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Non-differentiability of devil's staircases and dimensions of subsets of Moran sets

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Abstract

Let C be the homogeneous Cantor set invariant for $x \to ax$ and $x \to 1 - a + ax$. It has been shown by Darst that the Hausdorff dimension of the set of non-differentiability points of the distribution function of uniform measure on C equals $(\dim_H C)^2 = (\log 2/\log a)^2$. In this paper we generalize the essential ingredient of the proof of this result. Let $\Omega = \{0, 1, \ldots, r\}$. Let F be a Moran set associated with $\{0 < a_i < 1, i \in \Omega\}$ and $\Omega^\omega = \Omega \times \Omega \times \cdots$. Let ϕ be the associated coding map from Ω^ω onto F. Fix a non-empty set $\Gamma \subseteq \Omega$ with $\Gamma^c \neq \varnothing$ and let $z(\sigma, n)$ denote the position of the nth occurrence of the elements of Γ in $\sigma \in \Omega^\omega$. For given $0 \leqslant \xi \leqslant 1$, let

$$\Lambda = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{z(\sigma, n+1)}{z(\sigma, n)} = \xi^{-1}\right\}, \quad F_\xi = \phi(\Lambda),$$

and

$$\Lambda^* = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{z(\sigma, n+1)}{z(\sigma, n)} \geqslant \xi^{-1}\right\}, \quad F^*_\xi = \phi(\Lambda^*).$$

We show that $\dim_P F_\xi = \dim_P F^*_\xi = \dim_B F_\xi = \dim_B F^*_\xi = s$ with $\sum_{j\in\Omega} {a_i}^s = 1$, and $\dim_H F_\xi = \dim_H F^*_\xi = \eta$ where η is such that

$$\xi \log \sum_{j\in\Omega} a_j^\eta + (1-\xi) \log \sum_{j\in\Gamma^c} a_j^\eta = 0.$$

1. *Introduction*

Let $h_i(x) = ax + i(1-a), i = 0, 1$ with $x \in [0, 1]$ and $0 < a < \frac{1}{2}$. Then there exists a unique non-empty compact set $C \subset [0, 1]$ such that

$$C = h_0(C) \bigcup h_1(C).$$

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It is well known that the Hausdorff dimension of C equals $\dim_H C = -(\log 2/\log a)$. Let μ be the uniform probability measure on C. Consider the distribution function which is often referred to as the Devil's staircase (for $a = \frac{1}{3}$):

$$F(x) = \mu([0, x]), \quad x \in [0, 1].$$

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0, 1]\backslash C$ and the upper derivative of $F(x)$ is infinite on C. Let S be the set of points at which $F(x)$ is not differentiable, i.e., the set of points in C at which the lower derivative of $F(x)$ is finite. S can be decomposed into

$$S = N^+ \bigcup N^- \bigcup \{t : t \text{ is an endpoint of } C\}, \tag{1}$$

where $N^+(N^-)$ is the set of non-end points of C at which the lower right (left) derivative of $F(x)$ is finite. Each $t \in C$ can be encoded in the usual way by a $0-1$ sequence, denoted by $\{t\}$. Now let $z(t, n)$ denote the position of the nth zero in $\{t\}$. The set N^+ (symmetrically for N^-) is characterized by [**1**] as follows:

[a] if $t \in N^+$, then $\limsup_{n\to\infty}(z(t, n+1))/z(t, n) \geqslant -(\log a/\log 2)$;

[b] if $\limsup_{n\to\infty}(z(t, n+1))/z(t, n) > -(\log a/\log 2)$, then $t \in N^+$.

By means of the above [a] and [b] [**1**] proves that

$$\dim_H S = \dim_H N^+ = \left[\frac{\log 2}{\log a}\right]^2 = (\dim_H C)^2.$$

One of the results of this paper is that S is not a regular set: we will prove that the packing dimension of S, $\dim_P S = \dim_H C$.

For non-homogeneous Cantor sets C it is much more difficult to determine $\dim_H S$. The most studied non-homogeneous Cantor set in $\mathbf{R}$ is the set C satisfying $C = \bigcup_{j=0}^{r} h_j(C)$ with a disjoint union, where the h_js are similitude mappings with ratios $0 < a_j < 1$. It is well known that $\dim_H C = s$ with $\sum_{j=0}^{r} a_j^s = 1$. Consider the self-similar probability measure μ on C corresponding to the probability vector $(a_0^s, a_1^s, \ldots, a_r^s)$. The direct motivation of this paper is trying to determine dimensions of the set S of non-differentiability of points in C. The dimension formula given in this paper can be employed to obtain the same results as for the homogeneous Cantor set case, i.e. $\dim_H S = (\dim_H C)^2 = s^2$ and $\dim_P S = \dim_B S = \dim_H C = s$ (see [**5**]).

