt{2\pi}\, d_j^{d_j+\frac{1}{2}} e^{-d_j} e^{\frac{\theta_j}{12 d_j}}} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} \\
&\leq K_1 \frac{m^{m+\frac{1}{2}}}{\prod_{j=1}^{n} d_j^{d_j+\frac{1}{2}}} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} \\
&= K_1 \frac{m^{m+\frac{1}{2}}}{\prod_{j=1}^{n_1} (m\lambda_j - \delta_j)^{d_j+\frac{1}{2}} \prod_{j=n_1+1}^{n} (m\epsilon - \delta_j)^{d_j+\frac{1}{2}}} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} \\
&= K_1 \frac{m^{-\frac{1}{2}(n-1)}}{\prod_{j=1}^{n_1} \left(\lambda_j - \frac{\delta_j}{m}\right)^{m\lambda_j - \delta_j + 1/2} \prod_{j=n_1+1}^{n} \left(\epsilon - \frac{\delta_j}{m}\right)^{m\epsilon - \delta_j + 1/2}} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} \\
&\leq K_2 \frac{C_*(\epsilon)^m m^{-\frac{1}{2}(n-1)}}{\prod_{j=1}^{n_1} \lambda_j^{m\lambda_j} \prod_{j=n_1+1}^{n} \epsilon^{m\epsilon}} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} \\
&\leq K_3 \cdot m^{-\frac{1}{2}(n-1)} \cdot C_*(\epsilon)^m C_{**}(\epsilon)^m \prod_{j=1}^{n_1} \left(\frac{\sum_{i\in\Lambda_j} a_i^s}{\lambda_j}\right)^{m\lambda_j} \cdot \prod_{j=n_1+1}^{n} \left(\frac{\sum_{i\in\Lambda_j} a_i^s}{\epsilon}\right)^{m\epsilon},
\end{aligned} \tag{16}$$

where K_1, K_2 and K_3 are appropriate positive constants. Note that for $1 \leq i \leq n$, $1 \leq j \leq n_1$,

$$\frac{d_i + 1}{d_j} \leq \frac{m(\lambda_i + \epsilon) + 1}{m(\lambda_j - \epsilon)} \leq \frac{r + (r+1) \max_{1\leq j\leq n_1} \lambda_j}{(r-1) \min_{1\leq j\leq n_1} \lambda_j}, \tag{17}$$

by (12) and (15). Let

$$Q_1 = \frac{m!}{\prod_{j=1}^{n} d_j!} \cdot \prod_{j=1}^{n} \left(\sum_{i\in\Lambda_j} a_i^s\right)^{d_j} \tag{18}$$

and Q_2 be the term of the sum (14) for which in (18) the d_{j_1} and d_{j_2}, for some $j_1 \neq j_2$ with $1 \leq j_1 \leq n$, $1 \leq j_2 \leq n_1$, are replaced by $d_{j_1} + 1$ and $d_{j_2} - 1$ respectively and the other d_j's are kept fixed. Thus

$$\frac{Q_1}{Q_2} = \frac{d_{j_1} + 1}{d_{j_2}} \cdot \frac{\sum_{i\in\Lambda_{j_2}} a_i^s}{\sum_{i\in\Lambda_{j_1}} a_i^s} \leq K, \tag{19}$$

by (11) and (17). Since (19) implies that for any term Q of the sum (14)

