ON THE INTERSECTION OF TRANSLATION OF MIDDLE-α CANTOR SETS

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Abstract

Let Γ_α be the middle-α Cantor set and let $\Gamma_\alpha + t = \{x + t : x \in \Gamma_\alpha\}$ for $-1 \le t \le 1$. Let $H_p = \{t \in \Gamma_\alpha : \dim_H(\Gamma_\alpha \bigcap(\Gamma_\alpha + t)) = \dim_B(\Gamma_\alpha \bigcap(\Gamma_\alpha + t)) = -p\frac{\log 2}{\log \beta}\}, 0 < p < 1$ and $H_p^* = \{t \in \Gamma_\alpha - \Gamma_\alpha : \dim_H(\Gamma_\alpha \bigcap(\Gamma_\alpha + t)) = \dim_B(\Gamma_\alpha \bigcap(\Gamma_\alpha + t)) = -p\frac{\log 2}{\log \beta}\}, 0 < p < 1$. We give the dimensions of $\Gamma_\alpha \bigcap(\Gamma_\alpha + t)$, H_p and H_p^*. In addition the characteristic of H_p and H_p^* are described, respectively, by means of some measures μ and μ^* supported by Γ_α and $\Gamma_\alpha - \Gamma_\alpha$ respectively.

1 Introduction

The middle-α Cantor sets are a straightforward generalization of the classical middle third Cantor set. To define a middle-α Cantor set in the interval [0,1], first choose $\alpha \in (0, 1)$, let $\beta = (1 - \alpha)/2$ and then define the similarity maps

$$F_i(x) = \beta x + i(1 - \beta), \quad i = -1, 0, 1.$$

The middle-α Cantor set Γ_α in the interval [0,1] is defined as the unique invariant nonempty compact set under maps F_0 and F_1, i.e.

$$\Gamma_\alpha = F_0(\Gamma_\alpha) \bigcup F_1(\Gamma_\alpha). \tag{1}$$

As we know, in dynamical systems there exists much more subtle and complicated dynamical behavior associated with a homoclinic tangency. For example, the unfolding of homoclinic tangencies provide examples on how one can pass directly from a simple nonchaotic system to the chaotic region, without going through cascades of period doubling bifurcations (see [2],[10]). A homoclinic tangency is associated with a saddle fixed or periodic point of a diffeomorphism. This saddle may or not be part of a larger invariant set, say a horseshoe,i.e. a Cantor set. Generally the Hausdorff dimension of such a Cantor set is a number strictly between zero and one. Consider a zero-dimensional hyperbolic invariant set Λ in the plane. The stable and unstable manifolds, $W^s(\Lambda)$ and $W^u(\Lambda)$, are each locally the product of a Cantor set with an interval. The question is what conditions guarantee that there will be a point of tangency between two of the segments in these sets. If we regard $W^u(\Lambda)$ as a Cantor set of horizontal lines, then the question is whether some curve in $W^s(\Lambda)$

has a point with a horizontal tangency which is contained in $W^u(\Lambda)$. Naively, it would appear that if this is the case, then a slight vertical shift of $W^s(\Lambda)$ would separate the Cantor set at which $W^s(\Lambda)$ has a horizontal tangent from $W^u(\Lambda)$, as happens when Λ is a point or a periodic orbit. However, for general hyperbolic sets, Newhouse observed that this is not always possible. And the creation of horseshoes have led to the problem of analyzing the difference sets $\{t : \Gamma \bigcap (\Gamma' + t) \neq \emptyset\}$ of Cantor sets Γ and Γ' embedded in the real line (see [6][8][9]).

Now taking $\Gamma = \Gamma' = \Gamma_\alpha$, it seems very interesting to analyze the difference set $\Gamma_\alpha - \Gamma_\alpha$. Kraft[5] applyed cardinality and Lebesgue measure to the difference set $\Gamma_\alpha - \Gamma_\alpha$ to distinguish between different middle-α Cantor sets. The name and notation for the difference set come from the definition,

$$\Gamma_\alpha - \Gamma_\alpha := \{x - y : x, y \in \Gamma_\alpha\}. \tag{2}$$

Here is a more "dynamical" way of defining the difference set,

$$\Gamma_\alpha - \Gamma_\alpha = \{t : \Gamma_\alpha \bigcap (\Gamma_\alpha + t) \neq \emptyset\} \tag{3}$$

where $\Gamma_\alpha + t := \{x + t : x \in \Gamma_\alpha\}$. Notice that for any choice of $\alpha \in (0,1)$, $\Gamma_\alpha \subset \Gamma_\alpha - \Gamma_\alpha \subseteq [-1,1]$. Now we hope to apply the fractal dimensions to $\Gamma_\alpha - \Gamma_\alpha$ and the related sets. We shall try to answer the following questions:

Question 1. What are the fractal dimensions of $\Gamma_\alpha - \Gamma_\alpha$ for $\alpha \in (0,1)$?.
Question 2. What are the fractal dimensions of $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ for $t \in \Gamma_\alpha - \Gamma_\alpha$?.
Question 3. Since $\Gamma_\alpha \bigcap (\Gamma_\alpha + t) \subseteq \Gamma_\alpha$, we have $0 \leq \dim_H(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) \leq \dim_H \Gamma_\alpha = -\frac{\log 2}{\log \beta}$. Then for any given $0 \leq p \leq 1$, is there $t \in \Gamma_\alpha - \Gamma_\alpha$ such that $\dim_H(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = -p\frac{\log 2}{\log \beta}$?.
Question 4. For $0 < p < 1$ define

$$H_p := \left\{ t \in \Gamma_\alpha : \dim_H(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = \dim_B(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = -p\frac{\log 2}{\log \beta} \right\},$$

$$H_p^* := \left\{ t \in \Gamma_\alpha - \Gamma_\alpha : \dim_H(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = \dim_B(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = -p\frac{\log 2}{\log \beta} \right\}.$$

What are the fractal dimensions of H_p and H_p^* ?. Can we characterize H_p and H_p^* by means of some measures μ and μ^* supported by Γ_α and $\Gamma_\alpha - \Gamma_\alpha$ respectively ?.

