Perverse sheaves 4
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IV Decomposition Theorem.

<u>THM</u> (Deligne) Let $f: X \to Y$ be a proper smooth (surjective) algebraic map

$$Rf_* \mathbb{Q}_X = \bigoplus_{0 \le i \le d} \mathcal{H}^i(Rf_* \mathbb{Q}_X)[-i] \qquad d = \dim f^{-1}(y) \quad \forall y \in Y.$$

and $\mathcal{H}^i(Rf_* \mathbb{Q}_X)$ is a semi-simple local system. which is dual $\mathcal{H}^{2d-i}(Rf_* \mathbb{Q}_X)$

$$H^*(X, \mathbb{Q}) = H^*(Y, Rf_* \mathbb{Q}_X) = \bigoplus H^{*-i}(Y, \mathcal{H}^i(Rf_* \mathbb{Q}_X)).$$

The spectral sequence $E_2^{p,q} = \mathcal{H}^p(Y, \mathcal{H}^q(Rf_* \mathbb{Q}_X)) \Longrightarrow \mathcal{H}^{p+q}(X)$ degenerate at E_2

Hint Exercise 1.7.3 in de Cataldo's paper "perverse sheaves and the topology of alg var."
Dimca "sheaves in topology" section 5.4.

PD Poincare duality

HL Hard Lefschetz Theorem.

X smooth projective variety Let $\eta \in H^2(X)$ denote the Chern class of an ample line bundle

Then $\mathcal{H}^i(X) \xrightarrow[\sim]{\wedge \eta^{n-i}} \mathcal{H}^{2n-i}(X)$ $\qquad n = \dim X$

HR Hodge Riemann Relation You can define the primitive part of $\mathcal{H}^i(X)$

by Hodge decomposition. $\alpha \in \mathcal{H}^{p,q}_{prim}(X)$. $\qquad p+q+i = n$

$$Q(\alpha, \bar\alpha) := \int \eta^i \wedge \alpha \wedge \bar\alpha \quad \text{is positive definite. or negative definite}$$

<u>rough proof</u>. $R^q_{prim} f_* \mathbb{Q}_X$ is the sub-local system of $\mathcal{H}^q(Rf_* \mathbb{Q}_X)$ with stalk being $H^q_{prim}(f^{-1}(y))$

for $q < \dim f^{-1}(y) = d$

$$\begin{array}{ccc} H^p(Y, R^q_{prim} f_* \mathbb{Q}_X) & \xrightarrow{d_2} & H^{p+2}(Y, R^{q-1}_{prim} f_* \mathbb{Q}_X) \\ \Big\downarrow {\scriptstyle \wedge \eta^{d-q+1} = 0} & & \Big\downarrow {\scriptstyle \wedge \eta^{d-q+1}} \\ H^p(Y, R^{2d-2q+2}_{prim} f_* \mathbb{Q}_X) & \longrightarrow & \mathcal{H}^{p+2}(Y, R^{2d-2q+1}_{prim} f_* \mathbb{Q}_X) \end{array}$$

"f being proper" is necessary.

<u>Example</u> $X = \{([x, y, z], \lambda) \in \mathbb{P}^2 \times \mathbb{C} \setminus \{0,1\} \mid y^2 z = x(x-z)(x-\lambda z)\}$

$\downarrow f$

Example $\quad X = \{([x.y.z], \lambda) \in \mathbb{P} \times (\mathbb{C}\setminus\{0,1\}) \mid yz = x(x-z)(x-\lambda z)\}$

$$\begin{array}{c} X \\ \downarrow f \\ \mathbb{C}\setminus\{0,1\} = Y \end{array}$$

$\mathcal{H}^0(Rf_*\mathbb{Q}_X) = \mathbb{Q}_Y$

$\mathcal{H}^1(Rf_*\mathbb{Q}_X) = \mathcal{L} \quad \mathcal{L}$ is a rank 2 local system. (semi-simple)

$\mathcal{H}^2(Rf_*\mathbb{Q}_X) = \mathbb{Q}_Y \qquad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Example

$$\begin{array}{c} \mathbb{C}^* \searrow \\ \mathbb{C}^2\setminus\{0\} = X \\ \downarrow f \\ \mathbb{P}^1 = Y \end{array}$$

$\mathbb{CP}^1$ is simply connected.