$$\frac{Q}{Q_0} < K^{2nm\epsilon}, \tag{20}$$

it follows that when m is big enough

$$\begin{aligned}\sum_{\sigma\in\Omega_{\vec\lambda}^{-}(m,\epsilon)}|J_{\sigma|m}|^s &= |J_\emptyset|^s\sum\frac{m!}{\prod_{j=1}^n d_j!}\cdot\prod_{j=1}^n\left(\sum_{i\in\Lambda_j}a_i^s\right)^{d_j}\\ &< Q_0\cdot K^{2nm\epsilon}\cdot(2m\epsilon)^n\cdot|J_\emptyset|^s\\ &< K_4\cdot m^{\frac{n+1}{2}}\cdot\left[C_*(\epsilon)C_{**}(\epsilon)\prod_{j=1}^{n_1}\left(\frac{\sum_{i\in\Lambda_j}a_i^s}{\sum_{i\in\Lambda_j}a_i^{t_0}}\right)^{\lambda_j}\cdot\prod_{j=n_1+1}^{n}\left(\frac{\sum_{i\in\Lambda_j}a_i^s}{\epsilon}\right)^{\epsilon}\cdot K^{2n\epsilon}\right]^m\\ &\le K_4\cdot m^{\frac{n+1}{2}}K_*^m\end{aligned}$$

by (16), (20) and (13), where K_4 is an appropriate positive constant. Therefore, for any fixed $0<\epsilon\le\epsilon_0$ we

$$\sum_{m=m_0}^{\infty}\sum_{\sigma\in\Omega_{\vec\lambda}^{-}(m,\epsilon)}|J_{\sigma|m}|^s\le K_4\sum_{m=m_0}^{\infty}\cdot m^{\frac{n+1}{2}}K_*^m\to 0,\qquad\text{as } m_0\to+\infty.$$

This completes the proof. □

3. Proofs

In this section, we give the proof of Theorem 1.1. We choose a partition, say $\Lambda_j, j=1,\dots,n$, of Ω such that each Λ_j is contained in some Γ_i. For example, the Λ_is can be taken as the join of the partitions $\{\Gamma_i,\Gamma_i^c\}$, $i=1,2,\dots,k$, i.e.,

$$\{\Lambda_1,\Lambda_2,\dots,\Lambda_n\}:=\{A_1\cap A_2\cap\cdots\cap A_k\ne\emptyset : A_i\in\{\Gamma_i,\Gamma_i^c\}\}.$$

Set $\mathcal{I}_i=\{j:\Lambda_j\subseteq\Gamma_i, 1\le j\le n\}$, $1\le i\le k$. Then $\Gamma_i=\bigcup_{j\in\mathcal{I}_i}\Lambda_j$ and $\{\Lambda_j:j\in\mathcal{I}_i\}$ is a partition of Γ_i. Let

$$H\overset{\text{def}}{=}\left\{\vec\lambda=(\lambda_1,\dots,\lambda_n)\in\Xi_n:\left(\sum_{j\in\mathcal{I}_1}\lambda_j,\dots,\sum_{j\in\mathcal{I}_k}\lambda_j\right)\in A\right\}. \tag{21}$$

In (4), with A and Γ_js replaced, respectively, by H and Λ_is we can define $\widehat{M}_H$ in the same way. Clearly, $\widehat{M}_H\ne\emptyset$ if and only if $H\ne\emptyset$ (just simply by Law of Large Number). We claim that $\widehat{M}_A\ne\emptyset$ if and only if $H\ne\emptyset$. The sufficiency is clear since $\widehat{M}_A\supseteq\widehat{M}_H$ (in general, the inclusion is proper). On the other hand, take $\sigma\in\widehat{M}_A(\ne\emptyset)$ and denote $\lambda_j^{(m)}=f(\sigma,\Lambda_j,m)$, $j=1,2,\dots,n$. Then

$$(\lambda_1^{(m)},\dots,\lambda_n^{(m)})\in\Xi_n\quad\text{and}\quad\lim_{m\to\infty}\sum_{j\in\mathcal{I}_i}\lambda_j^{(m)}=\lim_{m\to\infty}f(\sigma,\Gamma_i,m)=c_i,\quad i=1,2,\dots,k,$$

where $(c_1,\dots,c_k)\in A$. Thus, for any $\epsilon>0$ the compact set $L_\epsilon:=\{(\lambda_1,\dots,\lambda_n)\in\Xi_n : c_i-\epsilon\le\sum_{j\in\mathcal{I}_i}\lambda_j\le c_i+\epsilon, 1\le i\le k\}$ is not empty. Therefore, $\cap_{\epsilon>0}L_\epsilon=\{(\lambda_1,\dots,\lambda_n)\in\Xi_n:\sum_{j\in\mathcal{I}_i}\lambda_j=c_i, 1\le i\le k\}$ is not empty. For each $(\lambda_1,\dots,\lambda_n)\in H$, let

