The dimension of subsets of Moran sets determined by the success run behaviour of their codings

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Abstract

Let $F \subset \mathbf{R}^n$ be a Moran set associated with the set $\{0 < a_j < 1, j = 0, 1, \cdots, r\}$. Let Γ be a non-empty subset of $\{0, 1, 2, \cdots, r\}$ with non-empty complement. Associated with the behaviour of success run of symbols from Γ in the coding space $\{0, 1, \cdots, r\}^{\mathbf{N}}$ is a decomposition of F such that

$$F = \bigcup_{t \in [0, +\infty]} F_t.$$

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Depending on F this might be a partition of F or almost a partition of F in the sense that $\sup_{x\in F} \#\{t : x \in F_t\} < +\infty$. We prove that each F_t is dense in F, and $\dim_H F_t = \dim_P F_t = \dim_B F_t = \dim_H F = \dim_P F = \dim_B F = s$ with $\sum_{j=0}^{r} a_j{}^s = 1$. For $\mathcal{L}^1$-a.e. $t \in [0, +\infty]$, $\mathcal{H}^s(F_t) = 0$ and F_t is an s-set when $t = -(\log \sum_{j\in\Gamma} a_j^s)^{-1}$. Moreover, associated with this decomposition $\{F_t : t \in [0, +\infty]\}$ of F is a measurable function Y such that each F_t is a level set of Y. The fractal dimensions of the graph of Y are also determined.

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1 Introduction

A basic task in Fractal geometry is to determine or estimate the various dimensions of fractal sets. Fractal dimensions are introduced to measure the sizes of fractal sets and are employed in many different disciplines. Unfortunately, it is very difficult to determine the exact fractal dimensions of general fractal sets. Many results on fractal dimensions are obtained for fractal sets with a special structure. Among them is a typical fractal structure called Moran set or Moran fractal. In order to describe Moran fractals let us introduce the following notations. Denote $\Omega = \{0, 1, \cdots, r\}$. Here r is a positive integer.

(i) $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \cdots) : \sigma(i) \in \Omega\}$;

(ii) $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) : \sigma(i) \in \Omega\}$ for $k \in \mathbf{N}$, and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$;

(iii) $|\cdot|$ is used to denote the length of word. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \cdots, \sigma(|\sigma|), \tau(1), \cdots, \tau(|\tau|))$. For any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$ write $\tau * \sigma = (\tau(1), \cdots, \tau(|\tau|), \sigma(1), \sigma(2), \cdots)$;

(iv) $\sigma|k = (\sigma(1), \sigma(2), \cdots, \sigma(k))$ for $\sigma \in \Omega^\omega$ and $k \in \mathbf{N}$;

(v) For $\sigma \in \Omega^k$, the cylinder set $C(\sigma)$ with base σ is defined as $C(\sigma) = \{\tau \in \Omega^\omega : \tau|k = \sigma\}$ for $k \in \mathbf{N}$;

(vi) Let $h_j : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, $0 \le j \le r$. Denote $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \Omega^k$ and $x \in \mathbf{R}^n$.

Fixing a nonempty compact set $J \subset \mathbf{R}^n$ with $\overline{\text{int}J} = J$ and positive real numbers $0 < a_j < 1, j = 0, 1, \cdots, r$, the related Moran set (or Moran fractal) is defined in the following way.

Step 1. For each $\sigma \in \Omega^k, k \in \mathbf{N}$, construct a compact set $J_\sigma \subset \mathbf{R}^n$ by induction:

• A family $\{J_j : j \in \Omega\}$ of nonoverlapping nonempty compact subsets of J is chosen such that $\overline{\text{int}J_j}{=}J_j$ and $|J_j| = a_j|J|$ where $|\cdot|$ denotes the diameter of set.

• Suppose that J_σ is given for some $\sigma \in \Omega^k$. Take a family $\{J_{\sigma*j} : j \in \Omega\}$ of nonoverlapping nonempty compact subsets of J_σ such that $\overline{\text{int}J_{\sigma*j}} = J_{\sigma*j}$, $|J_{\sigma*j}| = a_j|J_\sigma|$ and $J_{\sigma*j}$ contains an open ball of diameter $c|J_{\sigma*j}|$ where c is a positive constant independent of $\sigma * j$.

Step 2. The Moran fractal F associated with $\{0 < a_j < 1, j \in \Omega\}$ and the $J_\sigma, \sigma \in \Omega^*$ is defined as the nonempty compact set

$$F = \bigcap_{k=1}^{\infty} \bigcup_{\sigma\in\Omega^k} J_\sigma. \tag{1}$$

We shall refer to J_σ as a k-th level component set of F if $\sigma \in \Omega^k$. Define ϕ: $\Omega^\omega \longrightarrow \mathbf{R}^n$ by

$$\{\phi(\sigma)\} = \bigcap_{k=1}^{\infty} J_{\sigma|k}. \tag{2}$$

It is easy to see that $\phi(\Omega^\omega) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (1). But ϕ may not be an injection. An important property of ϕ is that there is positive constant c_1, independent of $x \in F$, such that

$$\sup_{x\in F} \#\{\phi^{-1}(x)\} < c_1 \tag{3}$$

by means of Lemma 9.2 in [5]. Let ρ be the usual metric on Ω^ω defined by

$$\rho(\sigma, \tau) = 2^{-\min\{i:\sigma(i)\neq\tau(i)\}},$$

with the convention $\rho(\sigma, \sigma) = 0$. Let F be equiped with the Euclidean metric. Then ϕ is continuous. Thus each $x \in F$ can be encoded via ϕ: $\sigma \in \Omega^\omega$ is called a location code of $x \in F$ if $\phi(\sigma) = x$ (Note that there are multiple location codes for some $x \in F$). Therefore ϕ is also called the coding map and Ω^ω is called the code space (or symbolic space).