$\mathcal{H}^0(Rf_*\mathbb{Q}_X) = \mathbb{Q}_Y$

$\mathcal{H}^1(Rf_*\mathbb{Q}_X) = \mathbb{Q}_Y$

But $\quad Rf_*\mathbb{Q}_X \neq \mathbb{Q}_X \oplus \mathbb{Q}_X[-1]$

$0 = H^1(X, \mathbb{Q}_X) = H^1(Y, Rf_*\mathbb{Q}_X) = H^1(Y, \mathbb{Q}_X \oplus \mathbb{Q}_X[-1]) = \mathbb{Q}$.

Decomposition THM $\quad$ Let $f: X \to Y$ be a proper algebraic map between two varieties. with X being smooth.

Then $\quad Rf_*\mathbb{Q}_X[\dim X] = \bigoplus_i {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[\dim X])[-i]$

where ${}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[\dim X])$ is a semi-simple perverse sheaves. and dual to ${}^p\mathcal{H}^{-i}(Rf_*\mathbb{Q}_X[\dim X])$ $\quad$ (since $Rf_* = Rf_!$)

$H^*(X, \mathbb{Q}) = \bigoplus$ some intersection cohomology.

MacPherson: "It contains as special cases the deepest homological properties of algebraic maps that we know."

3 different proof
1. BBD
2. Mixed Hodge module $\quad$ Saito.
3. Induction on semi-smallness by de Cataldo, Migliorini

(Williamson. The Hodge theory of the decomposition theorem. Bourbaki report)

Remark. · $\mathbb{Q}_X$ can be replaced by any semi-simple local system.

· If X is singular. $\mathbb{Q}_X[\dim X]$ can be replaced by $j_{!*}\mathbb{Q}_U[\dim X]$, where U is smooth Zariski open in X.

Example $\quad \mathbb{C} \xrightarrow{f} \mathbb{C} \qquad j: \mathbb{C}^* \to \mathbb{C}$

Zariski open in X.

Example $\quad \mathbb{C} \xrightarrow{f} \mathbb{C}$, $\quad x \mapsto x^k \qquad j: \mathbb{C}^* \to \mathbb{C}$

$Rf_* \mathbb{Q}_{\mathbb{C}^*} = \mathcal{L}$ rank k local system $\quad \pi_1(\mathbb{C}^*) \to \begin{pmatrix} 0 & & 1 \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}$

$Rf_* \mathbb{Q}_{\mathbb{C}}$ is semi-simple. In particular. $Rf_* \mathbb{Q}_{\mathbb{C}}[1] = j_{!*} \mathcal{L}[1]$

In fact $\mathcal{L} = \mathbb{Q}_{\mathbb{C}^*} \oplus \mathcal{L}'$ $\qquad j_{!*} \mathbb{Q}_{\mathbb{C}^*}[1] = \mathbb{Q}_{\mathbb{C}}[1] \qquad j_{!*} \mathcal{L}'[1] = j_! \mathcal{L}'[1]$ where $\mathcal{L}' \oplus \mathbb{Q}_{\mathbb{C}^*} = \mathcal{L}$

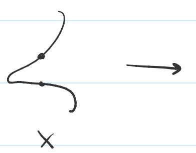
Example $\quad \dim Y = 2$ a smooth variety

$$\begin{array}{c} X = Bl_y Y \\ \downarrow f \\ Y \end{array} \qquad Rf_* \mathbb{Q}_X[2] = \mathbb{Q}_Y[2] \oplus \mathbb{Q}_y$$

$\dim Y = n \geq 3 \qquad Rf_* \mathbb{Q}_X[n] = \mathbb{Q}_Y[n] \oplus \mathbb{Q}_y[-n+2] \oplus \mathbb{Q}_y[-n+4] \oplus \cdots \oplus \mathbb{Q}_y[n-2]$

$\mathbb{C}^m \hookrightarrow \mathbb{C}^n \qquad \begin{array}{c} X = Bl_{\mathbb{C}^m} \mathbb{C}^n \\ \downarrow \\ Y = \mathbb{C}^n \end{array}$ $\qquad$ Exercise: What is $Rf_* \mathbb{Q}_X[n]$

Example



$Rf_* \mathbb{Q}_X[1] = j_{!*} \mathbb{Q}_{Y \setminus \{o\}}[1]$

Example Let X be a smooth projective surface. $\{C_i\}_{1 \le i \le k}$ some smooth irreducible different curve in X. Say $C = \bigcup_{1 \le i \le k} C_i$ is connected.

