THE DIMENSION OF SETS DETERMINED BY THEIR CODE BEHAVIOR

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Abstract. By prescribing their code run behavior, we consider some subsets of Moran fractals. Fractal dimensions of these subsets are exactly obtained. Meanwhile an interesting decomposition of Moran fractals are given.

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1. Introduction

Fractal dimensions such as Hausdorff, box and packing dimensions are introduced to measure the sizes of fractal sets and are employed in many different disciplines. Unfortunately, it is very difficult to determine the exact fractal dimensions of general fractal sets. Some results on fractal dimensions are obtained for those fractal sets with a special structure. Among them is a typical fractal structure termed as Moran set or Moran fractal. The Moran fractal is an extension of the self-similar set with separation property. The latter is in fact a map-specified Moran fractal. As a fractal set the various dimensions of the Moran fractal have been determined. One turns to studying its various subsets. These subsets generally play an important part in the study of fractals, e.g., they act as the multifractals (ref. [1]), the set of normal numbers and the set of the real numbers in $[0,1]$ for which the digits in their decimal expansion has a prescribed frequency (ref. [4]), the set of non-differentiability points of Cantor function (ref. [2], [3]), etc. Moran fractal is constructed in a good way so that it can be well encoded.

Let $\Omega = \{0, 1, \cdots, r\}$ where r is a positive integer. We write $\Omega^\omega = \{\sigma = (\sigma(1), \sigma(2), \cdots) : \sigma(j) \in \Omega\}$ and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$ with $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) : \sigma(j) \in \Omega\}$ for $k \in \mathbf{N}$. $|\sigma|$ is used to denote the length of word $\sigma \in \Omega^*$. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \cdots, \sigma(|\sigma|), \tau(1), \cdots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \cdots, \tau(|\tau|), \sigma(1), \sigma(2), \cdots)$ for any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$. For $\sigma \in \Omega^k$, let $C(\sigma) =$

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$\{\tau \in \Omega^\omega : \tau|k = \sigma\}$ where $\tau|k = (\tau(1), \tau(2), \cdots, \tau(k))$. $C(\sigma)$ is termed as the cylinder set with base σ. Fixing a nonempty compact set $J \subset \mathbf{R}^n$ with $\overline{\mathrm{int}J} = J$ and positive real numbers $0 < a_j < 1, j = 0, 1, \cdots, r$. First choose a family $\{J_j : j \in \Omega\}$ of nonoverlapping nonempty compact subsets of J such that $\overline{\mathrm{int}J_j} = J_j$ and $|J_j| = a_j|J|$ where $|\cdot|$ denotes the diameter of set. Suppose that J_σ is given for some $\sigma \in \Omega^k$. Take a family $\{J_{\sigma*j} : j \in \Omega\}$ of nonoverlapping nonempty compact subsets of J_σ such that $\overline{\mathrm{int}J_{\sigma*j}} = J_{\sigma*j}$, $|J_{\sigma*j}| = a_j|J_\sigma|$. We assume that there is a constant $0 < c < 1$ such that each J_σ contains an open ball of diameter $c|J_\sigma|$. The Moran fractal F associated with $\{0 < a_j < 1, j \in \Omega\}$ and the $J_\sigma, \sigma \in \Omega^*$ is defined as the nonempty compact set

$$F = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \Omega^k} J_\sigma. \tag{1}$$

We shall refer to J_σ as a k-th level component set of F if $\sigma \in \Omega^k$. Define ϕ: $\Omega^\omega \longrightarrow \mathbf{R}^n$ by

$$\{\phi(\sigma)\} = \bigcap_{k=1}^{\infty} J_{\sigma|k}. \tag{2}$$

It is easy to see that $\phi(\Omega^\omega) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (1) and (2). But ϕ may not be an injection. An important property of ϕ is that there is positive constant c_1, independent of $x \in F$, such that

$$\sup_{x \in F} \#\{\phi^{-1}(x)\} < c_1, \tag{3}$$

by means of Lemma 9.2 in [5]. Thus each $x \in F$ can be encoded via ϕ: $\sigma \in \Omega^\omega$ is called a location code of $x \in F$ if $\phi(\sigma) = x$. Note that each $x \in F$ has at most c_1 codes by (3). In this paper sometimes we use $(x(1), x(2), \cdots) \in \Omega^\omega$ to denote a specified location code of $x \in F$ if no confusion arises.