Some comments about Moran fractals are listed below.

(C1) The two crucial requirements in Step 1 (nonoverlapping and open ball condition – an analogue of the open set condition for self-similar sets) make the structure of the Moran set simpler, close to that of self-similar sets and lead to the exclusion of the non-self-similar self-affine sets (ref. [2],[7],[13],[16],[17] and [18],etc.).

(C2) The Moran fractals considered here are more general than those defined in [3] where $J_{\sigma * i}$ is geometrically similar to J_σ and [19] where the open balls are required to be centered at the points on F.

(C3) A Moran fractal is termed map-specified if there exist similitude contractions $h_j, j = 0, 1, \cdots, r$, such that $J_\sigma = h_\sigma(J)$ for any $\sigma \in \Omega^*$. In this case F is actually the self-similar set determined by $\{h_j : j \in \Omega\}$, which satisfies the open set condition with respect to the open set $O = \mathrm{int} J$ (i.e. $\bigcup_{j=0}^r h_j(O) \subset O$ with a disjoint union on the left) and the coding map ϕ in (2) can be changed into

$$\{\phi(\sigma)\} = \bigcap_{k=1}^{\infty} h_{\sigma|k}(\overline{O}) = \{\lim_{k\to\infty} h_{\sigma|k}(0)\}. \tag{4}$$

(C4) Moran fractals F are regular fractals in the sense that it has been proven that $\dim_H F = \dim_P F = \dim_B F = s$ and F is an $s-$set with $\sum_{j\in\Omega} a_j{}^s = 1$ (see [12], [14], [15] & [19]).

(C5) The more general Moran fractal structure proposed by Zhiying Wen can be produced by a similar method where the code space $\Omega^\omega = \prod_{i=1}^{\infty} \Omega_i$ and corresponding to different Ω_i there are different scaling coefficients $\{0 < a_{i,j} < 1, j = 0, 1, \cdots, r_i\}$.

Some fractal dimension results of this generalized Moran sets can be found in [8],[9], [10],[11] and [14], etc. The class of generalized Moran sets clearly contains the class of Moran sets, and in fact is far larger than the class of Moran sets, since a generalized Moran set often has different fractal dimensions and is not an $s-$set.

For any $E \subseteq F$ there exists $\Lambda \subset \Omega^\omega$ such that $E = \phi(\Lambda)$. For certain Λ, it is possible to determine the fractal dimensions of the projections $E = \phi(\Lambda)$. Up to now solutions which depend on the structure of Λ are obtained at least in the following cases:

(S1) Let $D = (d_{i,j})$ be a $(r+1) \times (r+1)$ matrix with entries 0 and 1, and $\Lambda = \{\sigma \in \Omega^\omega : d_{\sigma(k),\sigma(k+1)} = 1 \text{ for all } k \in \mathbf{N}\}$ (see [1], [15] & [19]);

(S2) $\Lambda = \prod_{i=1}^{\infty} \Omega_i$, where Ω_i is a non-empty subset of Ω for $i = 1, 2, \cdots$ (see [10] & [11]);

(S3) Let F be map-specified and Λ compact and invariant under the left-shift (see [7]).

Now let $\Gamma \subseteq \Omega = \{0, 1, \cdots, r\}$ be non-empty such that $\Gamma^c \neq \emptyset$. For $n \in \mathbf{N}$ and $\sigma \in \Omega^\omega$, we define $N_n(\sigma)$, the length of the Γ-run starting at n, by

$$N_n(\sigma) = \begin{cases} 0, & \text{if } \sigma(n) \notin \Gamma \\ k, & \text{if } \sigma(n) \in \Gamma, \sigma(n+1) \in \Gamma, \cdots, \sigma(n+k-1) \in \Gamma \text{ and } \sigma(n+k) \notin \Gamma \\ +\infty, & \text{if } \sigma(k) \in \Gamma \text{ for } k \geq n. \end{cases}$$

Let $(Q(n))$ be a sequence of real numbers such that $Q(n) \uparrow +\infty$ and $Q(n) = o(n)$ as $n \longrightarrow +\infty$. Let

$$\Lambda = \left\{ \sigma \in \Omega^\omega : \limsup_{n \to \infty} \frac{N_n(\sigma)}{Q(n)} = 1 \right\}.$$

The Hausdorff dimension $\dim_H(\cdot)$, box dimension $\dim_B(\cdot)$ and packing dimension $\dim_P(\cdot)$ of the projection $M = \phi(\Lambda)$ will be determined in Section 2.