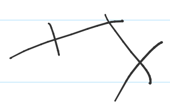
Can we contract C to a point such that the new variety Y is smooth?

$$\begin{array}{ccc} C & \subseteq & X \\ f \downarrow & & \downarrow f \\ y & \in & Y \end{array} \qquad Rf_* \mathbb{Q}_X[2] = \mathbb{Q}_Y[2] \oplus \mathbb{Q}_y^k \oplus \cancel{\mathbb{Q}_y^b[1]}$$

Stratification $\qquad H^*(f^{-1}(y), \mathbb{Q}) = H^*(C) = \begin{cases} \mathbb{Q} & i=0 \\ \mathbb{Q}^k & i=2 \\ \mathbb{Q}^b & i=1. \end{cases}$ $\quad C$ is connected

$\Rightarrow H^1(C) = 0$ i.e. $b = 0$

Then C_i has to be $\mathbb{P}^1$ and no circles.



Computations (搭积木)

$\dim Y = 0$. Y is just a point. PD.

$\dim Y = 1 \qquad Y = Y_1 \sqcup Y_0 \qquad \dim Y_1 = 1 \quad \dim Y_0 = 0.$

(In general we take Y_1 s.t. $f|_{f^{-1}(Y_1)} : f^{-1}(Y_1) \to Y_1$ is a smooth map

on Y_1 we get local systems $\mathcal{H}^i(Rf_* \mathbb{Q}_{f^{-1}(Y_1)}) = \mathcal{L}^i$

say $\dim X = n$ $\quad \mathcal{L}^0 \quad \mathcal{L}^1 \quad \cdots \quad \mathcal{L}^{2n-2}$

$\quad\quad\quad\quad\quad\quad -n \quad -n+1 \quad\quad n-2$

For $y \in Y_0$ $\quad H^0(f^{-1}(y)) \quad H^1(f^{-1}(y)) \quad \cdots \quad H^{2n-2}(f^{-1}(y))$

To be simple say $n=2$. $\quad \mathcal{L}^0 \quad \mathcal{L}^1 \quad \mathcal{L}^2 \quad\quad Y_1 \xrightarrow{j} Y$

$$Rf_* \mathbb{Q}_X[2] = j_{!*}\mathcal{L}^0[2] \oplus j_{!*}\mathcal{L}^1[1] \oplus j_{!*}\mathcal{L}^2 \oplus \cancel{\mathbb{Q}_{y_0}^a[2]} \oplus \cancel{\mathbb{Q}_{y_0}^b[1]} \oplus \mathbb{Q}_{y_0}^c$$

by duality.

$\mathcal{H}^{-1}(j_{!*}\mathcal{L}^0[1])_{y_0} = H^0(f^{-1}(y_0))$

$\mathcal{H}^{-1}(j_{!*}\mathcal{L}^1[1])_{y_0} = H^1(f^{-1}(y_0)) \quad\quad H^0(U_{y_0}, \mathcal{L}^2)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

$\quad\quad \mathcal{H}^2(f^{-1}(y_0)) \simeq \oplus\, H^{-1}(j_{!*}\mathcal{L}^2[1]) \oplus \mathbb{Q}^c$

Example $\quad X = \{ ([x.y.z], \lambda) \in \mathbb{P}^2 \times \mathbb{C} \mid y^2 z = x(x-z)(x-\lambda z) \}$

$\quad\quad \downarrow f$

$\quad\quad Y = \mathbb{C}$.

(X is not smooth but X is a rational homology manifold.
hence $\mathbb{Q}_X[\dim X]$ is a simple perverse sheaves.)

$\mathcal{L}^0 = \mathbb{Q}_{\mathbb{C}\setminus\{0,1\}} \quad\quad \mathcal{L}^2 = \mathbb{Q}_{\mathbb{C}\setminus\{0,1\}} \quad\quad \mathcal{L}^1$ rank 2 local system. $\quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$\quad \mathbb{Q}_{\mathbb{C}} \quad\quad\quad\quad \mathbb{Q}_{\mathbb{C}} \quad\quad\quad \mathcal{H}^{-1}(j_{!*}\mathcal{L}^1[1])_{0,1} = \mathbb{Q}$.