A Moran fractal is termed map-specified if there exist similitude contractions $h_j(x)$: $\mathbf{R}^n \longrightarrow \mathbf{R}^n, j = 0, 1, \cdots, r$, such that $J_\sigma = h_\sigma(J)$ for any $\sigma \in \Omega^*$ where $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(|\sigma|)}(x)$. In this case F is actually the self-similar set determined by $\{h_j : j \in \Omega\}$, which satisfies the open set condition with respect to the open set $O = \mathrm{int}J$, i.e., $\bigcup_{j=0}^r h_j(O) \subseteq O$ with a disjoint union on the left, and the coding map ϕ in (2) can be rewritten as

$$\{\phi(\sigma)\} = \bigcap_{k=1}^{\infty} h_{\sigma|k}(\overline{O}) = \{\lim_{k\to\infty} h_{\sigma|k}(0)\}. \tag{4}$$

Now let $\Gamma \subseteq \Omega = \{0, 1, \cdots, r\}$ be non-empty such that $\Gamma^c \neq \emptyset$. For $n \in \mathbf{N}$ and $\sigma \in \Omega^\omega$, we define $N_n(\sigma)$, the length of the Γ-run starting at n, by

$$N_n(\sigma) = \begin{cases} 0, & \text{if } \sigma(n) \notin \Gamma \\ k, & \text{if } \sigma(j) \in \Gamma \text{ for } j = n, n+1, \cdots, n+k-1, \text{ and } \sigma(n+k) \notin \Gamma \\ +\infty, & \text{if } \sigma(k) \in \Gamma \text{ for } k \geq n. \end{cases}$$

Hence for each $\sigma \in \Omega^\omega$ the sequence $\{N_n(\sigma)\}$ characterize its behavior. Let $\{Q(n)\}$ be a sequence of real numbers such that $Q(n) \uparrow +\infty$ as $n \longrightarrow +\infty$ and $\lim_{n\to+\infty} \frac{Q(n)}{n} = \alpha \in [0, +\infty]$. Let

$$\Lambda = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{N_n(\sigma)}{Q(n)} = 1\right\} \text{ and } M = \phi(\Lambda). \tag{5}$$

It is easy to see that M is dense in F since Λ is dense in Ω^ω. The Hausdorff dimension of M is expected depending on α. For the extreme case when $\alpha = 0$, it was proved in [8] that

$$\dim_H M = \dim_P M = \dim_B M = \dim_H F = s,$$

where s is given by

$$\sum_{j\in\Omega} a_j^s = 1. \tag{6}$$

Throughout this paper the positive real number s is always assumed giving in this equation. While for the general case when $\alpha \in (0, +\infty)$, we have

$$\dim_H M = \tilde{\eta} \text{ and } \dim_P M = \dim_B M = \dim_H F = s,$$

where $\tilde{\eta}$ is determined by

$$\frac{1}{1+\alpha} \log \sum_{j\in\Omega} a_j^{\tilde{\eta}} + \frac{\alpha}{1+\alpha} \log \sum_{j\in\Gamma} a_j^{\tilde{\eta}} = 0. \tag{7}$$

This can be easily derived from the result in [9]. In this paper we discuss the fractal dimensions of M for the another extreme case when $\alpha = +\infty$. We obtain

Theorem. *Let $\{Q(n)\}$ be a sequence of real numbers such that $Q(n) \uparrow +\infty$ as $n \longrightarrow +\infty$ and $\lim_{n\to+\infty} \frac{Q(n)}{n} = +\infty$. Let M be defined by (5). Let η be given by*

$$\sum_{j\in\Gamma} a_j^\eta = 1. \tag{8}$$

Then $\dim_H M = \eta$ with $\mathcal{H}^\eta(M) = +\infty$, $\dim_B M = \dim_P M = \dim_H F = s$ with s given in (6), and M is dense in F.

Summarizing above, the $\dim_H M$ for all $\alpha \in [0, +\infty]$ can be given by the unified formula (7) if we adopt the convention that $\frac{1}{\infty} = 0$ and $\frac{\infty}{\infty} = 1$.
Taking $Q(n) = tn^2$ in (5), we can get $F_t \subseteq F$ for $t \in (0, +\infty)$ by

$$\Lambda_t = \left\{\sigma \in \Omega^\omega : \limsup_{n\to\infty} \frac{N_n(\sigma)}{n^2} = t\right\} \text{ and } F_t = \phi(\Lambda_t). \tag{9}$$

Then we can obtain a decomposition of F, i.e., $F = \bigcup_{t\in[0,+\infty]} F_t$ by allowing $t = 0, +\infty$ in (9). These F_t have the following properties and will be verified at the end of next section:

(P1) $F = \bigcup_{t\in[0,+\infty]} F_t$ with $\sup_{x\in F} \#\{t : x \in F_t\} < c_1$ by (3), i.e., each $x \in F$ belongs to at most c_1 distinct F_t;

(P2) Each F_t is dense in F;

(P3) $\dim_H F_t = \eta$ for $t \in (0, \infty]$, $\dim_H F_0 = s$. $\dim_P F_t = \dim_B F_t = s$ for $t \in [0, \infty]$;

(P4) F_0 is an s-set, while $\mathcal{H}^\eta(F_t) = \infty$ for $t \in (0, \infty]$;