On the other hand, one can also consider the problem of non-differentiability for higher dimensional Cantor sets. For example, in the two-dimensional case let the non-empty compact set C be defined by $C = \bigcup_{j=0}^{3} h_j(C)$, where $h_j(x, y) = (x, y)A + b_j$ with the matrix $A = \operatorname{diag}(a, a)$, $0 < a < \frac{1}{2}$, $b_0 = (0, 0)$, $b_1 = (1-a, 0)$, $b_2 = (1-a, 1-a)$ and $b_3 = (0, 1-a)$. Let $F(x, y)$ be the distribution function of the uniform probability measure on C. Consider the set S of points in C which are not partially differentiable in the x-direction. A similar decomposition as in (1) can be made for S. Here each $t \in C$ can be encoded by a sequence with symbols from $\{0, 1, 2, 3\}$. Now we let $z(t, n)$ denote the position of the nth occurrence of the elements of the set $\{0, 3\}$. One can check that N^+ can also be characterized by the above [a] and [b]. In this special case our formula (8) yields $\dim_P N^+ = \dim_B N^+ = -(\log 4/\log a)$ and $\dim_H N^+ = (\log 2/\log a)^2 - (\log 2/\log a)$. We remark that even although N^+ is not a regular set, Marstrand's product theorem does apply, and so this result might also be obtained in that way.

The sets considered above are special cases of a class of subsets of general Moran fractals. We will focus on calculating the Hausdorff, packing and box dimension of subsets of these Moran fractals which are characterized by location codes like that in $[a]$ and $[b]$ above.

Denote $\mathbf{\Omega} = \{0, 1, \ldots, r\}$, where r is a positive integer. We use the following notation:

(i) $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \ldots) : 0 \leqslant \sigma(j) \leqslant r\}$;

(ii) $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k)) : 0 \leqslant \sigma(j) \leqslant r\}$ for $k \in \mathbf{N}$ and $\mathbf{\Omega}^* = \bigcup_{k=1}^{\infty} \mathbf{\Omega}^k$;

(iii) $|\cdot|$ is used to denote the length of word. For any $\sigma, \tau \in \mathbf{\Omega}^*$ write $\sigma * \tau = (\sigma(1), \ldots, \sigma(|\sigma|), \tau(1), \ldots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \ldots, \tau(|\tau|), \sigma(1), \sigma(2), \ldots)$ for any $\tau \in \mathbf{\Omega}^*$, $\sigma \in \mathbf{\Omega}^\omega$;

(iv) $\sigma|k = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ for $\sigma \in \mathbf{\Omega}^\omega$ and $k \in \mathbf{N}$;

(v) for $\sigma \in \mathbf{\Omega}^k$, the cylinder set $C(\sigma)$ with base σ is defined as $C(\sigma) = \{\tau \in \mathbf{\Omega}^\omega : \tau|k = \sigma\}$ for $k \in \mathbf{N}$.

Fixing a non-empty compact set $J \subset \mathbf{R}^n$ with $\overline{\text{int}\, J} = J$ and positive real numbers $0 < a_i < 1, i = 0, 1, \ldots, r$, the related Moran set (or Moran fractal) is defined in the following way. Let also a number $c > 0$ be given.

Step 1. For each $\sigma \in \mathbf{\Omega}^k, k \in \mathbf{N}$, construct a compact set $J_\sigma \subset \mathbf{R}^n$ by induction:

(i) A family $\{J_j : j = 0, 1, \ldots, r\}$ of non-overlapping non-empty compact subsets of J is chosen for $k = 1$ such that $\overline{\text{int} J_j} = J_j$, $|J_j| = a_j |J|$ where $|\cdot|$ denotes the diameter of a set.

(ii) Suppose that J_σ is given for some $\sigma \in \mathbf{\Omega}^k$. Take a family $\{J_{\sigma * i} : i = 0, 1, \ldots, r\}$ of non-overlapping non-empty compact subsets of J_σ such that $\overline{\text{int}\, J_{\sigma * i}} = J_{\sigma * i}$, $|J_{\sigma * i}| = a_i |J_\sigma|$ and $J_{\sigma * i}$ contains an open ball of diameter $c|J_{\sigma * i}|$.

Step 2. The Moran fractal F associated with $\{0 < a_i < 1, i = 0, 1, \ldots, r\}$ and the $J_\sigma, \sigma \in \mathbf{\Omega}^*$ is defined as the non-empty compact set

$$F = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \mathbf{\Omega}^k} J_\sigma. \tag{2}$$

It is well known that $\dim_H F = \dim_P F = \dim_B F = s$ and F is an s-set, where

$$\sum_{j \in \mathbf{\Omega}} a_j^s = 1. \tag{3}$$

Define ϕ: $\mathbf{\Omega}^\omega \to \mathbf{R}^n$ by

$$\{\phi(\sigma)\} = \bigcap_{k=1}^{\infty} J_{\sigma|k}. \tag{4}$$

It is easy to see that $\phi(\mathbf{\Omega}^\omega) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (2) and ϕ is a continuous surjection. Each $x \in F$ can be encoded via ϕ. An infinite sequence σ is called a location code of $x \in F$ if $\phi(\sigma) = x$. Here we would like to point out that there may be multiple location codes for some $x \in F$. However the number of location codes for any $x \in F$ is bounded by a positive constant independent of $x \in F$. For convenience we often use $x(k)$ to denote the kth component of a location code of $x \in F$.