Throughout this paper, log denotes the logarithm to base e. Now let us recall a probability result attributed to D.J. Newman (see [4], p.61):

Proposition *In a sequence of independent Bernoulli trials* $\{X_n\}$ *with success probability* $p \in (0,1)$*, define* $\widetilde{N}_n$ *to be the length of the maximal success run commencing at trial* n*, that is*

$$\{\widetilde{N}_n = j\} = \{X_{n+j} = 0, X_i = 1, n \leq i < n+j\}, \quad j \geq 0.$$

Then

$$P\left\{\limsup_{n\to\infty} \frac{\log p}{\log n} \widetilde{N}_n = -1\right\} = 1.$$

Associated with this proposition a class of special subsets F_t, $t \in [0, +\infty]$, of F can be defined by letting

$$\Lambda_t = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{N_n(\sigma)}{\log n} = t\right\} \text{ and } F_t = \phi(\Lambda_t). \tag{5}$$

The following properties of F_t will be proved in Section 2:

(P1) $F = \bigcup_{t\in[0,+\infty]} F_t$ with $\sup_{x\in F} \#\{t : x \in F_t\} < +\infty$;

(P2) Each F_t is dense in F;

(P3) $\dim_H F_t = \dim_P F_t = \dim_B F_t = s$ where s is defined by $\sum_{j\in\Omega} {a_j}^s = 1$;

(P4) Moreover, if F is map-specified with similitudes $h_j, j = 0, 1, \cdots, r$, i.e., a self-similar set, then each F_t is invariant under $\{h_j : j = 0, 1, \cdots, r\}$, i.e.,

$$F_t = \bigcup_{j=0}^{r} h_j(F_t).$$

On the other hand, the F_t can be considered as the level sets of the function $Y(\sigma) = \limsup_{n\to\infty} \frac{N_n(\sigma)}{\log n}$ defined on the probability space $(\Omega^\omega, \mathcal{G}^\omega, P)$. This will be discussed in Section 3 and the following results will be obtained (let s be as in P3):

(P5) For $\mathcal{L}^1$-a.e. $t \in [0, +\infty]$, $\mathcal{H}^s(F_t) = 0$, but when $t = \hat{t} = -(\log \sum_{j\in\Gamma} a_j^s)^{-1}$, $\mathcal{H}^s(F_{\hat{t}}) > 0$, from this it follows directly that $F_{\hat{t}}$ is an s-set;

(P6) Let G be the graph of Y. Then $\dim_H G = \dim_P G = \dim_B G = 1 + s$.

2 Dimension of subsets of Moran sets

In this section, a general dimension result on a class of subsets of Moran sets is first obtained. Then this result will be applied to give a decomposition of Moran sets. The following proposition will be employed.

Proposition 2.1 *Let* $M = \phi(\prod_{i=1}^{\infty} \Omega_i)$ *where the* Ω_i *are non-empty subsets of* $\Omega = \{0, 1, \cdots, r\}$, $i \in \mathbf{N}$. *Let* $d(k)$ *be such that*

$$\prod_{i=1}^{k} (\sum_{j \in \Omega_i} a_j^{d(k)}) = 1.$$

Then $\dim_H M = \liminf_{k\to\infty} d(k) \triangleq \underline{d}$ *(see [10]).*

Proof This result can be found in [10] for the more general Moran fractal structure (see (C5)). Here a simplified proof is given for this special case. For any $d > \underline{d}$, there exists a sequence $\{n_k : k = 1, 2, \cdots\}$ such that $d(n_k) < d$. For any $\delta > 0$, we can take k large enough such that $\{J_\sigma : \sigma \in \prod_{i=1}^{n_k} \Omega_i\}$ is a δ-covering of M. Then

$$\mathcal{H}_\delta^d(M) \le \sum_{\sigma \in \prod_{i=1}^{n_k} \Omega_i} |J_\sigma|^d = |J|^d \prod_{i=1}^{n_k} (\sum_{j \in \Omega_i} a_j^d) \le |J|^d \prod_{i=1}^{n_k} (\sum_{j \in \Omega_i} a_j^{d(n_k)}) = |J|^d,$$

which implies that $\dim_H M \le d$. So we have $\dim_H M \le \underline{d}$.

Now we turn to prove that $\dim_H M \ge \underline{d}$. Let us suppose without loss of generality that $\underline{d} > 0$. Then we only need to prove that for any fixed $0 < d < \underline{d}$, $\dim_H M \ge d$. Let us construct a probability measure $\widetilde{\mu}$ on $S^\omega \triangleq \prod_{i=1}^{\infty} \Omega_i$ such that for any $\sigma \in S^k \triangleq \prod_{i=1}^{k} \Omega_i, k = 1, 2, \cdots$

$$\widetilde{\mu}(C^*(\sigma)) = \frac{\prod_{i=1}^{k} a_{\sigma(i)}^d}{\prod_{i=1}^{k} (\sum_{j \in \Omega_i} a_j^d)},$$

where $C^*(\sigma) \triangleq \{\theta \in S^\omega : \theta|k = \sigma\}$ is a cylinder in S^ω with base σ. Let μ be the image measure under ϕ defined in (2) and restricted to S^ω here. For $\epsilon > 0$, write $S_\epsilon = \{\sigma \in \bigcup_{k=1}^{\infty} S^k : a_\sigma \triangleq \prod_{i=1}^{|\sigma|} a_{\sigma(i)} \le \epsilon$ and $a_{\sigma|(|\sigma|-1)} > \epsilon\}$. For any $x \in M$, $\epsilon > 0$, by $B(x, \epsilon)$ we denote the closed ball with center at x and radius ϵ. Let $L_{x,\epsilon} \triangleq \{\sigma \in S_\epsilon : J_\sigma \bigcap B(x,\epsilon) \neq \emptyset\}$. Since $M = \bigcup_{\sigma \in S_\epsilon}(M \bigcap J_\sigma)$ and $J_\sigma,\ \sigma \in S_\epsilon$ are pairwise nonoverlapping, there exists a finite constant c independent of x, ϵ such that $1 \leq \# L_{x,\epsilon} \leq c$. Thus

$$\mu(B(x,\epsilon)) \leq \sum_{\sigma \in L_{x,\epsilon}} \widetilde{\mu}(C^*(\sigma)) = \sum_{\sigma \in L_{x,\epsilon}} \frac{\prod_{k=1}^{|\sigma|} a_{\sigma(k)}^d}{\prod_{i=1}^{|\sigma|}(\sum_{j\in\Omega_i} a_j^d)} \leq c\epsilon^d,$$

where for the last inequality we use that $\prod_{i=1}^{|\sigma|}(\sum_{j\in\Omega_i} a_j^d) \geq 1$ when ϵ is small enough, since $d < \underline{d}$. Then by Frostman's lemma (see [5]) we obtain $\mathcal{H}^d(M) > 0$ which implies that $\dim_H M \geq d$. **QED**