(P5) Moreover, if F is map-specified with similitudes $h_j, j = 0, 1, \cdots, r$, i.e., a self-similar set, then each F_t is invariant under $\{h_j : j = 0, 1, \cdots, r\}$, i.e.,

$$F_t = \bigcup_{j=0}^{r} h_j(F_t).$$

On the other hand, the F_t can be considered as the level sets of the measurable function $Y(\sigma) = \limsup_{n\to\infty} \frac{N_n(\sigma)}{n^2}$: $\Omega^\omega \longrightarrow [0, +\infty]$. It can be employed to construct some very interesting functions, for example, for any $0 < \eta < 1$ a measurable function $g(x) : [0, 1] \longrightarrow [0, +\infty]$ can be made such that for any $t \in (0, +\infty]$ the t-level set of $g(x)$ is dense in $[0, 1]$ and has Hausdorff dimension η.

2. Proofs

In this section, a general dimension result on a class of subsets of Moran sets is first obtained. Then this result will be applied to give a decomposition of Moran sets. The following proposition will be employed.

Proposition. *Let* $A = \phi(\prod_{i=1}^{\infty} \Omega_i)$ *where the* Ω_i *are non-empty subsets of* $\Omega = \{0, 1, \cdots, r\}$, $i \in \mathbf{N}$. *Let* $d(k)$ *be such that*

$$\prod_{i=1}^{k} (\sum_{j \in \Omega_i} a_j^{d(k)}) = 1.$$

Then $\dim_H A = \liminf_{k \to \infty} d(k) \triangleq \underline{d}$ (see [6]).

Proof This result can be found in [6] for the more general Moran fractal structure. Here a simplified proof is given for this special case. For any $d > \underline{d}$, there exists a sequence $\{n_k : k = 1, 2, \cdots\}$ such that $d(n_k) < d$. For any $\delta > 0$, we can take k large enough such that $\{J_\sigma : \sigma \in \prod_{i=1}^{n_k} \Omega_i\}$ is a δ-covering of A. Then

$$\mathcal{H}_\delta^d(A) \le \sum_{\sigma \in \prod_{i=1}^{n_k} \Omega_i} |J_\sigma|^d = |J|^d \prod_{i=1}^{n_k} (\sum_{j \in \Omega_i} a_j^d) \le |J|^d \prod_{i=1}^{n_k} (\sum_{j \in \Omega_i} a_j^{d(n_k)}) = |J|^d,$$

which implies that $\dim_H A \le d$. So we have $\dim_H A \le \underline{d}$.

Now we turn to prove that $\dim_H A \ge \underline{d}$. Let us suppose without loss of generality that $\underline{d} > 0$. Then we only need to prove that for any fixed $0 < d < \underline{d}$, $\dim_H A \ge d$. Let us construct a probability measure $\widetilde{\mu}$ on $S^\omega \triangleq \prod_{i=1}^{\infty} \Omega_i$ such that for any $\sigma \in S^k \triangleq \prod_{i=1}^{k} \Omega_i, k = 1, 2, \cdots$

$$\widetilde{\mu}(C^*(\sigma)) = \frac{\prod_{i=1}^{k} a_{\sigma(i)}^d}{\prod_{i=1}^{k} (\sum_{j \in \Omega_i} a_j^d)},$$

where $C^*(\sigma) \triangleq \{\theta \in S^\omega : \theta | k = \sigma\}$ is a cylinder in S^ω with base σ. Let μ be the image measure under ϕ defined in (2) and restricted to S^ω here. For $\epsilon > 0$, write $S_\epsilon = \{\sigma \in \bigcup_{k=1}^{\infty} S^k : a_\sigma \triangleq \prod_{i=1}^{|\sigma|} a_{\sigma(i)} \le \epsilon$ and $a_{\sigma|(|\sigma|-1)} > \epsilon\}$. For any $x \in A$, $\epsilon > 0$, by $B(x, \epsilon)$ we denote the closed ball with center at x and radius ϵ. Let $L_{x,\epsilon} \triangleq \{\sigma \in S_\epsilon : J_\sigma \bigcap B(x, \epsilon) \ne \emptyset\}$. Since $A = \bigcup_{\sigma \in S_\epsilon} (A \bigcap J_\sigma)$ and J_σ, $\sigma \in S_\epsilon$ are pairwise nonoverlapping, there exists a finite constant $\bar{c}$ independent of x, ϵ such that $1 \le \# L_{x,\epsilon} \le \bar{c}$. Thus

$$\mu(B(x, \epsilon)) \le \sum_{\sigma \in L_{x,\epsilon}} \widetilde{\mu}(C^*(\sigma)) = \sum_{\sigma \in L_{x,\epsilon}} \frac{\prod_{k=1}^{|\sigma|} a_{\sigma(k)}^d}{\prod_{i=1}^{|\sigma|} (\sum_{j \in \Omega_i} a_j^d)} \le \bar{c} \epsilon^d,$$

where for the last inequality we use that $\prod_{i=1}^{|\sigma|}(\sum_{j\in\Omega_i} a_j^d) \geq 1$ when ϵ is small enough, since $d < \underline{d}$. Then by Frostman's lemma (ref. [5]) we obtain $\mathcal{H}^d(A) > 0$ which implies that $\dim_H A \geq d$. **QED**