Note that the sets $\Gamma_\alpha - \Gamma_\alpha$, $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$, H_p and H_p^* can be described geometrically. Consider the set $\Gamma_\alpha \times \Gamma_\alpha$ in the xoy-plane. Let L_t denote the straight line $y = x + t$. Then $\Gamma_\alpha - \Gamma_\alpha$ consists of those t which make the $A_t := L_t \bigcap (\Gamma_\alpha \times \Gamma_\alpha) \neq \emptyset$. The set $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ is the orthogonal projection of A_t on the y-axis and $\dim_H(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = \dim_H A_t$, $\dim_B(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) = \dim_B A_t$. H_p^* consists of those t such that $\dim_H A_t = \dim_B A_t = -p\frac{\log 2}{\log \beta}$, while H_p consists of those t such that the intersection points of L_t with y-axis lie in $\Gamma_\alpha \times \Gamma_\alpha$ and $\dim_H A_t = \dim_B A_t = -p\frac{\log 2}{\log \beta}$.

It is not difficult to verify that

$$\Gamma_\alpha - \Gamma_\alpha = F_0(\Gamma_\alpha - \Gamma_\alpha) \bigcup F_{-1}(\Gamma_\alpha - \Gamma_\alpha) \bigcup F_1(\Gamma_\alpha - \Gamma_\alpha), \tag{4}$$

i.e. $\Gamma_\alpha - \Gamma_\alpha$ is a self-similar set generated by similarity maps F_0, F_{-1} and F_1. Now Let us introduce the following notations:

(i) $\Omega := \prod_1^\infty \{0,1\}$, $\Omega^{(n)} := \prod_1^n \{0,1\}$, $\Omega^* := \bigcup_{n=1}^\infty \Omega^{(n)}$; $\Omega_1 := \prod_1^\infty \{-1,0,1\}$, $\Omega_1^{(n)} := \prod_1^n \{-1,0,1\}$, $\Omega_1^* := \bigcup_{n=1}^\infty \Omega_1^{(n)}$;

(ii) For $\sigma = (\sigma(1), \sigma(2), \cdots) \in \Omega$ or Ω_1, $k \in \mathbf{N}$ let $\sigma|k := (\sigma(1), \cdots, \sigma(k))$;

(iii) Denote $F_\sigma := F_{\sigma(1)} \circ \cdots \circ F_{\sigma(n)}$ for $\sigma \in \Omega^{(n)}$ or $\Omega_1^{(n)}$.

Thus we have

$$\Gamma_\alpha = \bigcap_{n=1}^\infty \Big(\bigcup_{\sigma \in \Omega^{(n)}} F_\sigma([0,1]) \Big). \tag{5}$$

$$\Gamma_\alpha - \Gamma_\alpha = \bigcap_{n=1}^\infty \Big(\bigcup_{\sigma \in \Omega_1^{(n)}} F_\sigma([-1,1]) \Big). \tag{6}$$

Two natural continuous correspondences $\phi : \Omega \longmapsto \Gamma_\alpha$ and $\phi_1 : \Omega_1 \longmapsto \Gamma_\alpha - \Gamma_\alpha$ are given as follows respectively:

$$\begin{aligned} \phi(\sigma) : &= \bigcap_{n=1}^\infty F_{\sigma|n}([0,1]) = \sum_{i=0}^\infty \sigma(i+1)\beta^i(1-\beta) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sigma(i+1)\beta^i(1-\beta) = \lim_{n\to\infty} F_{\sigma|n}(0); \end{aligned} \tag{7}$$

$$\begin{aligned} \phi_1(\sigma) : &= \bigcap_{n=1}^\infty F_{\sigma|n}([-1,1]) = \sum_{i=0}^\infty \sigma(i+1)\beta^i(1-\beta) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sigma(i+1)\beta^i(1-\beta) = \lim_{n\to\infty} F_{\sigma|n}(0). \end{aligned} \tag{8}$$

We call the sequence $\{\sigma(1), \sigma(2), \cdots\}$ as the *address* of $\phi(\sigma) \in \Gamma_\alpha$, respectively, of $\phi_1(\sigma) \in \Gamma_\alpha - \Gamma_\alpha$. Clearly for each $x \in \Gamma_\alpha$ it has unique address in Ω. But for each $x \in \Gamma_\alpha - \Gamma_\alpha$ it may have multiple addresses in Ω_1 unless $0 < \beta < 1/3$ when the maps $F_i, i = -1, 0, 1$, satisfy the open set condition related to the open set $(-1, 1)$.