Let $h_i : \mathbf{R}^n \to \mathbf{R}^n, 0 \leqslant i \leqslant r$. Denote $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \mathbf{\Omega}^k$ and $x \in \mathbf{R}^n$. A Moran fractal is termed as map-specified if there exist similitude contractions $h_i, i = 0, 1, \ldots, r$, such that $J_\sigma = h_\sigma(J)$ for any $\sigma \in \Omega^*$. In this case F is actually the self-similar set determined by $\{h_i, 0 \leqslant i \leqslant r\}$, which satisfies the open set condition with respect to the open set $O = \operatorname{int} J$ (i.e. $\bigcup_{i=0}^r h_i(O) \subset O$ with a disjoint union on the left) and the coding map ϕ in (4) can be changed into

$$\{\phi(\sigma)\} = \bigcap_{k=1}^{\infty} h_{\sigma|k}(\overline{O}).$$

Now let $\Gamma \subseteq \Omega = \{0, 1, \ldots, r\}$ be non-empty such that $\Gamma^c \neq \varnothing$. Let $z(\sigma, n)$ denote the position of the nth occurrence of elements of Γ in $\sigma \in \Omega^\omega$:

$$z(\sigma, n) = \begin{cases} k, & \text{if } \sigma(k) \in \Gamma \text{ and } \#\{1 \leqslant i < k : \sigma(i) \in \Gamma\} = n - 1 \\ +\infty, & \text{if } \#\{1 \leqslant i < +\infty : \sigma(i) \in \Gamma\} < n. \end{cases} \tag{5}$$

For given $0 \leqslant \xi \leqslant 1$, let

$$\Lambda = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{z(\sigma, n+1)}{z(\sigma, n)} = \xi^{-1}\right\} \quad \text{and} \quad F_\xi = \phi(\Lambda), \tag{6}$$

and

$$\Lambda^* = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{z(\sigma, n+1)}{z(\sigma, n)} \geqslant \xi^{-1}\right\} \quad \text{and} \quad F_\xi^* = \phi(\Lambda^*), \tag{7}$$

where we adopt the convention that $0^{-1} = +\infty$ and $+\infty/+\infty = +\infty$. It is easy to check that if $\sigma \in \Lambda$ (or Λ^*) then for any $j \in \Omega$ we have $j * \sigma \in \Lambda$ (or Λ^*). This means that F_ξ and F_ξ^* are both dense in F. For any fixed $0 \leqslant \xi \leqslant 1$, define the function

$$T(x) = \xi \log \sum_{j\in\Omega} a_j^x + (1 - \xi) \log \sum_{j\in\Gamma^c} a_j^x.$$

It is easy to verify that $T(x)$ is strictly decreasing in $[0, s]$. Since $T(0) > 0$ and $T(s) \leqslant 0$, there exists a unique $0 < \eta \leqslant s$ such that $T(\eta) = 0$, i.e.

$$\xi \log \sum_{j\in\Omega} a_j^\eta + (1 - \xi) \log \sum_{j\in\Gamma^c} a_j^\eta = 0. \tag{8}$$

η is a function $\eta(\xi)$ of ξ in $[0, 1]$. It is easy to verify that $\eta(\xi)$ is strictly increasing and continuous and $\eta(0) \leqslant \eta(\xi) \leqslant \eta(1) = s$ with $\sum_{j\in\Gamma^c} a_j^{\eta(0)} = 1$.

In the present paper, we shall prove:
$\dim_H F_\xi = \dim_H F_\xi^* = \eta$ and $\dim_P F_\xi = \dim_P F_\xi^* = \dim_B F_\xi = \dim_B F_\xi^* = s$ where F_ξ, F_ξ^*, η and s are defined in (6), (7), (8) and (3), respectively.

Obviously this result can be employed to obtain Darst's result for $\dim_H N^+$. In fact, for any $\epsilon > 0$ and $\xi = -(\log 2/\log a)$ we have $F_{\xi+\epsilon}^* \subseteq N^+ \subseteq F_\xi^*$ if we take $r = 1$, $\Gamma = \{0\}$ and $a_i = a$ for $i = 0, 1$. Therefore from the continuity of η we obtain letting $\epsilon \downarrow 0$ that $\dim_H N^+ = \dim_H F_\xi^* = (\log 2/\log a)^2$. In addition we have $\dim_B N^+ = \dim_P N^+ = -(\log 2/\log a)$.

2. *Dimensions of F_ξ and F_ξ^**

In this section the dimensions of F_ξ and F_ξ^* are obtained. The following proposition will be employed. Part (A) can be found in [**2**] for the more general Moran fractal structure and a simplified proof is given for this special case in [**4**].

PROPOSITION 2·1. *Let* $M = \phi(\prod_{i=1}^{\infty} \Omega_i)$ *where* $\Omega_i \subseteq \Omega$, $i \in \mathbf{N}$. *Let* $d(k)$ *be such that*

$$\prod_{i=1}^{k}(\sum_{j\in\Omega_i} a_j^{d(k)}) = 1.$$

Then (A) $\dim_H M = \liminf_{k\to\infty} d(k)$ (*see* [**2**]),
and (B) $\overline{\dim}_B M = \dim_P M = \limsup_{k\to\infty} d(k)$ (*see* [**3**]).