Remark. *Some further results on the set* A *defined in Proposition are:* $\overline{\dim}_B A = \dim_P A = \limsup_{k\to\infty} d(k)$ (see [7]), *and that* $0 < \mathcal{H}^{\underline{d}}(A) < +\infty$ *if and only if* $0 < \liminf_{k\to\infty} \prod_{i=1}^{k}(\sum_{j\in\Omega_i} a_j^{\underline{d}}) < +\infty$ (see [6] & [10]).

Proof of Theorem. (I) We prove $\dim_H M = \eta$.

At first we show $\dim_H M \leq \eta$. To do this, we prove below that $\dim_H M \leq d$ for any $\eta < d < s$.

Now fix arbitrarily d with $\eta < d < s$. Then we have

$$\sum_{j\in\Gamma} a_j^d < 1 \quad \text{and} \quad \sum_{j\in\Omega} a_j^d > 1.$$

Since $\lim_{k\to\infty} \frac{Q(k)}{k} = \infty$, we can take k^* such that when $k \geq k^*$

$$\frac{[0.5Q(k)]}{k} \geq \frac{-2\log k}{k\log(\sum_{j\in\Gamma} a_j^d)} - \left(1 - \frac{1}{k}\right)\frac{\log(\sum_{j\in\Omega} a_j^d)}{\log(\sum_{j\in\Gamma} a_j^d)}, \tag{10}$$

where for a real number a, $[a]$ is the maximal integer not more than a throughout this paper. For $k \geq k^*$, take

$$u_k = k - 1 + [0.5Q(k)], \tag{11}$$

and

$$E_k = \{x \in F : x(i) \in \Gamma \text{ for } k \leq i \leq u_k\}. \tag{12}$$

Thus from the definitions of M and E_k in (5) and (12) it follows

$$M \subseteq \limsup_{k\to\infty} E_k = \bigcap_{m=k^*}^{\infty} \bigcup_{k\geq m} E_k \triangleq E^\infty.$$

Since each E_k can be covered with $\{J_{\sigma*\tau} : \sigma \in \Omega^{k-1} \text{ and } \tau \in \Gamma^{u_k-k+1}\}$, for any $m \geq k^*$ we have

$$\mathcal{H}^d(E^\infty) \leq \mathcal{H}^d(\bigcup_{k\geq m} E_k) \leq |J|^d \sum_{k\geq m}\left(\sum_{j\in\Omega} a_j^d\right)^{k-1}\left(\sum_{j\in\Gamma} a_j^d\right)^{u_k-k+1}.$$

Here we declare that for $k \geq k^*$ we have

$$\left(\sum_{j\in\Omega} a_j^d\right)^{k-1} \left(\sum_{j\in\Gamma} a_j^d\right)^{u_k-k+1} \leq k^{-2},$$

which is equivalent to

$$\frac{u_k}{k} \geq \frac{-2\log k}{k\log(\sum_{j\in\Gamma} a_j^d)} + \left(1-\frac{1}{k}\right) - \left(1-\frac{1}{k}\right)\frac{\log(\sum_{j\in\Omega} a_j^d)}{\log(\sum_{j\in\Gamma} a_j^d)}. \tag{13}$$

But (13) holds by (10) and (11). Therefore we have $\mathcal{H}^d(E^\infty) \leq |J|^d \sum_{k\geq m} k^{-2} \longrightarrow 0$ as $m \longrightarrow \infty$.

Without loss of generality we assume $\eta > 0$. To show $\dim_H M \geq \eta$, we prove below that for arbitrarily fixed $0 < d < \eta$, there exists a subset E of M such that $\dim_H E \geq d$.

Let the positive constant c_2 be defined by

$$c_2 \triangleq -\log \sum_{j\in\Gamma^c} a_j^s. \tag{14}$$

Consider strictly decreasing continuous function

$$G(x) = \log \sum_{j\in\Gamma} a_j^x, \quad 0 \leq x \leq s.$$

Let $0 < \epsilon < 1$ be defined by $\frac{c_2\epsilon}{1-\epsilon} = G(d)$.