It is known that $\dim_H \Gamma_\alpha = \dim_B \Gamma_\alpha = -\frac{\log 2}{\log \beta}$ since the maps F_0, F_1 satisfy the open set condition related to the open set $(0, 1)$.

The rest of this present paper will be arranged as follows. In §2 we first list some basic properties of sets $\Gamma_\alpha - \Gamma_\alpha$ and $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ which can be found in [5] but are proved in simpler way here. The dimensions of $\Gamma_\alpha - \Gamma_\alpha$, $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$, H_p and H_p^* are studied in §3 and §4 respectively.

2 Basic properties of $\Gamma_\alpha - \Gamma_\alpha$ and $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$

In this section we discuss some basic properties of $\Gamma_\alpha - \Gamma_\alpha$ and $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ which can be found in [5] but are proved in simpler way here.

Theorem 2.1 *If $\beta < \frac{1}{3}$, then $\Gamma_\alpha - \Gamma_\alpha$ is a Cantor set of Lebesque measure zero. If $\beta \geq \frac{1}{3}$, then $\Gamma_\alpha - \Gamma_\alpha = [-1,1]$.*

Proof When $\beta < 1/3$ the maps F_{-1}, F_0 F_1 satisfy the open set condition . Then $\dim_H(\Gamma_\alpha - \Gamma_\alpha) = -\frac{\log 3}{\log \beta} < 1$. So $\Gamma_\alpha - \Gamma_\alpha$ is a Cantor set of Lebesgue measure zero. When $\beta \geq 1/3$ we have $[-1,1] = F_0([-1,1]) \bigcup F_{-1}([-1,1]) \bigcup F_1([-1,1])$ which indicates $\Gamma_\alpha - \Gamma_\alpha = [-1,1]$ by (6).

Lemma 2.2 *Choose $\beta \in (0,1/2)$. If $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ is a single point, then $t \in \wedge_\alpha := \{\sum_{i=0}^{\infty} u_i \beta^i (1-\beta) : u_i \in \{-1,1\}$ for all $i\}$.*

Proof Let $\sigma \in \Omega_1$ be such that $\phi_1(\sigma) = t \in \Gamma_\alpha - \Gamma_\alpha$. Thus $\{\sigma(1), \sigma(2), \cdots\}$ is a address in Ω_1 of t and

$$t = \sum_{i=0}^{\infty} \sigma(i+1) \beta^i (1-\beta).$$

Suppose that there exits some $\sigma(k) = 0$. Take $\tau, \omega \in \Omega$ such that

$$(\tau(i), \omega(i)) = \begin{cases} (0,0) & \text{if } \sigma(i) = -1, 0 \text{ and } i \neq k; \\ (1,1) & \text{if } \sigma(i) = 1 \ ; \\ (0,1) & \text{if } i = k. \end{cases}$$

Thus we will get two different points $\phi(\tau), \phi(\omega) \in \Gamma_\alpha \bigcap (\Gamma_\alpha + t)$.

Theorem 2.3 *If $\beta \geq 1/3$, then $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ is a Cantor set for almost all $t \in [-1,1]$.*

Proof Let $\Delta := \{\sigma \in \Omega_1 :$ there is some $k \geq 0$ such that $\sigma(i) \neq 0$ for $i \geq k\}$. Thus

$$\phi_1(\Delta) = \wedge_\alpha \bigcup (\bigcup_{k=1}^{\infty} \bigcup_{\tau \in \Omega_1^{(k)}} (\beta^k \wedge_\alpha + \sum_{i=0}^{k-1} \tau(i+1) \beta^i (1-\beta))).$$

Therefore $\phi_1(\Delta) \subset \Gamma_\alpha - \Gamma_\alpha = [-1,1]$ has Lebesgue measure zero since $\wedge_\alpha$ has Lebesgue measure zero. So we only need to prove for any $t \in [-1,1] \setminus \phi_1(\Delta)$, $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ has not any isolated point. Now for any fixed $t \in [-1,1] \setminus \phi_1(\Delta)$ we have that for each $\sigma = (\sigma(1), \sigma(2), \cdots) \in \phi_1^{-1}(t)$ there are $n_i \longrightarrow \infty$ with $\sigma(n_i) = 0$. Thus for any $x \in \Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ with address $\{\tau(1), \tau(2), \cdots\}$ there exists some $\sigma \in \phi_1^{-1}(t)$ such that

$$x = \sum_{i=1}^{\infty} \tau(i) \beta^{i-1} (1-\beta), \qquad \tau(i) \in A(\sigma(i))$$

by Lemma 3.3. Let $n_i \longrightarrow \infty$ with $\sigma(n_i) = 0$. Therefore the points $x_i := x + (1 - 2\tau(n_i)) \beta^{(n_i - 1)} (1-\beta) \in \Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ will tend to x.