THEOREM 2·2. *We have* $\dim_H F_\xi = \dim_H F_\xi^* = \eta$ *and* $\dim_P F_\xi = \dim_P F_\xi^* = \dim_B F_\xi = \dim_B F_\xi^* = s$ *where* F_ξ, F_ξ^*, η *and* s *are defined in* (6), (7), (8) *and* (3), *respectively.*

Proof. Let $\mathscr{H}^d(\cdot)$ denote d-dimensional Hausdorff measure. The case $\xi = 1$ is simple, and left to the reader. We now consider the case $0 < \xi < 1$. We shall first give upper bounds for the Hausdorff dimension of F_ξ^*. This part of the proof is similar to that in [**1**].

Note that $0 < \eta < s$ when $0 < \xi < 1$. At first we will show $\dim_H F_\xi^* \leqslant \eta$.

Fix an arbitrary d with $s \geqslant d > \eta$. Note that $T(d) < 0$, i.e.

$$\xi \log \sum_{j\in\Omega} a_j^d + (1-\xi)\log \sum_{j\in\Gamma^c} a_j^d < 0. \tag{9}$$

Note that from (9) and $d \leqslant s$ it follows that

$$-\log \sum_{j\in\Gamma^c} a_j^d > 0. \tag{10}$$

Thus we get

$$\frac{\log \sum_{j\in\Omega} a_j^d}{-\log \sum_{j\in\Gamma^c} a_j^d} < \xi^{-1} - 1,$$

by (9) and (10). Let $t > 0$ be such that

$$\frac{\log \sum_{j\in\Omega} a_j^d}{-\log \sum_{j\in\Gamma^c} a_j^d} = \xi^{-1} - 1 - t. \tag{11}$$

We will define a positive integer n^* (depending on d); for $k \geqslant n^*$, we will specify u_k so that

(I) the limsup of the sets

$$E_k = \{x : x(i) \in \Gamma^c \ \text{ for } \ k < i \leqslant u_k\}, \quad k \geqslant n^*$$

satisfies the formula

$$F_\xi^* \subseteq \limsup_{k\to\infty} E_k = \bigcap_{m=n^*}^{\infty} \bigcup_{k\geqslant m} E_k \overset{\Delta}{=} E^\infty, \tag{12}$$

(II) the inequality $(\sum_{j\in\Omega} a_j^d)^k (\sum_{j\in\Gamma^c} a_j^d)^{u_k - k} \leqslant k^{-2}$ is satisfied.

Taking logs in (II) and using (10), we obtain the equivalent inequality:

$$\frac{\log \sum_{j\in\Omega} a_j^d}{-\log \sum_{j\in\Gamma^c} a_j^d} + \frac{2\log k}{-k \log \sum_{j\in\Gamma^c} a_j^d} + 1 \leqslant \frac{u_k}{k}. \tag{13}$$

Choose n^* large enough to assure that when $k \geqslant n^*$

$$\frac{2\log k}{-k\log\sum_{j\in\Gamma^c} a_j^d} < \frac{t}{2} \text{ and } \frac{1}{k} < \frac{t}{8}. \tag{14}$$

Now for each $k \geqslant n^*$ we can choose u_k such that

$$\xi^{-1} - \frac{t}{2} < \frac{u_k}{k} \text{ and } \frac{u_k - 1}{k} \leqslant \xi^{-1} - \frac{t}{2}. \tag{15}$$

By (15) and the second inequality of (14), we obtain

$$\xi^{-1} - \frac{t}{2} < \frac{u_k}{k} < \xi^{-1} - \frac{t}{4}. \tag{16}$$

According to (11), (14) and the first inequality in (16), for $k \geqslant n^*$

$$\frac{\log\sum_{j\in\Omega} a_j^d}{-\log\sum_{j\in\Gamma^c} a_j^d} + \frac{2\log k}{-k\log\sum_{j\in\Gamma^c} a_j^d} + 1 < \xi^{-1} - 1 - t + \frac{t}{2} + 1 < \frac{u_k}{k}.$$

So (13), i.e. the inequality in (II), is satisfied for $k \geqslant n^*$.

To verify (12), we need to show that for each point $x \in F_\xi^*$ with location code σ there exists a strictly increasing sequence $\{k_i : i \in \mathbf{N}\}$ of positive integers such that $x \in E_{k_i}$. By the definitions (5) and (7) of $z(\sigma, n)$ and F_ξ^*, for each $x \in F_\xi^*$ with a location code $\sigma \in \Lambda^*$ there are two different cases, i.e. $\#\{1 \leqslant i < +\infty : \sigma(i) \in \Gamma\} = +\infty$ or $\#\{1 \leqslant i < +\infty : \sigma(i) \in \Gamma\} < +\infty$. In the former case, there exists a strictly increasing sequence $\{n_i : i \in \mathbf{N}\}$ of positive integers such that $z(\sigma, n_1) \geqslant n^*$ and

$$\frac{z(\sigma, n_i + 1)}{z(\sigma, n_i)} > \xi^{-1} - \frac{t}{4}. \tag{17}$$

Taking $k_i = z(\sigma, n_i)$ and using (17) as well as the second inequality in (16), we have $z(\sigma, n_i + 1) > u_{k_i}$, which implies that $x \in E_{k_i}$. In the latter case, it is easy to see that $x \in E_k$ for all $k \geqslant \max\{1 \leqslant i < +\infty : \sigma(i) \in \Gamma\}$.