Now define a sequence of positive integers b_i, $i \in \mathbf{N}$, by

$$b_{i+1} = b_i + [Q(b_i+1)]. \tag{15}$$

Here we take b_1 large enough to ensure $Q(k) \geq 2$ whenever $k \geq b_1$. Thus $b_i > i$. So the b_i increase strictly and tend to $+\infty$. Construct a set E as follows:

$$E = \{x \in F : \text{ all } \ x(k) \in \Gamma \ \text{ but } \ x(b_i) \in \Gamma^c, \ i = 1, 2, \cdots, \}.$$

Now for any $k \in \mathbf{N}$ there is an i with $b_i \leq k < b_{i+1}$. Thus for any $\sigma \in \Omega^\omega$ with $\phi(\sigma) \in E$, because $N_k(\sigma) \leq N_{b_i+1}(\sigma) = b_{i+1} - b_i - 1$, $N_{b_i}(\sigma) = 0$ and Q is non-decreasing we have

$$\limsup_{k\to\infty} \frac{N_k(\sigma)}{Q(k)} \leq \limsup_{i\to\infty} \frac{N_{b_i+1}(\sigma)}{Q(b_i+1)} = \limsup_{i\to\infty} \frac{b_{i+1}-b_i-1}{Q(b_i+1)} = 1$$

by (15). Therefore $E \subseteq M$ and by Proposition $\dim_H E = \liminf_{k\to\infty} d(k)$ where $d(k)$ with $b_i \le k < b_{i+1}$ is determined by

$$\left(\sum_{j\in\Gamma^c} a_j^{d(k)}\right)^i \left(\sum_{j\in\Gamma} a_j^{d(k)}\right)^{k-i} = 1. \tag{16}$$

Thus $0 \le d(k) \le s$ Note that

$$\lim_{i\to\infty} \frac{i}{b_i} = \lim_{i\to\infty} \frac{i-(i-1)}{b_i - b_{i-1}} = \lim_{i\to\infty} \frac{1}{Q(b_{i-1}+1)} = 0.$$

Thus we can take i^* such that when $i \ge i^*$ we have $\frac{i}{b_i} < \epsilon$. As a result, when $k \ge b_{i^*}$ from (16) and (14) it follows

$$G(d(k)) = \log \sum_{j\in\Gamma} a_j^{d(k)} = \frac{-i}{k-i} \log \sum_{j\in\Gamma^c} a_j^{d(k)} \le \frac{c_2 i}{k-i} \le \frac{c_2\epsilon}{1-\epsilon} = G(d).$$

So we have $d(k) \ge d$, implying $\dim_H E = \liminf_{k\to\infty} d(k) \ge d$.

(II) We prove $\dim_B M = \dim_P M = s$.

The $\dim_B M = \dim_B F = s$ can be derived from the density result proved later. Thus it suffices to prove for any given $0 < d < s$, $\dim_P M \ge d$. Let the positive constant c_3 be defined by

$$c_3 \triangleq \max\{\log \#\Gamma, \log \#\Gamma^c, |\log \sum_{j\in\Gamma} a_j^s|, |\log \sum_{j\in\Gamma^c} a_j^s|\}. \tag{17}$$

Consider the non-negative strictly decreasing function

$$G^*(x) = \log \sum_{j\in\Omega} a_j^x, \quad 0 \le x \le s.$$

Let $0 < \epsilon < 1$ be defined by $\frac{3c_3\epsilon}{1-\epsilon} = G^*(d)$.

Take the sequence of integers $0 < k_1 < u_1 < u_{1,1} < \cdots < u_{1,n_1} < k_2 < u_2 < \cdots < k_i < u_i < u_{i,1} < u_{i,2} < \cdots < u_{i,n_i} < k_{i+1} < u_{i+1} < \cdots$ as $\{b_i, i = 1, 2, \cdots\}$ defined in (15) and construct a set E as follows:

$$E = \{x \in F : x(b_i) \in \Gamma^c \text{ and } x(k) \in \Gamma \text{ for } k_i < k < u_i, \ i \ge 1\}. \tag{18}$$

Obviously for any choice of $\{n_i, i = 1, 2, \cdots\}$, the set E is a closed set with $E \subseteq M \subseteq F$ and is a generalized Moran fractal described in Proposition since $E = \phi(\prod_{i=1}^\infty \Omega_i)$ with $\Omega_{b_i} = \Gamma^c$, $\Omega_k = \Gamma$ when $k_i < k < u_i$ and $\Omega_i = \Omega$ for the rest. Denote

$$N_\Omega(k) \triangleq \#\{i : \Omega_i = \Omega, 1 \le i \le k\}, \quad N_\Gamma(k) \triangleq \#\{i : \Omega_i = \Gamma, 1 \le i \le k\}$$

and

$$N_{\Gamma^c}(k) \triangleq \#\{i : \Omega_i = \Gamma^c, 1 \le i \le k\}.$$

Thus we have that $N_\Omega(k) + N_\Gamma(k) + N_{\Gamma^c}(k) = k$ for $k \in \mathbf{N}$.
Note that for $k_i - 1 \le k \le u_i$, $N_\Omega(k) = N_\Omega(k_i - 1)$. For convenience we put