Remark 2.2 *One can show furthermore that* $\overline{\dim}_B M = \dim_P M = \limsup_{k\to\infty} d(k)$ *(see* [11]*), and that* $0 < \mathcal{H}^{\underline{d}}(M) < +\infty$ *if and only if* $0 < \liminf_{k\to\infty} \prod_{i=1}^k (\sum_{j\in\Omega_i} a_j^{\underline{d}}) < +\infty$ *(see* [10] & [14]*).*

Theorem 2.3 *Suppose* $Q(n) \uparrow +\infty$ *and* $Q(n) = o(n)$ *as* $n \longrightarrow +\infty$*. Let* s *be such that* $\sum_{j\in\Omega} a_j^s = 1$*. Let* $M = \phi(\Lambda)$ *where*

$$\Lambda = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{N_n(\sigma)}{Q(n)} = 1\right\}.$$

Then $\dim_H M = \dim_P M = \dim_B M = s$*, and* M *is dense in* F*.*

Proof It suffices to prove $\dim_H M \geq s$ since $\dim_H M \leq \dim_P M \leq s$ and $\dim_H M \leq \underline{\dim}_B M \leq \overline{\dim}_B M \leq s$ always hold. In the following, we shall complete our proof by proving that for any given $0 < d < s$ there exists a subset $E = E_d$ of M such that $\dim_H E \geq d$.

Let the constant c be defined by

$$c \triangleq \max\{\log \#\Gamma, \log \#\Gamma^c, |\log \sum_{j\in\Gamma} a_j^s|, |\log \sum_{j\in\Gamma^c} a_j^s|\}. \tag{6}$$

Consider the non-negative strictly decreasing function

$$G(x) = \log \sum_{j\in\Omega} a_j^x, \quad 0 \leq x \leq s.$$

Clearly $G(x) \downarrow 0$ as $x \uparrow s$. Now let ϵ be defined by $\frac{6c\epsilon}{1-\epsilon} = G(d)$. Then the solutions of the inequality

$$0 \leq G(x) = \log \sum_{j \in \Omega} a_j^x \leq \frac{6c\epsilon}{1-\epsilon}$$

will lie in $[d, s]$.

Given any sequence of integers $0 < k_1 < u_1 < u_{1,1} < \cdots < u_{1,n_1} < k_2 < u_2 < \cdots < k_i < u_i < u_{i,1} < u_{i,2} < \cdots < u_{i,n_i} < k_{i+1} < u_{i+1} < \cdots$, we construct a set E as follows:

$$E = \{x \in F : x(k_i), x(u_i), x(u_{i,j}) \in \Gamma^c \text{ and } x(k) \in \Gamma \text{ for } k_i < k < u_i, \ \ i \geq 1, 1 \leq j \leq n_i\}.$$

Here for convenience we use $x(k)$ to denote the k-th component of a location code of $x \in F$. The set E is a closed subset of F. In the definition of E, when $k_i < k < u_i$, we call $x(k)$ a Γ-choice; when $k = k_i, u_i$ and $u_{i,j}$, $x(k)$ is called a Γ^c-choice; and the rest of the $x(k)$ are called Ω-choices.

By respectively $N_\Omega(k), N_\Gamma(k)$ and $N_{\Gamma^c}(k)$ we denote the total number of Ω-, Γ- and Γ^c-choices among the first k entries in a location code of a point in E . Thus we have that $N_\Omega(k) + N_\Gamma(k) + N_{\Gamma^c}(k) = k$ for $k \in \mathbf{N}$.

Note that for $k_i - 1 \leq k \leq u_i$, $N_\Omega(k) = N_\Omega(k_i - 1)$. For convenience we put

$$f_i = N_\Omega(k_i - 1).$$

Now define a sequence of positive integers b_i, $i \in \mathbf{N}$, by

$$b_{i+1} = b_i + [Q(b_i + 1)], \tag{7}$$

where $[Q(b_i+1)]$ is the largest integer less than $Q(b_i+1)$. Here we take b_1 large enough to ensure $Q(b_1 + 1) \geq 2$. So the b_i increase strictly and tend to $+\infty$. We take the sequence of positive integers $k_1, u_1, u_{1,1}, \cdots, u_{1,n_1}, k_2, u_2, \cdots$ in the definition of E as the sequence $b_1, b_2, \cdots$. Now for any $n \in \mathbf{N}$ there is an i with $b_i \leq n < b_{i+1}$. Thus for any $\sigma \in \Omega^\omega$ with $\phi(\sigma) \in E$, because $N_n(\sigma) \leq b_{i+1} - b_i - 1$, $N_{b_i}(\sigma) = 0$ and Q is non-decreasing we have

$$\limsup_{n\to\infty} \frac{N_n(\sigma)}{Q(n)} \leq \limsup_{i\to\infty} \frac{b_{i+1} - b_i - 1}{Q(b_i + 1)} = 1$$

by (7). On the other hand, for any $\sigma \in \Omega^\omega$ with $\phi(\sigma) \in E$ we have