On the other hand, since each E_k can be covered with $\{J_{\sigma * \tau} : \sigma \in \mathbf{\Omega}^k$ and $\tau = (\tau(1), \tau(2), \ldots, \tau(u_k - k))$ with $\tau(j) \in \Gamma^c$ for $j = 1, \ldots, u_k - k\}$, for any $m \geqslant n^*$ we have

$$\mathscr{H}^d(E^\infty) \leqslant \mathscr{H}^d\Big(\bigcup_{k\geqslant m} E_k\Big) \leqslant |J|^d \sum_{k\geqslant m}\Big[\Big(\sum_{j\in\Omega} a_j^d\Big)^k \Big(\sum_{j\in\Gamma^c} a_j^d\Big)^{u_k - k}\Big] \leqslant |J|^d \sum_{k\geqslant m} k^{-2},$$

by (I) and (II). So we have $\mathscr{H}^d(E^\infty) = 0$ by letting $m \to \infty$. Consequently, $\mathscr{H}^d(F_\xi^*) = 0$ which implies $\dim_H F_\xi^* \leqslant \eta$.

We now turn to the second part of the proof of the case $0 < \xi < 1$, where we shall show $\dim_H F_\xi \geqslant \eta$ and $\dim_P F_\xi \geqslant s$ by proving that for any fixed $0 < d < \eta$ and $0 < d^* < s$ there exists a subset $E = E(d, d^*)$ of F_ξ such that $\dim_H E \geqslant d$ and $\dim_P E \geqslant d^*$. Let the constant c be defined by

$$c = \max\Big\{\log\#\Gamma, \log\#\Gamma^c, \Big|\log\sum_{j\in\Gamma} a_j^s\Big|, \Big|\log\sum_{j\in\Gamma^c} a_j^s\Big|\Big\}. \tag{18}$$

Note that both functions $T(x)$ and $G(x) \overset{\Delta}{=} \log\sum_{j\in\Omega} a_j^x$ are strictly decreasing with $T(\eta) = 0$ and $G(s) = 0$. Therefore we can choose a $0 < \epsilon < 1$ which satisfies:

[ϵ1] the solutions x of the following inequality will be in $[d, +\infty)$

$$T(x) = \xi \log \sum_{j\in\Omega} a_j^x + (1-\xi)\log \sum_{j\in\Gamma^c} a_j^x \leqslant \frac{6c\epsilon}{1-\epsilon}; \tag{19}$$

[ϵ2] the solutions x of the following inequality will be in $[d^*, +\infty)$

$$\log \sum_{j\in\Omega} a_j^x \leqslant \frac{5c\epsilon}{1-\epsilon}; \tag{20}$$

[ϵ3] with t being such that $\sum_{j\in\Gamma^c} a_j^t = 1$,

$$\frac{4c\epsilon}{1-\epsilon} < \log(1 + \sum_{j\in\Gamma} a_j^t). \tag{21}$$

Given any sequence of integers $0 < k_1 < u_1 < u_{1,1} < \cdots < u_{1,n_1} < k_2 < u_2 < \cdots < k_i < u_i < u_{i,1} < u_{i,2} < \cdots < u_{i,n_i} < k_{i+1} < u_{i+1} < \cdots$, we construct a set E as follows:

$$E = \{x\colon x(k_i), x(u_i), x(u_{i,j}) \in \Gamma \text{ and } x(k) \in \Gamma^c \text{ for } k_i < k < u_i, i \geqslant 1, 1 \leqslant j \leqslant n_i\}.$$

The set E is a closed subset of F. In the definition of E, when $k_i < k < u_i$, $x(k)$ is a restricted Γ^c-choice; when $k = k_i, u_i$ and $u_{i,j}$, $x(k)$ is a restricted Γ-choice; and the rest of $x(k)$ is a free choice, i.e. Ω-choice. In the following we specify $k_i, u_i, u_{i,1}, \ldots, u_{i,n_i}$ depending on ϵ to construct the set $E = E_\epsilon$.

By $N_\Omega(k)$, $N_\Gamma(k)$ and $N_{\Gamma^c}(k)$ we denote the number of Ω-, Γ- and Γ^c-choices among the first k entries in a location code of a point in E respectively. Thus we have $N_\Omega(k) + N_\Gamma(k) + N_{\Gamma^c}(k) = k$ for $k \in \mathbf{N}$. Note that for $k_i - 1 \leqslant j \leqslant u_i$, $N_\Omega(j) = N_\Omega(k_i)$. Now define a sequence of positive integers b_i, $i \in \mathbf{N}$, by

$$b_{i+1} = b_i\xi^{-1} + \theta_i, \quad \text{where } i \in \mathbf{N} \text{ and } 0 \leqslant \theta_i < 1, \tag{22}$$

with $b_1 = 6$. Thus the b_i increase strictly and tend to $+\infty$. We take the sequence of positive integers $k_1, u_1, u_{1,1}, \ldots, u_{1,n_1}, k_2, u_2, \ldots$, in the definition of E as the sequence $b_1, b_2, \ldots$. From the definition of E and (22) it follows that $E \subseteq F_\xi$ since

$$\xi^{-1} = \lim_{i\to\infty} \frac{u_i}{k_i} \leqslant \limsup_{n\to\infty} \frac{z(\sigma, n+1)}{z(\sigma, n)} \leqslant \limsup_{i\to\infty} \frac{b_{i+1}}{b_i} = \xi^{-1}.$$