Theorem 2.4 *If $\beta \in (\sqrt{2} - 1, 1/2)$, then $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ contains a Cantor set for all $t \in (-1,1)$.*

Proof First we will prove that for $t \in (-1,1)$ it has a address $\{u_1, u_2, \cdots\}$ such that there are infinitely many $u_i = 0$. This address $\{u_1, u_2, \cdots\}$ can be obtained as follows. If $t \in [-\beta, \beta]$ then take $u_1 = 0$. If $t \in (-1, -\beta)$ since $F_{-1}(\beta) > -\beta$ and

$$F_{-1}^n(F_0(-1)) = -\beta^{n+1} - \sum_{i=0}^{n-1} \beta^i(1-\beta) \longrightarrow -\sum_{i=0}^{\infty} \beta^i(1-\beta) = -1 \quad (n \to \infty)$$

where F_{-1}^n denotes the n times iterations of F_{-1}, there exists some n such that $t \in F_{-1}^n \circ F_0([-1,1]) \setminus \bigcup_{i=1}^{n-1} F_{-1}^i \circ F_0([-1,1])$. At this case we take $u_1 = \cdots = u_n = -1$ and $u_{n+1} = 0$. If $t \in (\beta, 1)$ the similar argument as above shows that there exits some n such that $t \in F_1^n \circ F_0([-1,1]) \setminus \bigcup_{i=1}^{n-1} F_1^i \circ F_0([-1,1])$. And at this case we take $u_1 = \cdots = u_n = 1$ and $u_{n+1} = 0$. Consequently according to the above for $t \in (-1,1)$ we can get $n \in \{0\} \bigcup \mathbf{N}$ such that $u_1 = \cdots = u_n = -1$ or 1 and $u_{n+1} = 0$. Now denote $t_1 = \beta^{-(n+1)}(t - \sum_{i=0}^{n} u_{i+1}\beta^i(1-\beta))$. Then $t_1 \in (-1,1)$. For t_1 repeating the above process we can get some $n_1 \in \{0\} \bigcup \mathbf{N}$ such that $t_1 \in F_1^{n_1} \circ F_0([-1,1]) \setminus \bigcup_{i=1}^{n_1-1} F_1^i \circ F_0([-1,1])$ or $t_1 \in F_{-1}^{n_1} \circ F_0([-1,1]) \setminus \bigcup_{i=1}^{n_1-1} F_{-1}^i \circ F_0([-1,1])$. Now we take $u_{n+2} = \cdots = u_{n+n_1+1} = 1$ or -1, $u_{n+n_1+2} = 0$. We can repeat the above process for $t_2 := \beta^{-(n+n_1+2)}(t - \sum_{i=0}^{n+n_1+1} u_{i+1}\beta^i(1-\beta))$ to further determine consequent u_i's.

For the address $\{u_1, u_2, \cdots\}$ chosen above of t let

$$Q := \{x = \sum_{i=0}^{\infty} \sigma(i+1)\beta^i(1-\beta) : \sigma(i) = 0 \quad \text{if } u_i = -1, \sigma(i) = 1 \text{ if } u_i = 1,$$
$$\sigma(i) = 0 \text{ or } 1 \text{ if } u_i = 0\}.$$

Thus $Q \subset \Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ and Q is a generalized Moran set which is a Cantor set (see §3).

3 The dimensions of $\Gamma_\alpha - \Gamma_\alpha$ and $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$

In this section we will apply the Hausdorff, box and packing dimensions to the middle-α Cantor sets, the difference sets $\Gamma_\alpha - \Gamma_\alpha$ and $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ to distinguish between different middle-α Cantor sets. To do this let us recall the generalized Moran set and related results.

A class of generalized Moran sets is introduced [3,4,7] and studied which is defined as follows:

Let $\{n_k\}_{k \geq 1}$ be sequence of integers and let $T_k = \{0 < t_j^{(k)} < 1 : 1 \leq j \leq n_k\}$ for $k \in \mathbf{N}$. Besides, the following notations are used:

(i) $S = \{\sigma = (\sigma(1), \sigma(2), \cdots) : 1 \leq \sigma(j) \leq n_j\}$;

(ii) $S^k = \{\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) : 1 \leq \sigma(j) \leq n_j\}$ for $k \in \mathbf{N}$ and $S^* = \bigcup_{k=1}^{\infty} S^k$;

(iii) $t(\sigma) = t_{\sigma(1)}^{(1)} t_{\sigma(2)}^{(2)} \cdots t_{\sigma(k)}^{(k)}$ for $\sigma \in S^k$;

(iv) $\sigma * i = (\sigma(1), \sigma(2), \cdots, \sigma(k), i)$ where $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S^k$ and $1 \leq i \leq n_{k+1}$;

(v) $\sigma|k = (\sigma(1), \sigma(2), \cdots, \sigma(k))$ for $\sigma \in S$ and $k \in \mathbf{N}$.

Now for each $\sigma \in S^*$ we construct compact set $J_\sigma \subset \mathbf{R}^d$. Given nonempty compact set $J \subset \mathbf{R}^d$ with $\overline{\mathrm{int} J} = J$, we take a family of nonoverlapping nonempty compact subset $\{J_i : 1 \le j \le n_1\}$ of J such that $\overline{\mathrm{int} J_j} = J_j$ and $|J_j| = t_j^{(1)}|J|$. Suppose that J_σ is given for some $\sigma \in S^k$. Take a family of nonoverlapping nonempty compact subsets $\{J_{\sigma * i} : 1 \le i \le n_{k+1}\}$ of J_σ such that $\overline{\mathrm{int} J_{\sigma * i}} = J_{\sigma * i}$ and $|J_{\sigma * i}| = t_i^{(k+1)}|J_\sigma|$. The nonempty compact set

$$E = \bigcap\nolimits_{k=1}^{\infty} \bigcup\nolimits_{\sigma \in S^k} J_\sigma \tag{9}$$

is called generalized Moran set. The following results have been proved in [3,4].