$$\limsup_{i\to\infty} \frac{N_{k_i+1}(\sigma)}{Q(k_i+1)} = \limsup_{i\to\infty} \frac{u_i - k_i - 1}{Q(k_i+1)} = 1$$

by (7). Therefore $E \subseteq M$. Note that this holds for any choice of $n_1, n_2, \cdots$. We shall now make a choice for this sequence, based on the previously defined ϵ. Suppose that the n_ℓ are defined for $\ell = 1, 2, \cdots, i-1$, then also k_i and u_i are determined. Letting n_i vary, we have

$$\begin{aligned}
\lim_{n_i\to\infty} \frac{N_\Omega(u_{i,n_i})}{u_{i,n_i}} &= \lim_{n_i\to\infty} \frac{N_\Omega(u_i) + u_{i,n_i} - u_i - n_i}{u_{i,n_i}} \\
&= 1 - \lim_{n_i\to\infty} \frac{n_i}{u_{i,n_i}} = 1 - \lim_{n_i\to\infty} \frac{n_i - (n_i - 1)}{u_{i,n_i} - u_{i,n_i-1}} \\
&= 1 - \lim_{n_i\to\infty} \frac{1}{[Q(u_{i,n_i-1}+1)]} = 1.
\end{aligned}$$

Here we use Stolz's theorem for the third equality above. Therefore we can choose n_i such that

$$f_{i+1} = N_\Omega(k_{i+1} - 1) \geq (1-\epsilon)k_{i+1}. \tag{8}$$

According to Proposition 2.1 we have $\dim_H E = \liminf_{k\to\infty} d(k)$ where $d(k)$ satisfies

$$\Big(\sum_{j\in\Omega} a_j^{d(k)}\Big)^{N_\Omega(k)} \Big(\sum_{j\in\Gamma^c} a_j^{d(k)}\Big)^{N_{\Gamma^c}(k)} \Big(\sum_{j\in\Gamma} a_j^{d(k)}\Big)^{N_\Gamma(k)} = 1. \tag{9}$$

Taking logs in (9), and using $N_\Omega(k) + N_\Gamma(k) + N_{\Gamma^c}(k) = k$ we get

$$\log \sum_{j\in\Omega} a_j^{d(k)} = -\frac{N_{\Gamma^c}(k)}{N_\Omega(k)} \log \sum_{j\in\Gamma^c} a_j^{d(k)} - \frac{k - N_\Omega(k) - N_{\Gamma^c}(k)}{N_\Omega(k)} \log \sum_{j\in\Gamma} a_j^{d(k)}. \tag{10}$$

We shall show that there exist an i^* such that $d(k) \geq d$ when $k \geq k_{i^*}$. Then $\dim_H E \geq d$ by Proposition 2.1.

Now by the properties of $(Q(n))$ we can take i^* such that for all $i \geq i^*$

$$\frac{[Q(k_i+1)]}{k_i} \leq \epsilon \quad \text{and} \quad [Q(u_i+1)] - 1 > \frac{-\log \sum_{j\in\Gamma^c} a_j^d}{\log \sum_{j\in\Omega} a_j^d}. \tag{11}$$

We shall consider two cases for the k with $k \geq k_{i^*}$.

CASE 1. For some $i \geq i^*$ one has $k_i \leq k \leq u_i$.

In this case $N_\Omega(k) = N_\Omega(u_i) = f_i$ and hence the equality (10) can be written as

$$\log \sum_{j\in\Omega} a_j^{d(k)} = -\frac{N_{\Gamma^c}(k)}{f_i}(\log \sum_{j\in\Gamma^c} a_j^{d(k)} - \log \sum_{j\in\Gamma} a_j^{d(k)}) - \frac{k-f_i}{f_i} \log \sum_{j\in\Gamma} a_j^{d(k)}. \tag{12}$$

Note that

$$0 \leq \frac{N_{\Gamma^c}(k)}{f_i} = \frac{k - N_\Omega(k) - N_\Gamma(k)}{f_i} \leq \frac{k-f_i}{f_i}, \tag{13}$$

and

$$\begin{aligned} 0 \leq \frac{k-f_i}{f_i} &\leq \frac{u_i}{f_i} - 1 = \frac{k_i + [Q(k_i+1)]}{f_i} - 1 = \frac{k_i}{f_i}\left(1 + \frac{[Q(k_i+1)]}{k_i}\right) - 1 \\ &\leq \frac{1}{1-\epsilon}\left(1 + \frac{[Q(k_i+1)]}{k_i}\right) - 1 = \frac{\epsilon}{1-\epsilon} + \frac{1}{1-\epsilon} \cdot \frac{[Q(k_i+1)]}{k_i}, \end{aligned} \tag{14}$$

by (8). Note that $|\log \sum_{j\in\Gamma^c} a_j^{d(k)}| \leq c$, $|\log \sum_{j\in\Gamma} a_j^{d(k)}| \leq c$ by (6). Therefore when $i \geq i^*$ and $k_i \leq k \leq u_i$

$$\log \sum_{j\in\Omega} a_j^{d(k)} \leq \frac{2\epsilon}{1-\epsilon} \cdot 2c + \frac{2\epsilon}{1-\epsilon} \cdot c = \frac{6c\epsilon}{1-\epsilon},$$

by (12), (13), (14) and the first inequality of (11), which means that $d(k) \geq d$.

CASE 2. For some $i \geq i^*$ one has $u_i < k < k_{i+1}$.