$\lambda = 0 \quad y^2 = x^2(x-1)$

$H^0(f^{-1}(0)) = \mathbb{Q} \quad H^1(f^{-1}(0)) = \mathbb{Q} \quad H^2(f^{-1}(0)) = \mathbb{Q}$

$\lambda = 1 \quad y^2 = x(x-1)^2$

$\mathcal{H}^0(f^{-1}(1)) = \mathbb{Q} \quad \mathcal{H}^1(f^{-1}(1)) = \mathbb{Q} \quad \mathcal{H}^2(f^{-1}(1)) = \mathbb{Q}$

$\dim Y = 2 \quad\quad Y = Y_2 \sqcup Y_1 \sqcup Y_0$

To be simple assume $\dim X = 2$

By computing $H^*(f^{-1}(y))$ for $y \in Y_2$ one gets a local system $\mathcal{L}$ on Y_2

Compute $j_{!*}\mathcal{L}[2]$ where $j : Y_2 \hookrightarrow Y$.

Compute $\mathcal{H}^*(f^{-1}(y))$ for $y \in Y_1$. one gets a $\mathcal{L}'$ on Y_1

Compare $j_{!*}\mathcal{L}[2]$ with $\mathcal{L}'$ $\quad\quad$ ($f^{-1}(y)$ are just points for $y \in Y_1$)

Compare $j_{!*}\mathcal{L}[2]$ with $\mathcal{L}'$. ($f^{-1}(y)$ are just points for $y \in Y_1$)

In fact $j_{!*}\mathcal{L}[2]$ "contains" $\mathcal{L}'[2]$ by duality.

$$Rf_* \mathbb{Q}_X[2] = j_{!*}\mathcal{L}[2] \oplus \cancel{\mathbb{Q}^a_{y_0}[2]} \oplus \cancel{\mathbb{Q}^b_{y_0}[1]} \oplus \mathbb{Q}^c_{y_0}$$

by duality.

$$\mathbb{Q}^c_{y_0} = H^2(f^{-1}(y_0))$$

Application $\underline{\text{Local invariant cycle thm}}$

Thm Let $f: X \to Y$ be a proper alg map with X being smooth.

For any point $y \in Y$. B_y y' a generic point in B_y (back to previous example)

Then $H^i(f^{-1}(y)) \twoheadrightarrow H^i(f^{-1}(y'))^{\text{Inv}}$ for any i.

Proof. $U \subseteq Y$ is the biggest stratum and $y' \in U$ $j: U \hookrightarrow Y$

$$H^0(j_{!*}\mathcal{L}^i[\dim Y])_y \twoheadleftarrow H^i(f^{-1}(y)).$$

$$\parallel$$

$$H^0(U \cap B_y, \mathcal{L}^i) = H^i(f^{-1}(y'))^{\text{Inv}}$$ Invariant part w.r.t. rep for local system $\mathcal{L}^i|_{U \cap B_y}$.

Global invariant cycle theorem

Let f be a proper map from $X \to Y$ with X being smooth. (f is surjective)

Let F be a generic fiber of f. i.e. $\exists$ stratification of Y with U being the Zariski open stratum s.t. $F = f^{-1}(y)$ for $y \in U$.

Then we have the following surjective map

$$\begin{tikzcd} H^i(X) \arrow[rr, two heads] & & H^i(F)^{\text{Inv}} \\ & H^i(\overline{X}) \arrow[ul] \arrow[ur, two heads] & \end{tikzcd}$$

Inv with respect the monodromy by $\pi_1(U)$

$\overline{X}$ is any smooth compactification of X.

Def semi-small. $f: X \to Y$ proper with Whitney stratification $X = \coprod X_\alpha$, $Y = \coprod Y_\beta$

f is called semi-small if $2\dim f^{-1}(y) \leq \dim X - \dim Y_\beta$ for any $y \in Y_\beta$

Prop If f is semi-small, then $Rf_* \mathbb{Q}_X[\dim X]$ is perverse.

If $f: X \to Y$ is proper and $\dim X = \dim Y = 2$, then f is semi-small.

Observation. If f is semi-small, then $Rf_* \mathbb{Q}_X$ is much easier to compute.

In fact we only need $H^*(f^{-1}(y))$ for top dim cohomology.

Use Check def of semi-small + stalk for simple perverse sheaves.

In fact we only need $H^*(f^{-1}(y))$ for top dim cohomology.

Hint. Check def of semi-small + stalk for simple perverse sheaves.