Theorem 3.1 *Assume (1) there is a positive number c such that for all* $\sigma \in S^*$, J_σ *contains a ball with diameter* $c|J_\sigma|$; *(2)* $\inf_{k,j} t_j^{(k)} > 0$ *and (3)*$\lim_{k \to \infty} \sup_{\sigma \in S^k} t(\sigma) = 0$. *Then we have*

$$\dim_H E = \underline{s}, \qquad \dim_P E = \overline{\dim}_B E = \overline{s}$$

where $\underline{s} = \underline{\lim}_{k \to \infty} s_k$, $\overline{s} = \overline{\lim}_{k \to \infty} s_k$ *and* s_k *satisfy* $\prod_{i=1}^{k} [\sum_{j=1}^{n_i} (t_j^{(i)})^{s_k}] = 1$.

Example Let $\{\Phi_k\}_{k \ge 1} = \{\phi_1^{(k)}, \cdots, \phi_{n_k}^{(k)}\}_{k \ge 1}$ be a family of similar maps in $\mathbf{R}^d$ and the similar ratio for $\phi_j^{(k)}$ be $t_j^{(k)}$. Let $\{\Phi_k\}_{k \ge 1}$ satisfy the uniformly open set condition,i.e. there is a bounded nonempty open set $V \subset \mathbf{R}^d$ such that for each $k \in \mathbf{N}$ and $1 \le j \le n_k$, $\phi_j^{(k)}(V) \subset V$ and $\phi_i^{(k)}(V) \bigcap \phi_j^{(k)}(V) = \emptyset$ for $1 \le i \ne j \le n_k$. Now take $J = \overline{V}$ and $J_\sigma = \phi_{\sigma(1)}^{(1)} \circ \phi_{\sigma(2)}^{(2)} \circ \cdots \circ \phi_{\sigma(k)}^{(k)}(J)$ for each $k \in \mathbf{N}$ and $\sigma \in S^k$. Thus by (9) we can get a class of generalized Moran set generated by the similarity contraction maps family $\{\Phi_k\}_{k \ge 1}$. It is easy to see that this kind of generalized Moran sets satisfy the condition (1) of Theorem 3.1.

On the sets $\Gamma_\alpha - \Gamma_\alpha$, from the Theorem 2.1 it follows

Theorem 3.2 *When* $0 < \beta < 1/3$, $\dim_H(\Gamma_\alpha - \Gamma_\alpha) = \dim_B(\Gamma_\alpha - \Gamma_\alpha) = -\frac{\log 3}{\log \beta}$. *When* $1/3 \le \beta < 1/2$, $\dim_H(\Gamma_\alpha - \Gamma_\alpha) = \dim_B(\Gamma_\alpha - \Gamma_\alpha) = 1$.

Let $A := \{0, 1\}$ and $A + i := \{i + j : j \in A\}, i = -1, 0, 1$. Let $A(i) := A \bigcap (A + i), i = -1, 0, 1$. To get the dimensions of sets $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ we need the following

Lemma 3.3 *For* $t \in \Gamma_\alpha - \Gamma_\alpha$

$$\Gamma_\alpha \bigcap (\Gamma_\alpha + t) = \bigcup_{\sigma \in \phi_1^{-1}(t)} \left\{ \sum_{i=1}^{\infty} \tau(i) \beta^{i-1}(1 - \beta) : \tau(i) \in A(\sigma(i)) \right\}.$$

Proof For any $x \in \Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ there exists $u \in \Omega$ such that $\phi(\tau) - \phi(u) = t$ where $\tau = \phi^{-1}(x)$, i.e.

$$t = \phi(\tau) - \phi(u) = \sum_{i=1}^{\infty} (\tau(i) - u(i))\beta^{i-1}(1-\beta).$$

Taking $\sigma(i) = \tau(i) - u(i)$ then $\sigma \in \phi_1^{-1}(t)$ and $\tau(i) \in A(\sigma(i))$.

On the other hand for any $x \in \bigcup_{\sigma \in \phi_1^{-1}(t)} \{\sum_{i=1}^{\infty} \tau(i)\beta^{i-1}(1-\beta) : \tau(i) \in A(\sigma(i))\}$ there is some $\sigma \in \phi_1^{-1}(t)$ and $\tau \in \Omega$ with $\tau(i) \in A(\sigma(i))$ such that

$$x = \sum_{i=1}^{\infty} \tau(i)\beta^{i-1}(1-\beta),$$

which means $x \in \Gamma_\alpha$. Taking $u(i) = \tau(i) - \sigma(i)$ then we have $u \in \Omega$ and $x = \phi(u) + t \in (\Gamma_\alpha + t)$.

Theorem 3.4 *Let $0 < \beta \leq 1/3$. Then for any $t \in \Gamma_\alpha - \Gamma_\alpha$*

$$\begin{aligned} \dim_H(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) &= \underline{\dim}_B(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) \\ &= \underline{\lim}_{k\to\infty} -\frac{\log(\prod_{j=1}^k m(\sigma(j)))}{k} \cdot \frac{1}{\log\beta} = -\frac{\log 2}{\log\beta}\underline{t}; \end{aligned}$$

$$\begin{aligned} \dim_P(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) &= \overline{\dim}_B(\Gamma_\alpha \bigcap (\Gamma_\alpha + t)) \\ &= \overline{\lim}_{k\to\infty} -\frac{\log(\prod_{j=1}^k m(\sigma(j)))}{k} \cdot \frac{1}{\log\beta} = -\frac{\log 2}{\log\beta}\overline{t}, \end{aligned}$$

where $\sigma = (\sigma(1), \sigma(2), \cdots) \in \phi_1^{-1}(t)$, $m(\sigma(j)) := \mathrm{Card}A(\sigma(j))$ and $\overline{t}$ (resp. $\underline{t}$) is the upper (resp. lower) density of 0 occuring in $\phi_1^{-1}(t)$.