First note that if $d(k) \geq d$ and $x(k+1)$ is a Ω-choice, then $N_\Omega(k+1) = N_\Omega(k) + 1$, and hence we see directly from (9) that also $d(k+1) \geq d$ since $\sum_{j\in\Omega} a_j^{d(k)} \geq 1$.

To finish Case 2, it suffices therefore to show that for each $k = u_{i,\ell}$, $\ell = 1, \cdots, n_i$ we have $d(u_{i,\ell}) \geq d$. Putting $u_{i,0} = u_i$, this will be done by showing that $d(u_{i,\ell}) \geq d$ implies $d(u_{i,\ell+1}) \geq d$ for $\ell = 0, \cdots, n_i - 1$ (by Case 1 we have already $u_{i,0} \geq d$). In fact, taking $k = u_{i,\ell}$ and $k = u_{i,\ell+1}$ in (9), we obtain

$$(\sum_{j\in\Omega} a_j^{d(u_{i,\ell})})^{N_\Omega(u_{i,\ell})} (\sum_{j\in\Gamma^c} a_j^{d(u_{i,\ell})})^{N_{\Gamma^c}(u_{i,\ell})} (\sum_{j\in\Gamma} a_j^{d(u_{i,\ell})})^{N_\Gamma(u_{i,\ell})} = 1, \tag{15}$$

and

$$(\sum_{j\in\Omega} a_j^{d(u_{i,\ell+1})})^{N_\Omega(u_{i,\ell+1})}(\sum_{j\in\Gamma^c} a_j^{d(u_{i,\ell+1})})^{N_{\Gamma^c}(u_{i,\ell+1})}(\sum_{j\in\Gamma} a_j^{d(u_{i,\ell+1})})^{N_\Gamma(u_{i,\ell+1})} = 1. \tag{16}$$

Now suppose that $d(u_{i,\ell+1}) < d$. Then, since $N_\Gamma(u_{i,\ell+1}) = N_\Gamma(u_{i,\ell})$, $N_{\Gamma^c}(u_{i,\ell+1}) = N_{\Gamma^c}(u_{i,\ell}) + 1$ and $N_\Omega(u_{i,\ell+1}) = N_\Omega(u_{i,\ell}) + [Q(u_{i,\ell}+1)] - 1$, it follows from (15) and (16) that

$$\left(\sum_{j\in\Omega} a_j^{d(u_{i,\ell+1})}\right)^{[Q(u_{i,\ell}+1)]-1}\left(\sum_{j\in\Gamma^c} a_j^{d(u_{i,\ell+1})}\right) < 1. \tag{17}$$

But since $Q(n)$ is increasing, and $\sum_{j\in\Omega} a_j^{d(u_{i,\ell+1})} > 1$, this implies

$$[Q(u_i+1)] - 1 \leq [Q(u_{i,\ell}+1)] - 1 \leq \frac{-\log\sum_{j\in\Gamma^c} a_j^{d(u_{i,\ell+1})}}{\log\sum_{j\in\Omega} a_j^{d(u_{i,\ell+1})}} \leq \frac{-\log\sum_{j\in\Gamma^c} a_j^d}{\log\sum_{j\in\Omega} a_j^d},$$

which is impossible by the second inequality of (11). So we must have $d(u_{i,\ell+1}) \geq d$, and we have finished the proof of Case 2. Hence we have obtained that $\dim_H E \geq d$, and completed the proof of the dimension result on M.

Finally, the density result is derived directly from the fact that if $\sigma \in \Lambda$ then for any $k \in \mathbf{N}$ those $\tau \in \Omega^\omega$ with $\tau(i) = \sigma(i)$, $i \geq k$ will lie in Λ. **QED**

As a Corollary to Theorem 2.3 we can now prove (P1), (P2), (P3) and (P4) of the previous section.

Proof (of (P1)–(P4)). By (3) and the definition (5) of Λ_t we have $\bigcup_{t\in[0,+\infty]}\Lambda_t = \Omega^\omega$ which leads to (P1). (P2) and (P3) can be directly derived from Theorem 2.3 by taking $Q(n) = t\log n$ for $t \in (0,+\infty)$, $Q(n) = \log\log n$ for $t = 0$ and $Q(n) = (\log n)^2$ for $t = +\infty$. Note that if $\sigma \in \Lambda_t$ then for any $j \in \Omega$ we have $j * \sigma \in \Lambda_t$ since

$$\limsup_{n\to\infty}\frac{N_n(j*\sigma)}{\log n} = \limsup_{n\to\infty}\left[\frac{N_{n-1}(\sigma)}{\log(n-1)}\cdot\frac{\log(n-1)}{\log n}\right] = t.$$

Thus $\Lambda_t = \bigcup_{j=0}^r j * \Lambda_t$ where $j * \Lambda_t = \{j * \sigma : \sigma \in \Lambda_t\}$, and we get (P4) by (4). **QED**

3 Level sets and Hausdorff measure

Let $(p_0, p_1, \cdots, p_r)$ be a probability vector with each $p_j > 0$ and let $(\Omega^\omega, \mathcal{G}^\omega, P)$ be the probability space so that $\mathcal{G}^\omega$ is the σ-algebra generated by all cylinder sets of Ω^ω and the probability measure P is defined for any cylinder set $C(\tau)$ by

$$P(C(\tau)) = \prod_{i=1}^{|\tau|} p_{\tau(i)}. \tag{18}$$

A sequence of independent Bernoulli trials $\{X_n\}$ on the probability space $(\Omega^\omega, \mathcal{G}^\omega, P)$ with success probability p can be defined by letting $\sum_{j\in\Gamma} p_j = p$ and $X_n(\sigma) = 1$, if $\sigma(n) \in \Gamma$ and $X_n(\sigma) = 0$, for $\sigma(n) \in \Gamma^c$. Therefore, taking $t = -\frac{1}{\log p}$ in (5) we have $P(\Lambda_t) = 1$ by the Proposition in Section 1. For each $t \in (0, +\infty)$, let $(p_0, p_1, \cdots, p_r)$ be any fixed probability vector such that $\sum_{j\in\Gamma} p_j = e^{-1/t}$ holds, then we define $\mu_t(B) = P(\phi^{-1}(B))$ for each Borel subset $B \subseteq F$. Then $\mu_t(F_t) = 1$ where F_t is defined in (5).