We have not yet specified indices n_i. Suppose that the n_j are defined for $j = 1, 2, \ldots, i-1$, then also k_i and u_i are determined. Letting n_i vary, we have (using Stolz's theorem in the second line)

$$\begin{aligned}\lim_{n_i\to\infty} \frac{N_\Omega(u_{i,n_i})}{u_{i,n_i}} &= \lim_{n_i\to\infty} \frac{N_\Omega(u_i) + u_{i,n_i} - u_i - n_i}{u_{i,n_i}} \\ &= 1 - \lim_{n_i\to\infty} \frac{n_i}{u_{i,n_i}} = 1 - \lim_{n_i\to\infty} \frac{n_i - (n_i - 1)}{u_{i,n_i} - u_{i,n_i-1}} \\ &= 1 - \lim_{n_i\to\infty} \frac{1}{(\xi^{-1}-1)u_{i,n_i-1}} = 1.\end{aligned}$$

Therefore noting that we can identify k_{i+1} with u_{i,n_i+1}, we can choose n_i such that

$$N_\Omega(k_{i+1}) \geqslant (1-\epsilon)k_{i+1}. \tag{23}$$

Now take $E \overset{\Delta}{=} E_\epsilon$ corresponding to the choice of n_i satisfying (23). According to Proposition 2·1 we have $\dim_H E = \liminf_{k\to\infty} d(k)$ where $d(k)$ satisfies

$$\left(\sum_{j\in\Omega} a_j^{d(k)}\right)^{N_\Omega(k)} \left(\sum_{j\in\Gamma} a_j^{d(k)}\right)^{N_\Gamma(k)} \left(\sum_{j\in\Gamma^c} a_j^{d(k)}\right)^{k-N_\Omega(k)-N_\Gamma(k)} = 1. \tag{24}$$

Taking logs in (24) we get after some algebra

$$\xi \log \sum_{j\in\Omega} a_j^{d(k)} + (1-\xi)\log \sum_{j\in\Gamma^c} a_j^{d(k)} = \left(1 - \frac{\xi k}{N_\Omega(k)}\right)\log \sum_{j\in\Gamma^c} a_j^{d(k)} + \frac{\xi N_\Gamma(k)}{N_\Omega(k)}\left(\log \sum_{j\in\Gamma^c} a_j^{d(k)} - \log \sum_{j\in\Gamma} a_j^{d(k)}\right). \tag{25}$$

We shall show that there exists an i^* such that $d(k) \geqslant d$ when $i \geqslant i^*$ and $k_i \leqslant k < k_{i+1}$. Then $\dim_H E \geqslant d$ by Proposition 2·1. At first note that since $0 \leqslant d(k) \leqslant s$, we have $\sum_{j\in\Omega} a_j^{d(k)} \geqslant 1$ and therefore by (24) we have $d(u_{i,j}) \leqslant d(k)$ when $u_{i,j} < k < u_{i,j+1}$, $0 \leqslant j \leqslant n_i$, with $u_{i,0} \overset{\Delta}{=} u_i$ and $u_{i,n_i+1} \overset{\Delta}{=} k_{i+1}$. So we only need to consider $d(k)$ for $k_i \leqslant k \leqslant u_i$ and $k = u_{i,j}$. When $k_i \leqslant k \leqslant u_i$, then $N_\Omega(k) = N_\Omega(k_i)$ and hence the equality (25) can be written as

$$\begin{aligned} T(d(k)) &= \xi \log \sum_{j\in\Omega} a_j^{d(k)} + (1-\xi)\log \sum_{j\in\Gamma^c} a_j^{d(k)} \\ &= \left(1 - \frac{\xi k}{N_\Omega(k_i)}\right)\log \sum_{j\in\Gamma^c} a_j^{d(k)} \\ &\quad + \frac{\xi N_\Gamma(k)}{N_\Omega(k_i)}\left(\log \sum_{j\in\Gamma^c} a_j^{d(k)} - \log \sum_{j\in\Gamma} a_j^{d(k)}\right). \end{aligned} \tag{26}$$

Note that by (22) and (23)

$$(1-\epsilon)k_i \leqslant N_\Omega(k_i) \leqslant k_i \quad \text{and} \quad u_i = \xi^{-1}k_i + \theta \text{ for some } 0 \leqslant \theta < 1. \tag{27}$$