$$Z(x)=\sum_{j=1}^n\lambda_j\log\sum_{i\in\Lambda_j}a_i^x-\sum_{j=1}^n\lambda_j\log\lambda_j, \tag{22}$$

with the convention $0\log 0=0$, as before. It is easy to see that the function $Z(x)$ has a unique zero, denoted by $x(\lambda_1,\dots,\lambda_n)$, in $[0,\xi]$ where ξ is defined by $\sum_{i=1}^r a_i^\xi=1$, since $Z(x)$ is strictly decreasing with $Z(0)\ge 0$ and $Z(\xi)\le 0$. This allows us to define a continuous function $x(\lambda_1,\dots,\lambda_n)$ on H by the implicit function theorem. Denote

$$x^*=\sup_{(\lambda_1,\dots,\lambda_n)\in H}x(\lambda_1,\dots,\lambda_n). \tag{23}$$

The supremum can be reached when H is compact. Then, for $s>x^*$ we have

$$\prod_{j=1}^n\lambda_j^{\lambda_j}>\prod_{j=1}^n\left(\sum_{i\in\Lambda_j}a_i^s\right)^{\lambda_j}\qquad\text{for all }(\lambda_1,\dots,\lambda_n)\in H. \tag{24}$$

Lemma 3.1. *Let H and x^* be defined as in (21) and (23), respectively. Let H be compact. Then for any $s>x^*$ there exist finite probability vectors $\vec\lambda_i=(\lambda_1^{(i)},\dots,\lambda_n^{(i)})\in H$, $i=1,2,\dots,\hbar$ such that*

$$\widehat{M}_A \subseteq \bigcup_{m_0=1}^{\infty} \bigcap_{m=m_0}^{\infty} \bigcup_{i=1}^{\hbar} \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i)),$$

where $\epsilon_0(s, \vec{\lambda})$ is given in Lemma 2.2 for $\vec{\lambda} \in H$.

Proof. Since $\left\{ \times_{j=1}^{n} (\lambda_j - \epsilon_0(s, \vec{\lambda}), \lambda_j + \epsilon_0(s, \vec{\lambda})) : \vec{\lambda} \in H \right\}$ is an open covering of H, we can choose finite probability vectors, say $\vec{\lambda}_i = (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \in H$, $i = 1, 2, \ldots, \hbar$ such that

$$H^* \stackrel{\text{def}}{=} \bigcup_{i=1}^{\hbar} \left(\mathop{\times}_{j=1}^{n} (\lambda_j^{(i)} - \epsilon_0(s, \lambda_j^{(i)}), \lambda_j^{(i)} + \epsilon_0(s, \lambda_j^{(i)})) \right) \supseteq H.$$

For each $p \in \mathbb{N}$ and each $\vec{c} = (c_1, \ldots, c_k) \in A$, consider equations

$$\begin{cases} \lambda_j \geq 0 \\ \sum_{j=1}^{n} \lambda_j = 1 \\ c_i - \dfrac{1}{p} \leq \sum_{j \in \mathcal{I}_i} \lambda_j \leq c_i + \dfrac{1}{p}, \quad 1 \leq i \leq k. \end{cases} \tag{25}$$

By $T_{p,\vec{c}}$ we denote the set of solutions to (25). Then $H \subseteq \bigcup_{\vec{c} \in A} T_{p,\vec{c}}$. Note that the distance between H and $(H^*)^c$ is positive. Therefore, there exists a p_0 such that for $p \geq p_0$

$$H \subseteq \bigcup_{\vec{c} \in A} T_{p,\vec{c}} \subseteq H^*. \tag{26}$$

In the following, we fix such a p. Now we need to prove that for each $\sigma \in \widehat{M}_A$ there exists an $m_0(\sigma)$ such that

$$\sigma \in \bigcup_{i=1}^{\hbar} \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i)) \quad \text{for } m \geq m_0(\sigma). \tag{27}$$