$$f_i = N_\Omega(k_i - 1).$$

We shall now make a choice for $\{n_i, i = 1, 2, \cdots\}$, based on the previously defined ϵ. Suppose that the n_ℓ are defined for $\ell = 1, 2, \cdots, i-1$, then also k_i and u_i are determined. Letting n_i vary, we have

$$\begin{aligned}\lim_{n_i\to\infty} \frac{N_\Omega(u_{i,n_i})}{u_{i,n_i}} &= \lim_{n_i\to\infty} \frac{N_\Omega(u_i) + u_{i,n_i} - u_i - n_i}{u_{i,n_i}} \\ &= 1 - \lim_{n_i\to\infty} \frac{n_i}{u_{i,n_i}} = 1 - \lim_{n_i\to\infty} \frac{n_i - (n_i - 1)}{u_{i,n_i} - u_{i,n_i-1}} \\ &= 1 - \lim_{n_i\to\infty} \frac{1}{[Q(u_{i,n_i-1} + 1)]} = 1.\end{aligned}$$

Therefore we can choose n_i such that

$$f_{i+1} = N_\Omega(k_{i+1} - 1) \ge (1-\epsilon) k_{i+1}. \tag{19}$$

According to Proposition we have $\overline{\dim}_B E = \dim_P E = \limsup_{k\to\infty} d(k)$ where $d(k)$ satisfies

$$\left(\sum_{j\in\Omega} a_j^{d(k)}\right)^{N_\Omega(k)} \left(\sum_{j\in\Gamma^c} a_j^{d(k)}\right)^{N_{\Gamma^c}(k)} \left(\sum_{j\in\Gamma} a_j^{d(k)}\right)^{N_\Gamma(k)} = 1. \tag{20}$$

Also we have $0 \le d(k) \le s$. Taking logs in (20), and using $N_\Omega(k) + N_\Gamma(k) + N_{\Gamma^c}(k) = k$ we get

$$\begin{aligned}\log \sum_{j\in\Omega} a_j^{d(k)} &= -\frac{N_{\Gamma^c}(k)}{N_\Omega(k)} \log \sum_{j\in\Gamma^c} a_j^{d(k)} \\ &\quad - \frac{k - N_\Omega(k) - N_{\Gamma^c}(k)}{N_\Omega(k)} \log \sum_{j\in\Gamma} a_j^{d(k)}.\end{aligned} \tag{21}$$

Taking $k = k_i$ in (21), we have

$$\begin{aligned}\log \sum_{j\in\Omega} a_j^{d(k_i)} &= -\frac{N_{\Gamma^c}(k_i)}{f_i} \left(\log \sum_{j\in\Gamma^c} a_j^{d(k_i)} - \log \sum_{j\in\Gamma} a_j^{d(k_i)}\right) \\ &\quad - \frac{k_i - f_i}{f_i} \log \sum_{j\in\Gamma} a_j^{d(k_i)}.\end{aligned} \tag{22}$$

Note that

$$0 \le \frac{N_{\Gamma^c}(k_i)}{f_i} = \frac{k_i - N_\Omega(k_i) - N_\Gamma(k_i)}{f_i} \le \frac{k_i - f_i}{f_i} = \frac{k_i}{f_i} - 1 \le \frac{\epsilon}{1-\epsilon} \tag{23}$$

by (19). Note that $|\log\sum_{j\in\Gamma^c} a_j^{d(k_i)}| \le c_3$, $|\log\sum_{j\in\Gamma} a_j^{d(k_i)}| \le c_3$ by (17). Therefore

$$G^*(d(k_i)) = \log\sum_{j\in\Omega} a_j^{d(k_i)} \le \frac{\epsilon}{1-\epsilon}\cdot 2c_3 + \frac{\epsilon}{1-\epsilon}\cdot c_3 = \frac{3c_3\epsilon}{1-\epsilon} = G^*(d),$$

by (22) and (23), which means that $d(k_i) \ge d$. So we get $\overline{\dim}_B E = \dim_P E \ge d$.

(III) We prove $\mathcal{H}^\eta(M) = +\infty$.

Let E be defined in (18). We show that $\mathcal{H}^\eta(E) = +\infty$. Denote $\mathcal{G} \triangleq \prod_{i=1}^\infty \Omega_i$ and $\mathcal{G}^* \triangleq \{\sigma : \sigma \in \mathcal{G}_k, k \ge 1\}$ with $\mathcal{G}_k \triangleq \prod_{i=1}^k \Omega_i$. Let $\mathcal{Q} = \{J_\sigma : \sigma \in \mathcal{G}^*\}$. For $\alpha \ge 0$ define

$$\begin{aligned}\mathcal{H}^\alpha_{\mathcal{Q}}(E) \quad &\triangleq \quad \lim_{\delta\to 0^+}\inf\{\sum_\sigma |J_\sigma|^\alpha : \{J_\sigma\} \text{ is a finite nonoverlapping} \\ &\qquad \delta-\text{covering of} E \text{ and } J_\sigma \in \mathcal{Q}\}.\end{aligned} \tag{24}$$