So we have

$$\frac{N_\Gamma(k)}{N_\Omega(k_i)} = \frac{k - N_\Omega(k_i) - N_{\Gamma^c}(k)}{N_\Omega(k_i)} \leqslant \frac{k_i - N_\Omega(k_i) + 1}{N_\Omega(k_i)} \leqslant \frac{\epsilon}{1-\epsilon} + \frac{1}{N_\Omega(k_i)}, \tag{28}$$

and

$$\begin{aligned} 1 - \frac{k\xi}{N_\Omega(k_i)} &\geqslant 1 - \frac{u_i\xi}{N_\Omega(k_i)} = 1 - \frac{k_i + \xi\theta}{N_\Omega(k_i)} \geqslant 1 - \frac{1}{1-\epsilon} - \frac{1}{N_\Omega(k_i)} \\ &= -\frac{\epsilon}{1-\epsilon} - \frac{1}{N_\Omega(k_i)}, \end{aligned} \tag{29}$$

by (27). Now because $u_i \to \infty$, also $k_i \to \infty$ and by (23), we can take i^* by requiring that when $i \geqslant i^*$,

[i^*1] for $N_\Omega(k_i)$ we have

$$\frac{1}{N_\Omega(k_i)} \leqslant \frac{\epsilon}{1-\epsilon}; \tag{30}$$

$[i^*2]$ for u_i we have

$$(\xi^{-1}-1)u_i > 1 + \frac{\max\{|\log\sum_{j\in\Gamma} a_j^d|, \log(\#\Gamma)\}}{\log\sum_{j\in\Omega} a_j^d}. \tag{31}$$

Note that when $i \geqslant i^*$ and $k_i \leqslant k \leqslant u_i$, we have $\log\sum_{j\in\Gamma^c} a_j^{d(k)} < 0$. Otherwise suppose that $\log\sum_{j\in\Gamma^c} a_j^{d(k)} \geqslant 0$, i.e., $d(k) \leqslant t$ with $\sum_{i\in\Gamma^c} a_i^t = 1$. Thus from (26), (28), (18) and (30) it follows that

$$\begin{aligned}
\log\left(1+\sum_{j\in\Gamma} a_j^t\right) &= \log\sum_{j\in\Omega} a_j^t \leqslant \log\sum_{j\in\Omega} a_j^{d(k)} \\
&= \left(1-\frac{k}{N_\Omega(k_i)}\right)\log\sum_{j\in\Gamma^c} a_j^{d(k)} + \frac{N_\Gamma(k)}{N_\Omega(k_i)}\left(\log\sum_{j\in\Gamma^c} a_j^{d(k)} - \log\sum_{j\in\Gamma} a_j^{d(k)}\right) \\
&\leqslant \frac{N_\Gamma(k)}{N_\Omega(k_i)}\left(\log\sum_{j\in\Gamma^c} a_j^{d(k)} - \log\sum_{j\in\Gamma} a_j^{d(k)}\right) \leqslant \frac{4c\epsilon}{1-\epsilon},
\end{aligned}$$

which contradicts (21). Therefore when $i \geqslant i^*$ and $k_i \leqslant k \leqslant u_i$, from (26), (29), (18), (28) and (30) it follows that

$$\begin{aligned}
T(d(k)) &= \xi\log\sum_{j\in\Omega} a_j^{d(k)} + (1-\xi)\log\sum_{j\in\Gamma^c} a_j^{d(k)} \\
&= \left(1-\frac{\xi k}{N_\Omega(k_i)}\right)\log\sum_{j\in\Gamma^c} a_j^{d(k)} + \frac{\xi N_\Gamma(k)}{N_\Omega(k_i)}\left(\log\sum_{j\in\Gamma^c} a_j^{d(k)} - \log\sum_{j\in\Gamma} a_j^{d(k)}\right) \\
&\leqslant \left(-\frac{\epsilon}{1-\epsilon} - \frac{1}{N_\Omega(k_i)}\right)\log\sum_{j\in\Gamma^c} a_j^{d(k)} \\
&\quad + \frac{\xi N_\Gamma(k)}{N_\Omega(k_i)}\left(\log\sum_{j\in\Gamma^c} a_j^{d(k)} - \log\sum_{j\in\Gamma} a_j^{d(k)}\right) \leqslant \frac{6c\epsilon}{1-\epsilon},
\end{aligned}$$

i.e. $d(k) \geqslant d$ by (19).

This takes care of k with $k_i \leqslant k \leqslant u_i$. But actually, when $i \geqslant i^*$, $d(u_{i,j}) \geqslant d$ also holds for all $j = 1, 2, \ldots, n_i$. Otherwise, suppose $d(u_{i,j^*}) < d$ for some $1 \leqslant j^* \leqslant n_i$. Taking $k = u_i$ and u_{i,j^*} in (24), we get

$$\left(\sum_{j\in\Omega} a_j^{d(u_i)}\right)^{N_\Omega(u_i)} \left(\sum_{j\in\Gamma} a_j^{d(u_i)}\right)^{N_\Gamma(u_i)} \left(\sum_{j\in\Gamma^c} a_j^{d(u_i)}\right)^{N_{\Gamma^c}(u_i)} = 1, \tag{32}$$

and

$$\left(\sum_{j\in\Omega} a_j^{d(u_{i,j^*})}\right)^{N_\Omega(u_{i,j^*})} \left(\sum_{j\in\Gamma} a_j^{d(u_{i,j^*})}\right)^{N_\Gamma(u_{i,j^*})} \left(\sum_{j\in\Gamma^c} a_j^{d(u_{i,j^*})}\right)^{N_{\Gamma^c}(u_{i,j^*})} = 1. \tag{33}$$