Proof Note that maps $F_i, i = -1, 0, 1$, satisfy the open set condition with the open set $(-1, 1)$ and the set $\Gamma_\alpha - \Gamma_\alpha$ is invariant under $F_i, i = -1, 0, 1$. Thus for any $t \in \Gamma_\alpha - \Gamma_\alpha$ the set $\phi_1^{-1}(t)$ is a singleton when $0 < \beta < 1/3$, contains at most two elements when $\beta = 1/3$. In addition when $\phi_1^{-1}(t)$ contains exactly two elements, one has its all components equal to 1 with finite exception and the other has its all components equal to -1 with finite exception. Therefore when $\phi_1^{-1}(t)$ contains exactly two elements, $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ is a finite set by Lemma 3.3. In this case we have $\lim_{k\to\infty}[\log(\prod_{j=1}^k m(\sigma(j)))]/k = 0$ since $m(-1) = m(1) = 1$ which proves our results. When $\phi_1^{-1}(t)$ is a singleton the Lemma 3.3 and the Example tell us that $\Gamma_\alpha \bigcap (\Gamma_\alpha + t)$ is a generalized Moran set generated by the maps family $\{\Phi_k\}_{k\geq 1}$ where $\{\Phi_k\} = \{F_i : i \in A(\sigma(k)\}$. Let s_k be such that

$$\prod_{j=1}^{k} [\sum_{i \in A(\sigma(j))} \beta^{s_k}] = 1,$$

i.e. $s_k = -[\log(\prod_{j=1}^{k} m(\sigma(j)))]/[k\log\beta]$. Therefore by Theorem 3.1

$$\dim_H(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) = \underline{\lim}_{k\to\infty} -\frac{\log(\prod_{j=1}^{k} m(\sigma(j)))}{k}\cdot\frac{1}{\log\beta} = -\frac{\log 2}{\log\beta}\underline{t};$$

$$\begin{aligned}\dim_P(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) &= \overline{\dim}_B(\Gamma_\alpha\bigcap(\Gamma_\alpha+t))\\ &= \overline{\lim}_{k\to\infty} -\frac{\log(\prod_{j=1}^{k} m(\sigma(j)))}{k}\cdot\frac{1}{\log\beta} = -\frac{\log 2}{\log\beta}\overline{t}.\end{aligned}$$

Since all the maps $F_i, i=0,1$ have the same similarity ratio we have $\dim_H(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) = \underline{\dim}_B(\Gamma_\alpha\bigcap(\Gamma_\alpha+t))$ by Theorem 2.4 in [7].

Corollary 3.5 *Let $0<\beta\le 1/3$. For any $0\le p\le 1$ there exists $t\in\Gamma_\alpha-\Gamma_\alpha$ such that*

$$\dim_H(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) = \dim_B(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) = -p\frac{\log 2}{\log\beta}.$$

In addition the above t can be chosen from both Γ_α and $(\Gamma_\alpha-\Gamma_\alpha)\setminus\Gamma_\alpha$.

Proof When $p=0$ we can take $t=0\in\Gamma_\alpha$ or $t=(1-\beta)^2\in(\Gamma_\alpha-\Gamma_\alpha)\backslash\Gamma_\alpha$ with address $\{1,-1,0,0,\cdots\}$. When $p\neq 0$ let n_k be the maximal natural number such that $[n_k p]=k$ where $[n_k p]$ is the integer part of $n_k p$. So $n_{k+1}\le n_k+n_1$. Let $\sigma(n)=0$ if $n=n_k$ for some $k\in\mathbf{N}$, otherwise $|\sigma(n)|=1$. Taking $\sigma=(\sigma(1),\sigma(2),\cdots)$ we have

$$\lim_{k\to\infty}\frac{\log(\prod_{j=1}^{n_k} m(\sigma(j)))}{n_k} = \lim_{k\to\infty}\frac{(n_k-\sum_{j=1}^{n_k}|\sigma(j)|)\log 2}{n_k} = \lim_{k\to\infty}\frac{k\log 2}{n_k} = p\log 2.$$

On the other hand for any $n\in\mathbf{N}$ there exists $k\in\mathbf{N}$ such that $n_k\le n<n_{k+1}\le n_k+n_1$. Thus

$$\frac{n_k}{n}\cdot\frac{[n_k p]}{n_k}\le\frac{n-\sum_{i=1}^{n}|\sigma(i)|}{n}\le\frac{n_k}{n}\cdot\frac{[n_k p]+1}{n_k}.$$

Therefore $\lim_{n\to\infty}[\log(\prod_{j=1}^{n} m(\sigma(j)))]/n = p\log 2$ which means

$$\dim_H(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) = \dim_B(\Gamma_\alpha\bigcap(\Gamma_\alpha+t)) = -p\frac{\log 2}{\log\beta}$$

by Theorem 3.4 if taking $t=\phi_1(\sigma)$. In addition letting $\sigma(n)=1$ for all $n\neq n_k$ leads $t\in\Gamma_\alpha$ and letting $\sigma(n)=1$ for infinitely many $n\neq n_k$ and $\sigma(n)=-1$ for infinitely many $n\neq n_k$ will lead $t\in(\Gamma_\alpha-\Gamma_\alpha)\backslash\Gamma_\alpha$.