For a given $\sigma \in \widehat{M}_A$, there exists a $\vec{c} = (c_1, \ldots, c_k) \in A$ such that

$$f(\sigma, \Gamma_j) = c_j, \quad 1 \leq j \leq k.$$

Thus, there exists an $m_0(\sigma)$ such that for all $m \geq m_0(\sigma)$

$$c_j - \frac{1}{p} < \sum_{i \in \mathcal{I}_j} f(\sigma, \Lambda_i, m) < c_j + \frac{1}{p}, \quad 1 \leq j \leq k. \tag{28}$$

So $(f(\sigma, \Lambda_i, m))_{i=1}^{n} \in T_{p,\vec{c}}$, leading to (27) by (25), (26), (28) and (9). □

Remark. In fact, a stronger result holds, i.e., $\widehat{M}_A \subseteq \bigcup_{m_0=q}^{\infty} \bigcap_{m=m_0}^{\infty} \bigcup_{i=1}^{\hbar} \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i))$ for any positive integer q. We only need to choose $m_0(\sigma) \geq q$ for each $\sigma \in \widehat{M}_A$ in the above proof.

Lemma 3.2. *Let x^* be defined as in (23). If A is compact, then* $\dim_P M_A \leq x^*$.

Proof. It suffices to show $\dim_P M_A \leq s$ for any $s > x^*$. Note that H is compact by (21) and the compactness of A. Now for a fixed $s > x^*$, by Lemma 3.1 there exist $\hbar$ probability vectors $\vec{\lambda}_i = (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \in H$, $1 \leq i \leq \hbar$ such that

$$\widehat{M}_A \subseteq \bigcup_{m_0=1}^{\infty} \bigcap_{m=m_0}^{\infty} \bigcup_{i=1}^{\hbar} \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i)).$$

Denote $G_{m_0} = \bigcap_{m=m_0}^{\infty} \bigcup_{i=1}^{\hbar} \bigcup_{\tau \in \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i))} J_{\tau|m}$. Then for each $x \in G_{m_0}$ there exists a $\sigma \in \Omega^{\mathbb{N}}$ (one of its location codes) satisfying: for each $m \geq m_0$ there exists $1 \leq i \leq \hbar$ such that

$$\lambda_j^{(i)} - \epsilon_0(s, \vec{\lambda}_i) \leq f(\sigma, \Lambda_j, m) \leq \lambda_j^{(i)} + \epsilon_0(s, \vec{\lambda}_i) \quad \text{for all } 1 \leq j \leq n.$$

Therefore, G_{m_0} is increasing and $M_A \subseteq \bigcup_{m_0=1}^{\infty} G_{m_0}$. So we only need to prove $\dim_P G_{m_0} \leq s$ for each m_0. Let

$$\mathcal{J}_m \stackrel{\text{def}}{=} \{J_{\tau|m} : \tau \in \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i)), i = 1, \ldots, \hbar\}, \quad m \in \mathbb{N}.$$

Then for any $h \geq m_0$, $G_{m_0} \subseteq \bigcup_{J \in \mathcal{J}_h} J$. Moreover, $\mathcal{J}_{\geq h} \stackrel{\text{def}}{=} \{J \in \mathcal{J}_m : m \geq h\}$ is a Vitali covering of G_{m_0}.

Now we fix m_0 and consider the J-type packing of G_{m_0}. For $h \in \mathbb{N}$, denote $\delta_h = a_{\min}^h |J_\emptyset|$. Let $\mathcal{J} \overset{\text{def}}{=} \{J_\sigma : \sigma \in \mathcal{A}\}$ (recall $\mathcal{A} \subseteq \Omega^*$) be a J-type δ_h-packing of G_{m_0} with $h \geq m_0$. Note that $|\sigma| \geq h$ for each $\sigma \in \mathcal{A}$. We classify the elements of $\mathcal{J}$ into two classes, denoted by $\mathcal{J}_1$ and $\mathcal{J}_2$. For a $J_\sigma \in \mathcal{J}$, if $J_\sigma \in \mathcal{G}_{\geq h}$ then put it in $\mathcal{J}_1$, otherwise put it in $\mathcal{J}_2$. We extend the collection $\mathcal{G}_{\geq h}$ so that it contains all elements of $\mathcal{J}_2$. For each $J_{\tau|m} \in \mathcal{G}_{\geq h}$, let $\widetilde{\mathcal{J}}_{\tau|m}$ be the collection of component sets intersecting $J_{\tau|m}$ with diameter lying in $(|J_{\tau|m+1}|, |J_{\tau|m}|]$. Then