<u>Example</u> Say $f: X \to Y$ is semi-small. with $\dim X = 3$ and X being smooth.

$$Y = Y_3 \amalg Y_2 \amalg Y_1 \amalg Y_0$$

$$Rf_* \mathbb{Q}_X[3] = j_{!*} L_{Y_3}[3] \qquad (L_{Y_3})_y = \mathcal{H}^0(f^{-1}(y)), \; y \in Y_3$$

$j: Y_3 \hookrightarrow Y$ $\qquad \oplus \; \cancel{L_{Y_2}[2]}$

$j': Y_1 \hookrightarrow \overline{Y_1}$ $\qquad \oplus \; j'_{!*} L_{Y_1}[1] \qquad (L_{Y_1})_y = \mathcal{H}^2(f^{-1}(y)) \; y \in Y_1$

$\oplus \; \cancel{\;\;}$ by duality.

for $y \in Y_2$. $f^{-1}(y)$ has dim 0

$y \in Y_1$ $f^{-1}(y)$ has dim ≤ 1

$y \in Y_0$ $f^{-1}(y)$ $\cdots$ ≤ 1

<u>Def</u> Semi-smallness of f $\qquad X = \amalg X_\alpha \qquad Y = \amalg Y_\beta$

$$r(f) := \max_\beta \{ 2\dim f^{-1}(y_\beta) + \dim Y_\beta - \dim X \}$$

f is semi-small iff $r(f) = 0$

Rough idea about the proof of decomposition theorem by de Cataldo. Migliorini.

1. Induction on semi-smallness. reduce the proof to the case where f is semi-small.

2 By Lefschetz hyperplane section theorem. reduce the proof to show

$$Rf_* \mathbb{Q}_X = P \oplus \mathbb{Q}_y^c \quad (*) \text{ where point } y \text{ is a stratum in } Y$$

and $c = \dim \mathcal{H}^{2\dim f^{-1}(y)}(f^{-1}(y))$ if we assume $\dim X$ is even.

To prove $(*)$. one has to use the Hodge Riemann Relation.

for the image $H_n(F. \mathbb{Q}) \longrightarrow H_n(X. \mathbb{Q})$ $\qquad n = \dim X$ is even

(F may have serveral irreducible components)

One needs to use. <u>PD</u>. <u>HL</u>. <u>HR</u> and induction on semi-smallness.

Application. V is a finite dim vector space over some field k.

Say $V = \mathrm{span}\{v_1, \ldots, v_n\}$ and $\dim V = d$

Set $w_i = \# \{ i\text{-dim sub vector space spanned by some vectors in } \{v_1, \ldots, v_n\} \}$

Top heavy conjecture. (Proved by Huh Wang 2018).

$$w_i \le w_{d-i} \quad \text{for } i \le \frac{d}{2}$$

$$w_i \le w_{i+1} \quad \text{for } i \le \frac{d}{2}-1$$

proof by decomposition theorem.

They find a variety Y $\quad V \subseteq \mathbb{C}^n \subseteq \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n \text{ times}}$

$Y = \bar{V}$ $\quad \dim \mathcal{H}^{2i}(Y) = w_i$.

Y is not smooth. but

$$\begin{tikzcd}
\mathcal{H}^*(Y) \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{injective}"] & IH^*(Y) \arrow[r, hook] \arrow[d, "\wr\ HL"] & H^*(X) \\
H^{2d-*}(Y) \arrow[r] & IH^{2d-*}(Y) &
\end{tikzcd}$$

by decomposition theorem

X is some kind of resolution of singularities of Y.

How about more general?

Matroid. A finite set $S = \{v_1, \dots, v_n\}$

We order some subset of S are independent.

Axioms: 1. empty set is independent

2. subset of independent set is independent

3. if A, B are two independent sets and $\#(A) > \#(B)$ then there exists $x \in A \setminus B$ s.t. $B \cup \{x\}$ is independent.

One can also define $\{w_i\}$ for matroid.

How about the top heavy conjecture?

Yes by. Braden. Huh. Matherne. Proudfoot. Wang 2021

Kazhdan-Lusztig Positivity Conjecture by Elias. Williamson. 2014

Log-concave conjecture. (open) for vector space.

$w_i^2 \ge w_{i-1} \cdot w_{i+1}$. $\quad$ even stronger $w_i^2 \ge \frac{i+1}{i} w_{i-1} \cdot w_{i+1}$.