Open problem. What are the dimensions of $\Gamma_\alpha\bigcap(\Gamma_\alpha+t)$ for $1/3<\beta<1/2$?.

4 The Dimensions of H_p and H_p^*

We will apply the multifractal decomposition results for Moran sets to Γ_α and $\Gamma_\alpha - \Gamma_\alpha$ to get the dimension results of H_p and H_p^* respectively. Note that, as a special case, the middle-α Cantor set Γ_α is a pairwise disjoint, map specified Moran construction and so is the set $\Gamma_\alpha - \Gamma_\alpha$ when $0 < \beta < 1/3$. First let us recall the related results on the multifractal decomposition results of Moran sets applied to the case of middle-α Cantor sets.

In the definition of generalized Moran set in §3, if taking $n_k \equiv n$ and $t_j^{(k)} \equiv t_j$, the nonempty compact set E determined by (9) is called as Moran set. In [1] the authors gave an important results on the multifractal decompositions of these Moran sets. In the following we will list some results on the multifractal decompositions of the middle-α Cantor set Γ_α according to [1]. In this case we prefer to take $S = \Omega = \prod_{i=1}^{\infty}\{0,1\}$ with using index $0,1$ instead of index $1,2$ just for convenience.

Fix a probability vector (p_0, p_1) with each p_i positive, $p_0 \neq p_1$ and $p_0 + p_1 = 1$. Let $\hat{\mu}$ be the corresponding infinite product measure on S. Let μ be the image measure on Γ_α induced by ϕ defined by (7). So $\mu(E) = \hat{\mu}(\phi^{-1}(E))$, for $E \subset \mathbf{R}$. Let $t_0 = t_1 = \beta$. For $\sigma \in S$ write

$$p(\sigma|k) = p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(k)}; \qquad t(\sigma|k) = t_{\sigma(1)} t_{\sigma(2)} \cdots t_{\sigma(k)}$$

For each $q \in \mathbf{R}$ there is a unique number $\beta(q)$ such that

$$p_0^q \beta^{\beta(q)} + p_1^q \beta^{\beta(q)} = 1,$$

i.e. $\beta(q) = -(\log(p_0^q + p_1^q))/\log\beta$. Define

$$\alpha(q) = -\beta'(q) = \frac{p_0^q \log p_0 + p_1^q \log p_1}{p_0^q \log\beta + p_1^q \log\beta}; \quad f(q) = q\alpha(q) + \beta(q),$$

$$\hat{E}_q = \left\{\sigma \in S : \lim_{k\to\infty} \frac{\log p(\sigma|k)}{\log t(\sigma|k)} = \alpha(q)\right\}; \quad E_q = \phi(\hat{E}_q).$$

Note that $\alpha(q)$ is strictly decreasing because of $p_0 \neq p_1$ and satisfies

$$\lim_{q\to\infty} \alpha(q) = \min\left\{\frac{\log p_0}{\log\beta}, \frac{\log p_1}{\log\beta}\right\}; \quad \lim_{q\to-\infty} \alpha(q) = \max\left\{\frac{\log p_0}{\log\beta}, \frac{\log p_1}{\log\beta}\right\}.$$

According to [1], for middle-α Cantor set Γ_α the following multifractal decompositions results hold.

Theorem 4.1 *(1) For each $q \in \mathbf{R}, \dim_H E_q = \dim_P E_q = f(q)$;*
(2) $x \in E_q$ iff the pointwise dimension of μ at x is $\alpha(q)$:

$$\lim_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon} = \alpha(q),$$

where $B(x,\varepsilon)$ denotes the ball with center at x and radius ε.

Let $G(\lambda) = \log p_1/\log\beta + (\log p_0/\log\beta - \log p_1/\log\beta)\lambda$ which is strictly monotonic because of $p_0 \neq p_1$. Thus we can determine a function $q = Q(\lambda)$ from $(0,1)$ onto $\mathbf{R}$ by letting $G(\lambda) = \alpha(q)$.

It follows immediately from Theorem 3.4 and Corollary 3.5

Lemma 4.2 *Let $0 < \beta \leq 1/3$. For $0 < p < 1$*

$$H_p = \left\{ t \in \Gamma_\alpha : \lim_{k\to\infty} \frac{k - \sum_{i=1}^k \sigma(i)}{k} = p,\ \sigma = (\sigma(1), \sigma(2), \cdots) \in \phi^{-1}(t) \right\}.$$

Note that for $\sigma = (\sigma(1), \sigma(2), \cdots) \in \Omega$

$$\frac{\log p(\sigma|k)}{\log t(\sigma|k)} = \frac{\log p_1}{\log\beta} + \left(\frac{\log p_0}{\log\beta} - \frac{\log p_1}{\log\beta}\right) \frac{k - \sum_{i=1}^k \sigma(i)}{k}$$

Thus by the definitions of E_q and $Q(\lambda)$, Theorem 4.1 and Lemma 4.2 we can get

Theorem 4.3 *Let $0 < \beta \leq 1/3$. We have*
(1) $H_p = E_{Q(p)} = \{t \in \Gamma_\alpha : \lim_{\varepsilon\to 0} \log\mu(B(t,\varepsilon))/\log\varepsilon = \alpha(Q(p))\}$;
(2) $\dim_H H_p = \dim_P H_p = f(Q(p))$.