(P1) $J_{\tau|m} \in \widetilde{\mathcal{J}}_{\tau|m}$;

(P2) $\#\widetilde{\mathcal{J}}_{\tau|m} \leq \xi^*$ where the positive number ξ^* is independent of the choice of $J_{\tau|m}$. This can be seen by the discussion in Section 1, or by means of [9, Lemma 9.2].

The extended collection, denoted by $\widetilde{\mathcal{G}}_{\geq h}$, of $\mathcal{G}_{\geq h}$ is defined as the union of all $\widetilde{\mathcal{J}}_{\tau|m}$. We claim that $\mathcal{J}_2 \subseteq \widetilde{\mathcal{G}}_{\geq h}$. For each fixed $J_\sigma \in \mathcal{J}_2$, take an $x \in J_\sigma \cap G_{m_0}$(this intersection is not empty from the definition of a packing). For this x, it has a location code τ lying in $\bigcap_{m=m_0}^{\infty} \bigcup_{i=1}^{\hbar} \Omega_{\vec{\lambda}_i}(m, \epsilon_0(s, \vec{\lambda}_i))$. Since $J_{\tau|m} \in \mathcal{G}_{\geq h}$ for all $m \geq h$, there exists a unique $n \geq h$ such that $|J_\sigma| \in (|J_{\tau|n+1}|, |J_{\tau|n}|]$, leading to $J_\sigma \in \widetilde{\mathcal{J}}_{\tau|n}$ (note that $J_\sigma \cap J_{\tau|n} \supseteq \{x\}$ is not empty). Thus

$$\sum_{\sigma\in\mathcal{A}} |J_\sigma|^s \leq \sum_{J\in\widetilde{\mathcal{G}}_{\geq h}} |J|^s \leq \xi \sum_{J\in\mathcal{G}_{\geq h}} |J|^s = \xi \sum_{i=1}^{\hbar}\sum_{m=h}^{\infty} \sum_{\sigma\in\Omega_{\vec{\lambda}_i}(m,\epsilon_0(s,\vec{\lambda}_i))} \left|J_{\sigma|m}\right|^s .$$

By Lemma 2.2 and (24), we have $\widetilde{\mathcal{P}}^s_{\delta_h}(G_{m_0}) \leq 1$ when h is big enough. Thus

$$\widetilde{\mathcal{P}}^s(G_{m_0}) \leq \widetilde{\mathcal{P}}^s_0(G_{m_0}) \leq 1,$$

yielding $\dim_P G_{m_0} \leq s$ by Lemma 2.1. □

Proof of Theorem 1.1. We divide the proof into two cases.

Case 1. A is compact.

We first show $\dim_H M_A \geq t$. Without loss of generality, we suppose $t > 0$. Since A is compact, the supremum in (6) can be reached at some probability vector, say at $\vec{p} = (\bar{p}_1, \ldots, \bar{p}_r)$. Without loss of generality, we assume that $\bar{p}_i > 0$ for $1 \leq i \leq k_1$ and $\bar{p}_i = 0$ for $k_1 + 1 \leq i \leq r$, where $k_1 \leq r$. Denote $\Omega_1 = \{1, \ldots, k_1\} \subseteq \Omega$. Let $M = \phi(\widehat{M})$ where

$$\widehat{M} = \{\sigma \in \Omega_1^{\mathbb{N}} : f(\sigma, \{i\}) = \bar{p}_i, 1 \leq i \leq k_1\}. \tag{29}$$