On the other hand, define random variables $Y(\sigma)$, $Y_n(\sigma), n \in \mathbf{N}$ on $(\Omega^\omega, \mathcal{G}^\omega, P)$ as follows:

$$Y_n(\sigma) = \frac{N_n(\sigma)}{\log n} \quad \text{and} \quad Y(\sigma) = \limsup_{n\to\infty} Y_n(\sigma).$$

We have $\Lambda_t = \{\sigma \in \Omega^\omega : Y(\sigma) = t\}$, i.e., Λ_t is a level set of $Y(\sigma)$, and so the projection $F_t = \phi(\Lambda_t)$ of Λ_t is, in some sense, a level set of $Y(\sigma)$.

We consider the set

$$G = \{(\phi(\sigma), Y(\sigma)) : \sigma \in \Omega^\omega\} \subseteq \mathbf{R}^n \times (\mathbf{R} \cup \{\infty\}). \tag{19}$$

If ϕ is invertible, then letting $x = \phi(\sigma)$ we have $G = \{(x, Y(\phi^{-1}(x)) : x \in F\}$, i.e., G is the graph of the function $Y(\phi^{-1}(x))$ on F. Note that $G = \bigcup_{t\in[0,+\infty]}\{(x,t) : x \in F_t\}$ and $\dim_H F_t = s$. We shall show that $\dim_H G = 1+s$. The following proposition can be found in Proposition 7.9 of [5] where the proof is given for $n = 2$, which can easily be extended to the general case.

Proposition 3.1 *Let H be a Borel subset of $\mathbf{R}^n$, $n \geq 2$. If $1 \leq t \leq n$, then*

$$\int_{-\infty}^{\infty} \mathcal{H}^{t-1}(H \bigcap L_x)\mathrm{d}x \leq (n-1)^{\frac{t-1}{2}} \mathcal{H}^t(H),$$

where L_x denotes the hyperplane $\{(x_1, \cdots, x_n) \in \mathbf{R}^n : x_n = x\}$. **QED**

Theorem 3.2 *Let G be as in* (19)*. Then* $\dim_H G = \dim_P G = \dim_B G = 1 + s$*.*

Proof We have $\dim_H G \geq 1 + s$ by Proposition 3.1 and Theorem 2.3. In fact, for any $1 \leq t < 1 + s$,

$$n^{\frac{t-1}{2}} \mathcal{H}^t(G) \geq \int_{-\infty}^{\infty} \mathcal{H}^{t-1}(G \bigcap L_x)\mathrm{d}x = \int_0^{\infty} \mathcal{H}^{t-1}(F_x)\mathrm{d}x = +\infty.$$

On the other hand, the product formula 7.5 in [5] gives

$$\overline{\dim}_B G \leq \overline{\dim}_B (F \times [0, +\infty]) \leq \overline{\dim}_B F + \overline{\dim}_B([0, +\infty]) = 1 + s. \text{ } \mathbf{QED}$$

Since F is an s-set, $\mathcal{H}^s(F_t) \leq \mathcal{H}^s(F) < +\infty$, $t \in [0, +\infty]$. We will consider the problem whether F_t is an s-set, i.e., whether or not $\mathcal{H}^s(F_t) > 0$. By $\mathcal{L}^1$ we denote the one-dimensional Lebesgue measure on the real line.

Theorem 3.3 *For $\mathcal{L}^1$-a.e. $t \in [0, +\infty]$, $\mathcal{H}^s(F_t) = 0$, but when $t = \hat{t} = -(\log \sum_{j \in \Gamma} a_j^s)^{-1}$, $\mathcal{H}^s(F_{\hat{t}}) > 0$, i.e., $F_{\hat{t}}$ is an s-set.*

Proof We first prove the second part. Consider the probability space $(\Omega^\omega, \mathcal{G}^\omega, P)$ where P is defined by (18) with $p_j = a_j^s, j \in \Omega$. By the definition (5) of $F_{\hat{t}}$ and the Proposition in Section 1, we get $P(\Lambda_{\hat{t}}) = 1$. Let $\widehat{M}_{\hat{t}} = \phi(\widehat{\Lambda}_{\hat{t}})$ where

$$\widehat{\Lambda}_{\hat{t}} = \left\{ \sigma \in \Lambda_{\hat{t}} : \lim_{l \to \infty} \frac{\#\{1 \leq k \leq l : \sigma(k) = j\}}{l} = a_j^s, \ j \in \Omega \right\}.$$