Similar to H_p we can study the dimensions of H_p^* by means of the multifractal decomposition results on $\Gamma_\alpha - \Gamma_\alpha$. Note that $\Gamma_\alpha - \Gamma_\alpha$ is an invariant set under $F_i, i = -1, 0, 1$ which satisfy the open set condition if $0 < \beta \leq 1/3$. To do this we take a probability vector $\{p_{-1}, p_0, p_1\}$ with each positive, $p_0 \neq p_{-1} = p_1$ and $p_0 + 2p_1 = 1$. Let $\hat{\mu}^*$ be the corresponding infinite product measure on Ω_1. Let μ^* be the image measure on $\Gamma_\alpha - \Gamma_\alpha$ induced by ϕ_1 defined by (8). So $\mu^*(E) = \hat{\mu}^*(\phi_1^{-1}(E))$, for $E \subset \mathbf{R}$.

Here we take $S = \Omega_1 = \prod_{i=1}^\infty \{-1, 0, 1\}$ and $t_{-1} = t_0 = t_1 = \beta$. For $\sigma \in \Omega_1$ the $p(\sigma|k)$ and $t(\sigma|k)$ are defined in the same method as above.

For each $q \in \mathbf{R}$ there is a unique number $\beta^*(q)$ such that

$$p_0^q \beta^{\beta^*(q)} + 2p_1^q \beta^{\beta^*(q)} = 1,$$

i.e. $\beta^*(q) = -(\log(p_0^q + 2p_1^q))/\log\beta$. Define

$$\alpha^*(q) = -\beta^{*\prime}(q) = \frac{p_0^q \log p_0 + 2p_1^q \log p_1}{(p_0^q + 2p_1^q)\log\beta}; \quad f^*(q) = q\alpha^*(q) + \beta^*(q),$$

$$\hat{E}_q^* = \left\{ \sigma \in \Omega_1 : \lim_{k\to\infty} \frac{\log p(\sigma|k)}{\log t(\sigma|k)} = \alpha^*(q) \right\}; \quad E_q^* = \phi_1(\hat{E}_q^*).$$

According to [1] for set $\Gamma_\alpha - \Gamma_\alpha$ the following multifractal decompositions results hold.

Theorem 4.4 *(1) Let $0 < \beta \leq 1/3$. For each $q \in \mathbf{R}, \dim_H(E_q^*) = \dim_P(E_q^*) = f^*(q)$.*

(2) Let $0 < \beta < 1/3$. $x \in E_q^*$ iff the pointwise dimension of μ^* at x is $\alpha^*(q)$:

$$\lim_{\varepsilon \to 0} \frac{\log \mu^*(B(x,\varepsilon))}{\log \varepsilon} = \alpha^*(q).$$

Let $G^*(\lambda) = \log p_1/\log\beta + (\log p_0/\log\beta - \log p_1/\log\beta)\lambda$ which is strictly monotonic because of $p_0 \neq p_1$. Note that when $p_0 \neq p_1$ the function $\alpha^*(q)$ defined on $\mathbf{R}$ is strictly decreasing and satisfies

$$\lim_{q\to\infty} \alpha^*(q) = \min\left\{\frac{\log p_0}{\log\beta}, \frac{\log p_1}{\log\beta}\right\}; \quad \lim_{q\to-\infty} \alpha^*(q) = \max\left\{\frac{\log p_0}{\log\beta}, \frac{\log p_1}{\log\beta}\right\}.$$

Thus we can determine a function $q = Q^*(\lambda)$ from $(0,1)$ onto $\mathbf{R}$ by letting $G^*(\lambda) = \alpha^*(q)$. Therefore Theorem 3.4 and Corollary 3.5 mean

Lemma 4.5 *Let $0 < \beta \leq 1/3$. For $0 < p < 1$*

$$H_p^* = \left\{ t \in \Gamma_\alpha - \Gamma_\alpha : \lim_{k\to\infty} \frac{k - \sum_{i=1}^k |\sigma(i)|}{k} = p,\ \sigma = (\sigma(1), \sigma(2), \cdots) = \phi_1^{-1}(t) \right\}.$$

Note that for $\sigma = (\sigma(1), \sigma(2), \cdots) \in \Omega_1$

$$\frac{\log p(\sigma|k)}{\log t(\sigma|k)} = \frac{\log p_1}{\log\beta} + \left(\frac{\log p_0}{\log\beta} - \frac{\log p_1}{\log\beta}\right) \frac{k - \sum_{i=1}^k |\sigma(i)|}{k}.$$

Thus by the definitions of E_q^* and $Q^*(\lambda)$, Theorem 4.1 and Lemma 4.5 we can get

Theorem 4.6 *(1) Let $0 < \beta < 1/3$. We have*

$$H_p^* = E_{Q^*(p)}^* = \left\{ t \in \Gamma_\alpha - \Gamma_\alpha : \lim_{\varepsilon\to 0} \frac{\log \mu^*(B(t,\varepsilon))}{\log\varepsilon} = \alpha^*(Q^*(p)) \right\};$$

(2) Let $0 < \beta \leq 1/3$. We have $\dim_H H_p^* = \dim_P H_p^* = f^*(Q^*(p))$.

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