Then $\widehat{M} \subseteq \widehat{M}_A$ and so $M \subseteq M_A$. Construct a probability measure $\hat{\mu}$ on $\Omega_1^{\mathbb{N}}$ by defining for $\sigma \in \Omega_1^m$

$$\hat{\mu}(C(\sigma)) = \prod_{i=1}^{m} \bar{p}_{\sigma(i)},$$

where, as before, $C(\sigma) = \{\tau \in \Omega_1^{\mathbb{N}} : \tau|m = \sigma\}$ is the cylinder set with the base σ. Let μ on F be the image measure of $\hat{\mu}$ under ϕ. By Birkhoff's Ergodic theorem (cf. [18]) or Law of Large Number, we have for $\hat{\mu} - a.e.\sigma \in \Omega_1^{\mathbb{N}}$

$$f(\sigma, \{i\}) = \bar{p}_i, \quad 1 \leq i \leq k_1.$$

Therefore, $\hat{\mu}(\widehat{M}) = \mu(M) = 1$. Now for $\sigma \in \Omega_1^*$, write $a(\sigma) = \prod_{\ell=1}^{|\sigma|} a_{\sigma(\ell)}$ and $\bar{p}(\sigma) = \prod_{\ell=1}^{|\sigma|} \bar{p}_{\sigma(\ell)}$. Then for any $\sigma \in \widehat{M}$, it is easy to verify that

$$\lim_{n\to\infty} \frac{\log \bar{p}(\sigma|n)}{\log a(\sigma|n)} = \frac{\sum_{\ell=1}^{k_1} \bar{p}_\ell \log \bar{p}_\ell}{\sum_{\ell=1}^{k_1} \bar{p}_\ell \log a_\ell} = \frac{\sum_{\ell=1}^{r} \bar{p}_\ell \log \bar{p}_\ell}{\sum_{\ell=1}^{r} \bar{p}_\ell \log a_\ell} = t.$$

Fix $0 < \epsilon < t$. Let $M^{(m)} = \phi(\widehat{M}^{(m)})$ where

$$\widehat{M}^{(m)} = \left\{\sigma \in \widehat{M} : \frac{\log \bar{p}(\sigma|n)}{\log a(\sigma|n)} > t - \epsilon \text{ for all } n \geq m\right\}. \tag{30}$$

Then

$$1 = \hat{\mu}(\widehat{M}) = \lim_{m\to\infty} \hat{\mu}(\widehat{M}^{(m)}) \quad \text{and} \quad 1 = \mu(M) = \lim_{m\to\infty} \mu(M^{(m)}).$$

We fix an m such that $\mu(M^{(m)}) > 0$. Let $\hat{\mu}_m$ be the restriction of $\hat{\mu}$ to $M^{(m)}$ and let μ_m be the induced measure on $M^{(m)}$ of $\hat{\mu}_m$ by ϕ, i.e., for any Borel set $A \subseteq M^{(m)}$

$$\mu_m(A) = \hat{\mu}_m(\phi^{-1}(A)) = \hat{\mu}(\phi^{-1}(A) \cap \widehat{M}^{(m)}).$$

By $B_R(x)$ we denote the closed ball with center at x and radius R. Let $0 < R < a_{\min}^m$. For each $\sigma \in \widehat{M}^{(m)}$ there exists a positive integer $h(\sigma, R)$ such that

$$a_{\min}R < a(\sigma|h(\sigma, R)) \le R. \tag{31}$$

Note that $h(\sigma, R) > m$ and write $W = \{\sigma|h(\sigma, R) : \sigma \in \widehat{M}^{(m)}\}$. For any fixed $x \in M^{(m)}$ let $W^* = \{\tau \in W : J_\tau \cap B_R(x) \cap M^{(m)} \ne \emptyset\}$. Then there exists a finite positive constant ξ_1 independent of the R and x such that $\#W^* \le \xi_1$ by [9, Lemma 9.2]. So