Since $d(u_i) \geqslant d > d(u_{i,j^*})$, $N_{\Gamma^c}(u_{i,j^*}) = N_{\Gamma^c}(u_i)$, $N_\Gamma(u_{i,j^*}) = N_\Gamma(u_i)+j^*$ and $N_\Omega(u_{i,j^*}) = N_\Omega(u_i) + \sum_{k=0}^{j^*-1}((\xi^{-1}-1)u_{i,k} + \theta_{i,k} - 1)$ with $u_{i,0} \stackrel{\Delta}{=} u_i$, it follows from (32) and (33) that

$$\left(\sum_{j\in\Omega} a_j^{d(u_{i,j^*})}\right)^{\sum_{k=0}^{j^*-1}((\xi^{-1}-1)u_{i,k}+\theta_{i,k}-1)} \left(\sum_{j\in\Gamma} a_j^{d(u_{i,j^*})}\right)^{j^*} < 1, \tag{34}$$

and also since $d(u_{i,j^*}) < d < s$ we have

$$\sum_{j\in\Omega} a_j^{d(u_{i,j^*})} > 1. \tag{35}$$

Therefore (34), (35) and the monotone increasing of $u_{i,k}, k = 0, \ldots, j^*-1$, imply

$$\begin{aligned}(\xi^{-1}-1)u_i - 1 &\leqslant \frac{\sum_{k=0}^{j^*-1}((\xi^{-1}-1)u_{i,k} + \theta_{i,k} - 1)}{j^*} < \frac{-\log\sum_{j\in\Gamma} a_j^{d(u_{i,j^*})}}{\log\sum_{j\in\Omega} a_j^{d(u_{i,j^*})}} \\ &\leqslant \frac{|\log\sum_{j\in\Gamma} a_j^{d(u_{i,j^*})}|}{\log\sum_{j\in\Omega} a_j^{d(u_{i,j^*})}} \leqslant \frac{\max\{|\log\sum_{j\in\Gamma} a_j^d|, \log(\#\Gamma)\}}{\log\sum_{j\in\Omega} a_j^d},\end{aligned}$$

which is impossible by (31). Thus we complete the proof of $\dim_H E \geqslant d$ for the subset E of F_ξ. Since $d < \eta$ was arbitrary, and since $F_\xi \subseteq F_\xi^*$ it now follows by combining this result with the first part of the proof that $\dim_H F_\xi = \dim_H F_\xi^* = \eta$.

Finally taking $k = k_i$ in (25) and noting that $-(\epsilon/1-\epsilon) \leqslant 1 - (k_i/N_\Omega(k_i)) \leqslant 0$, then when $i \geqslant i^*$ we have

$$\begin{aligned}\log\sum_{j\in\Omega} a_j^{d(k_i)} &= \left(1 - \frac{k_i}{N_\Omega(k_i)}\right)\log\sum_{j\in\Gamma^c} a_j^{d(k_i)} \\ &\quad + \frac{N_\Gamma(k_i)}{N_\Omega(k_i)}\left(\log\sum_{j\in\Gamma^c} a_j^{d(k_i)} - \log\sum_{j\in\Gamma} a_j^{d(k_i)}\right) \leqslant \frac{5c\epsilon}{1-\epsilon},\end{aligned}$$

by (28), (30) and (18). It follows from (20) that $d(k_i) \geqslant d^*$ if $i \geqslant i^*$. By Proposition 2·1 it follows that $\dim_P E = \overline{\dim}_B E \geqslant d^*$. Consequently we get $\dim_P F_\xi^* = \overline{\dim}_B F_\xi^* = \dim_P F_\xi = \overline{\dim}_B F_\xi = s$. The denseness of Λ and Λ^* in Ω^ω implies that $\underline{\dim}_B F_\xi^* = \underline{\dim}_B F_\xi = \underline{\dim}_B F = s$.

This ends the proof of the case $0 < \xi < 1$. We now consider $\xi = 0$. In this case $F_0 = F_0^*$. That $\dim_H F_0 \geqslant \eta$, where η satisfies $\sum_{j\in\Gamma^c} a_j^\eta = 1$, follows again from Proposition 2·1 by constructing an M contained in F_0 of type $M = \phi(\prod_{i=1}^\infty \Omega_i)$ that has Hausdorff dimension η. In fact define M by requiring that $\Omega_i = \Gamma$ if $i = 2^{k^2}$ and $\Omega_i = \Gamma^c$ for $i \neq 2^{k^2}$. That $\dim_H F_0 \leqslant \eta$ follows from the case $0 < \xi < 1$ by using continuity of T and the fact that $F_0 \subseteq F_{1/k}^*$ for $k = 1, 2, \ldots$. To prove that $\dim_P F_0 = s$ one can proceed similarly to the case $0 < \xi < 1$, but now constructing $E = E_\epsilon = \{x: x(k_i) \in \Gamma \text{ and } x(k) \in \Gamma^c \text{ for } k_i < k \leqslant u_i,\ \ i \geqslant 1\}$, which has $u_i = (i+1)k_i$ and $N_\Omega(k_{i+1}) \geqslant (1-\epsilon)k_{i+1}$ for all i.

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