Then we have (see [6] [10])

$$c_\alpha \mathcal{H}^\alpha_{\mathcal{Q}}(E) \le \mathcal{H}^\alpha(E) \le \mathcal{H}^\alpha_{\mathcal{Q}}(E), \tag{25}$$

where c_α is a positive number depending on α. For $\sigma \in \mathcal{G}^*$ denote

$$C_{\mathcal{G}}(\sigma) \triangleq \{\tau \in \mathcal{G} : \tau||\sigma| = \sigma\}.$$

A finite nonoverlapping δ-covering $\{J_\sigma : \sigma \in \mathcal{T} \subseteq \mathcal{G}^*\}$ of E is called *full* if $\bigcup_{\sigma\in\mathcal{T}} C_{\mathcal{G}}(\sigma) = \mathcal{G}$. Note that a finite nonoverlapping covering of E may not be full. Let $\{J_\sigma : \sigma \in \mathcal{T} \subseteq \mathcal{G}^*\}$ be a finite nonoverlapping δ-covering of E. If it is not full, then for each $\sigma \in \mathcal{T}$ let

$$\mathcal{G}^*_\sigma = \{\tau \in \mathcal{G}^* : |J_\tau| \le |J_\sigma| \text{ but } |J_{\tau|(|\tau|-1)}| > |J_\sigma|,\ J_\tau \bigcap J_\sigma \ne \emptyset\}.$$

Then there is a positive constant q independent of σ such that $\#\mathcal{G}^*_\sigma < q$ by means of Lemma 9.2 in [5]. Take $\mathcal{D} = (\bigcup_{\sigma\in\mathcal{T}} \mathcal{G}^*_\sigma)$. By $\widetilde{\mathcal{T}}$ we denote the subset of $\mathcal{D}$ by deleting those $\tau \in \mathcal{D}$ for which there is a $\gamma \in \mathcal{D}$ with $|\gamma| < |\tau|$ and $\tau||\gamma| = \gamma$. Thus we get a full finite nonoverlapping δ-covering $\{J_\sigma : \sigma \in \widetilde{\mathcal{T}} \subseteq \mathcal{G}^*\}$ of E satisfying

$$\sum_{\sigma\in\widetilde{\mathcal{T}}} |J_\sigma|^\eta \le \sum_{\sigma\in\mathcal{T}}\sum_{\tau\in\mathcal{G}^*_\sigma} |J_\tau|^\eta \le q\sum_{\sigma\in\mathcal{T}} |J_\sigma|^\eta,$$

i.e., for each a finite nonoverlapping $\delta-$ covering $\{J_\sigma : \sigma \in \mathcal{T} \subseteq \mathcal{G}^*\}$ of E there is a full finite nonoverlapping $\delta-$ covering $\{J_\sigma : \sigma \in \tilde{\mathcal{T}} \subseteq \mathcal{G}^*\}$ of E such that

$$\sum_{\sigma \in \mathcal{T}} |J_\sigma|^\eta \geq q^{-1} \sum_{\sigma \in \widetilde{\mathcal{T}}} |J_\sigma|^\eta, \tag{26}$$

where the positive real number q is independent of the choice of the finite nonoverlapping $\delta-$ covering of E. On the other hand, for each full finite nonoverlapping δ-covering $\{J_\sigma : \sigma \in \widetilde{\mathcal{T}} \subseteq \mathcal{G}^*\}$ of E, let $k_1 = \max_{\sigma \in \widetilde{\mathcal{T}}} |\sigma|$ and $k_2 = \min_{\sigma \in \widetilde{\mathcal{T}}} |\sigma|$. In [6] and [7], it states that there is $k_2 \leq k \leq k_1$ such that

$$\sum_{\sigma \in \widetilde{\mathcal{T}}} |J_\sigma|^\eta \geq \sum_{\sigma \in \mathcal{G}_k} |J_\sigma|^\eta = |J|^\eta \prod_{i=1}^{k} \sum_{j \in \Omega_i} a_j^\eta. \tag{27}$$

From (24),(25),(26) and (27) it follows that we only need to prove

$$\liminf_{k \to \infty} \prod_{i=1}^{k} \sum_{j \in \Omega_i} a_j^\eta = +\infty. \tag{28}$$

By the definitions of E and η in (18) and (8) we have

$$\begin{aligned} \log \prod_{i=1}^{k} \sum_{j \in \Omega_i} a_j^\eta &= \log \left(\left(\sum_{j \in \Omega} a_j^\eta \right)^{N_\Omega(k)} \left(\sum_{j \in \Gamma^c} a_j^\eta \right)^{N_{\Gamma^c}(k)} \right) \\ &= N_\Omega(k) \left(\log \sum_{j \in \Omega} a_j^\eta + \frac{N_{\Gamma^c}(k)}{N_\Omega(k)} \log \sum_{j \in \Gamma^c} a_j^\eta \right). \end{aligned} \tag{29}$$