$$\mu_m(B_R(x)) \le \hat{\mu}_m\left(\bigcup_{\tau \in W^*} C(\tau)\right) \le \sum_{\tau \in W^*} \hat{\mu}(C(\tau)) = \sum_{\tau \in W^*} \bar{p}(\tau) \le \xi R^{t-\epsilon},$$

by (30) and (31). So we get

$$\liminf_{R \to 0} \frac{\log \mu_m(B_R(x))}{\log R} \ge t - \epsilon.$$

By [17, Theorem 1] or [8, Proposition 2.3(a)]

$$\dim_H M_A \ge \dim_H M \ge \dim_H M^{(m)} \ge t - \epsilon,$$

which implies $\dim_H M_A \ge t$ if letting $\epsilon \to 0$.

Now we turn to show $\dim_P M_A \le t$. By Lemma 3.2 we only need to check $t = x^*$. Suppose x^* is reached at $\vec{\lambda} = (\lambda_1^*, \ldots, \lambda_n^*) \in H$ (note that H is compact in this case), i.e., $x^* = x(\lambda_1^*, \ldots, \lambda_n^*)$. By (22),

$$\sum_{j=1}^n \lambda_j^* \log \sum_{i \in \Lambda_j} a_i^{x^*} - \sum_{j=1}^n \lambda_j^* \log \lambda_j^* = 0.$$

Recall $\Lambda_j, j = 1, \ldots, n$ is a partition of Ω. For each $\ell \in \Omega$, take

$$p_\ell = (a_\ell^{x^*} \lambda_j^*) / \sum_{m \in \Lambda_j} a_m^{x^*} \quad \text{if } \ell \in \Lambda_j.$$

Then $(p_1, \ldots, p_r)$ is a probability vector. Note that $\Gamma_i = \bigcup_{j \in \mathcal{I}_i} \Lambda_j, i = 1, 2, \ldots, k$. Thus

$$\sum_{\ell \in \Gamma_i} p_\ell = \sum_{j \in \mathcal{I}_i} \sum_{\ell \in \Lambda_j} p_\ell = \sum_{j \in \mathcal{I}_i} \lambda_j^* \quad \text{for } i = 1, 2, \ldots, k,$$

which gives that $\left(\sum_{\ell \in \Gamma_1} p_\ell, \ldots, \sum_{\ell \in \Gamma_k} p_\ell\right) = \left(\sum_{j \in \mathcal{I}_1} \lambda_j^*, \ldots, \sum_{j \in \mathcal{I}_k} \lambda_j^*\right) \in A$ by (21). However,

$$\begin{aligned}\sum_{\ell=1}^r p_\ell \log p_\ell &= x^* \sum_{\ell=1}^r p_\ell \log a_\ell + \sum_{j=1}^n \sum_{\ell \in \Lambda_j} p_\ell \left(\log \lambda_j^* - \log \sum_{m \in \Lambda_j} a_m^{x^*}\right) \\ &= x^* \sum_{\ell=1}^r p_\ell \log a_\ell + \sum_{j=1}^n \lambda_j^* \left(\log \lambda_j^* - \log \sum_{m \in \Lambda_j} a_m^{x^*}\right) \\ &= x^* \sum_{\ell=1}^r p_\ell \log a_\ell,\end{aligned}$$

implying that $x^* \le t$ by (6). The opposite inequality is direct since we have proved $t \le \dim_H M_A \le \dim_P M_A \le x^*$.

Case 2. A is not compact.

By $\bar{A}$ we denote the closure of A. The following fact is obvious:

$$\begin{aligned} t &= \sup\left\{\frac{\sum_{\ell=1}^r p_\ell \log p_\ell}{\sum_{\ell=1}^r p_\ell \log a_\ell} : \left(\sum_{\ell \in \Gamma_1} p_\ell, \ldots, \sum_{\ell \in \Gamma_k} p_\ell\right) \in A, p_\ell \ge 0 \text{ and } \sum_{\ell=1}^r p_\ell = 1\right\} \\ &= \sup\left\{\frac{\sum_{\ell=1}^r p_\ell \log p_\ell}{\sum_{\ell=1}^r p_\ell \log a_\ell} : \left(\sum_{\ell \in \Gamma_1} p_\ell, \ldots, \sum_{\ell \in \Gamma_k} p_\ell\right) \in \bar{A}, p_\ell \ge 0 \text{ and } \sum_{\ell=1}^r p_\ell = 1\right\}.\end{aligned}$$