Then $\widehat{M}_{\hat{t}} \subseteq F_{\hat{t}}$. Since $P(\Lambda_{\hat{t}}) = 1$, we have $P(\widehat{\Lambda}_{\hat{t}}) = 1$ by the Law of Large Numbers or the Ergodic Theorem. Let $\mu_{\hat{t}}$ be the image of P under ϕ as before. By $B_r(x)$ we denote the closed ball with centre at x and radius r. Let r be given small enough. Then for each $\sigma \in \widehat{\Lambda}_{\hat{t}}$ there exists a positive integer $h(\sigma, r)$ such that

$$r \cdot \min_{j\in\Omega} a_j \leq a(\sigma|h(\sigma, r)) \leq r, \tag{20}$$

where $a(\sigma) \triangleq \prod_{k=1}^{|\sigma|} a_{\sigma(k)}$ for $\sigma \in \Omega^*$. Let $W = \{\sigma|h(\sigma, r) : \sigma \in \widehat{\Lambda}_{\hat{t}}\}$. For any fixed $x \in \widehat{M}_{\hat{t}}$ let $W^* = \{\tau \in W : J_\tau \bigcap B_r(x) \bigcap \widehat{M}_{\hat{t}} \neq \emptyset\}$. Then there exists a finite positive constant ξ independent of r and x such that $\# W^* \leq \xi$ by Lemma 9.2 in [5]. Hence

$$\mu_{\hat{t}}(B_r(x)) \leq P(\bigcup_{\tau\in W^*} C(\tau)) \leq \sum_{\tau\in W^*} P(C(\tau)) = \sum_{\tau\in W^*} (a(\tau))^s \leq \xi r^s$$

by (18) and (20). So we get

$$\mathcal{H}^s(F_{\hat{t}}) \geq \mathcal{H}^s(\widehat{M}_{\hat{t}}) > 0$$

by Frostman's lemma.

Now we turn to prove the first part. Consider the product measure space $(F \times [0, +\infty], \mathcal{B}_1 \times \mathcal{B}_2, \mathcal{H}^s \times \mathcal{L}^1)$ of the σ-finite measure spaces $(F, \mathcal{B}_1, \mathcal{H}^s)$ and $([0, +\infty], \mathcal{B}_2, \mathcal{L}^1)$. Write $G = \bigcup_{t\in[0,+\infty]}\{(x, t) : x \in F_t\}$ as before. By χ we denote the indicator of G. By Fubini's Theorem we have

$$\int_F \int_{[0,+\infty]} \chi(x, t) \mathrm{d}\mathcal{L}^1(t) \mathrm{d}\mathcal{H}^s(x) = \int_{[0,+\infty]} \mathcal{H}^s(F_t) \mathrm{d}\mathcal{L}^1(t).$$

Note that each fixed $x \in F$, $\chi(x, t) \neq 0$ only for finitely many t by (3). Then the left integral equals 0. Noting that $\mathcal{H}^s(F_t) \geq 0$ for $t \in [0, +\infty]$, we have that for $\mathcal{L}^1$-a.e. $t \in [0, +\infty]$, $\mathcal{H}^s(F_t) = 0$. **QED**

We conjecture that F_t is an s-set if *and only if* $t = -(\log \sum_{j\in\Gamma} a_j^s)^{-1}$.

As an application of the preceding, we end with an example of a measurable function $T:\ [0,1] \longrightarrow [0,+\infty]$, such that each t-level set $T^{-1}(t) \subseteq [0,1]$, $t \in [0,+\infty]$ has Hausdorff dimension 1.

Example. Take $r = 2$, $\Gamma = \{0,2\}$ and $J = [0,1]$. Consider the map-specified Moran fractal F with $h_j(x) = \frac{1}{3}x + \frac{j}{3}$, $x \in \mathbf{R}^1$ and $j = 0,1,2$. Then we have $F = [0,1]$, $a_j = \frac{1}{3}, j = 0,1,2$ and $s = 1$. Let F_t, $t \in [0,+\infty]$, be defined by (5). Then we get a decomposition (F_t) of F satisfying the properties (P1)–(P4). Note that each $x \in F$, either has unique location code or it has only two location codes in $\Omega^\omega = \{0,1,2\} \times \{0,1,2\} \times \cdots$. In the former case, the unique location code has both infinitely many components in Γ and infinitely many components 1. So the corresponding x only lies in one of the sets F_t. But in the latter case, one of the two location codes only has components 0 except for finitely many components, the other only has components 2 except for finitely many components. So the corresponding x lies in F_∞. Hence (F_t) is a partition of F, satisfying the properties (P1)–(P4). Define the function $T:\ [0,1] \longrightarrow [0,\infty]$ by

$$T(x) = t \quad \text{if} \quad x \in F_t.$$

Then we get a measurable function $T:\ [0,1] \longrightarrow [0,+\infty]$, such that each t-level set $T^{-1}(t)(= F_t) \subseteq [0,1]$, $t \in [0,+\infty]$ has Hausdorff dimension 1 and the graph of $T(x)$ has Hausdorff dimension 2. Let $\hat{t} = 1/(\log 3 - \log 2)$. Then we have $\mathcal{H}^1(F_{\hat{t}}) > 0$ by Theorem 3.3. In fact, in this special case we have

$$\mathcal{H}^1(F_{\hat{t}}) \geq \mathcal{H}^1(\widehat{M}_{\hat{t}}) = \mathcal{L}^1(\widehat{M}_{\hat{t}}) = \mu_{\hat{t}}(\widehat{M}_{\hat{t}}) = 1,$$

since we always have $\mathcal{H}^1(\cdot) = \mathcal{L}^1(\cdot)$ and $\mu_{\hat{t}}$ is actually the one-dimensional Lebesgue measure in this special case. Thus we get $\mathcal{H}^1(F_{\hat{t}}) = 1$ and $\mathcal{H}^1(F_t) = 0$ for the other t. **QED**

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