Noting that $\lim_{k \to \infty} N_\Omega(k) = +\infty$, $\sum_{j \in \Omega} a_j^\eta > 1$ and $\lim_{k \to \infty} \frac{N_{\Gamma^c}(k)}{N_\Omega(k)} = 0$, thus (28) holds by (29). For the density result, it is derived directly from the fact that if $\sigma \in \Lambda$ then for any $k \in \mathbf{N}$ those $\tau \in \Omega^\omega$ with $\tau(i) = \sigma(i),\ i \geq k$ will lie in Λ. **QED**

As an application of the Theorem we can now prove (P1), (P2), (P3), (P4) and (P5) of the previous section.

Proof of (P1)–(P5). By (3) and the definition (9) of Λ_t we have $\bigcup_{t \in [0, +\infty]} \Lambda_t = \Omega^\omega$ which leads to (P1). (P2) can be got directly from the definition (9) of F_t. For $t \in (0, \infty)$, we have $\dim_H F_t = \eta$ with $\mathcal{H}^\eta(F_t) = \infty$ and $\dim_P F_t = \dim_B F_t = s$ by taking $Q(n) = tn^2$ in Theorem. If taking $Q(n) = n^3$ in Theorem, we have $F_{+\infty} \supseteq M$ which implies $\dim_H F_{+\infty} \geq \eta$ with $\mathcal{H}^\eta(F_{+\infty}) = \infty$ and $\dim_P F_{+\infty} = \dim_B F_{+\infty} = s$. If taking $Q(n) = n \log n$ in Theorem, we have $F_0 \supseteq M$, which implies $\dim_P F_0 = \dim_B F_0 = s$. $\dim_H F_{+\infty} \le \eta$ can be got from the fact $F_{+\infty} \subseteq \limsup_{k\to\infty} E_k$ with E_k being defined in (12) and $Q(k) = k^2$ in (11). Finally some results in [8] imply that F_0 is an s-set. Thus we get (P3) and (P4). Note that if $\sigma \in \Lambda_t$ then for any $j \in \Omega$ we have $j * \sigma \in \Lambda_t$ since

$$\limsup_{n\to\infty} \frac{N_n(j*\sigma)}{n^2} = \limsup_{n\to\infty} \left[\frac{N_{n-1}(\sigma)}{(n-1)^2} \cdot \frac{(n-1)^2}{n^2}\right] = t.$$

Thus $\Lambda_t = \bigcup_{j=0}^{r} j * \Lambda_t$ where $j * \Lambda_t = \{j * \sigma : \sigma \in \Lambda_t\}$, which get (P5) by (4). **QED**

In the following example we give a measurable function $g(x) : [0,1] \longrightarrow [0,+\infty]$ such that for any given $0 < \eta < 1$ the t-level set, $t \in (0,+\infty]$, of $g(x)$ is dense in $[0,1]$ and has Hausdorff dimension η.

Example. Take $r = 2$, $\Gamma = \{0,2\}$ and $J = [0,1]$. Take positive real numbers a_0, a_1 and a_2 such that $\sum_{i=0}^{2} a_i = 1$ and $a_0^\eta + a_2^\eta = 1$. Consider the map-specified Moran fractal F with $h_0(x) = a_0 x$, $h_1(x) = a_1 x + a_0$ and $h_2(x) = a_2 x + a_0 + a_1$, $x \in \mathbf{R}^1$. Then we have $F = [0,1]$ and $s = 1$. Note that each $x \in F$ either has unique location code or has only two location codes in $\Omega^\omega = \{0,1,2\} \times \{0,1,2\} \times \cdots$. In the former case, the corresponding x only lies in one of the sets F_t. In the latter case, one of the two location codes only has components 0 except for finitely many components, the other only has components 2 except for finitely many components. So the corresponding x only lies in F_∞. Then (F_t) with $t \in [0,+\infty]$, defined in (9), is a partition of $F(= [0,1])$, satisfying the properties of (P1)-(P5). Define function $g : [0,1] \longrightarrow [0,\infty]$ by

$$g(x) = t \quad \text{if} \quad x \in F_t.$$

Thus we have $g^{-1}(t) = F_t$. Then by Theorem we get a measurable function $g : [0,1] \longrightarrow [0,+\infty]$ satisfying:

- each t-level set $g^{-1}(t) \subseteq [0,1]$ is dense in $[0,1]$, $t \in [0,+\infty]$;
- the t-level set $g^{-1}(t)$ has Hausdorff dimension η and infinite Hausdorff measure in η for $t \in (0,+\infty]$, $g^{-1}(0)$ is a 1-set and $g^{-1}(t)$ have packing and box dimensions 1 for all $t \in [0,+\infty]$;
- $g^{-1}(t) = \bigcup_{i=0}^{2} h_i(g^{-1}(t))$.

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