Thus, $\dim_H M_A \le \dim_P M_A \le \dim_P M_{\overline{A}} = t$ by Case 1. On the other hand, for any $0 < \epsilon < t$ (we disdain considering the case $t = 0$), we take the probability vector $(p_1, \ldots, p_r)$ in the definition of t in (6) such that

$$\frac{\sum_{\ell=1}^{r} p_\ell \log p_\ell}{\sum_{\ell=1}^{r} p_\ell \log a_\ell} \ge t - \epsilon.$$

As done in (29), we can define a subset $M \subseteq M_A$ such that $\dim_H M \ge t - \epsilon$. □

Acknowledgments

The authors would like to thank Prof. P. Mattila for his valuable comments.

The first author was supported by the National Natural Science Foundation of China #10571058 and Shanghai Leading Academic Discipline Project #B407. The second author was supported by the Ph.D. Program Scholarship Fund of ECNU 2008.

References

[1] I.S. Baek, L. Olsen, N. Snigireva, Divergence points of self-similar measures and packing dimension, Adv. Math. 214 (2007) 267–287.
[2] L. Barreira, B. Saussol, J. Schmeling, Distribution of frequencies of digits via multifractal analysis, J, Number Theory 97 (2002) 410–438.
[3] L. Barreira, B. Saussol, Variational principles and mixed multifractal spectra, Trans. Amer. Math. Soc. 353 (2001) 3919–3944.
[4] L. Barreira, B. Saussol, J. Schmeling, Higher-dimensional multifractal analysis, J. Math. Pures Appl. 81 (2002) 67–91.
[5] L. Barreira, J. Schmeling, Sets of "non-typical" points have full topological entropy and full Hausdorff dimension, Israel J. Math. 116 (2000) 29–70.
[6] R. Cawley, R.D. Mauldin, Multifractal decomposition of Moran fractals, Adv. Math. 92 (1992) 196–236.
[7] H.G. Eggleston, The fractional dimension of a set defined by decimal properties, Quart. Journ. of Math. (Oxford) 20 (March) (1949) 31–36.
[8] K.J. Falconer, Techniques in Fractal Geometry, John Wiley & Sons Ltd, 1997.
[9] K.J. Falconer, Fractal Geometry–Mathematical Foundations and Applications, John Wiley & Sons Ltd, Chichester, 1990.
[10] W.X. Li, An equivalent definition of packing dimension and its application, Nonlinear Anal. RWA, in press (doi:10.1016/j.nonrwa.2008.02.004).
[11] W.X. Li, F.M. Dekking, Hausdorff dimension of subsets of Moran fractals with prescribed group frequency of their codings, Nonlinearity 16 (2003) 187–199.
[12] W.X. Li, L. Olsen, Z.Y. Wen, Hausdorff and packing dimensions of subsets of Moran fractals with prescribed mixed group frequency of their codings, Aequationes Math. (in press).
[13] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge University press, 1995.
[14] L. Olsen, Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages. IV: Divergence points and packing dimension, preprint..
[15] L. Olsen, S. Winter, Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages. II: Non-linearity, divergence points and Banach space valued spectra, Bull. Sci. Math. 131 (2007) 518–558.
[16] D.W. Spear, Measure and self-similarity, Adv. Math. 91 (1992) 143–157.
[17] C. Tricot, Two definitions of fractional dimension, Math. Proc. Cambridge Philos. Soc. 91 (1982) 57–74.
[18] P. Walters, An Introduction to Ergodic Theory in: GTM, vol. 79, Springer-Verlag.
[19] Z.Y. Wen, Moran sets and Moran classes, Chinese Sci. Bull. 46 (22) (2001) 1849–1856.