Perverse sheaves 3

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III simple perverse sheaves

What does (co)-support condition really mean?

Recall from yesterday. If X is a smooth variety
then $L[\dim X]$ is perverse for any local system L on X

So in this case $\mathcal{H}^i(L)_x = 0$ for any $i \neq -\dim X$ and any point $x \in X$.

How about general perverse sheaves? What can we say about its stalk cohomology from support condition?

Let P be a perverse sheaf on X (X could be singular)

Say $\mathrm{Supp}(P) = Y$. Then P can be viewed as a Perverse sheaf on Y
(since Y is closed in X)

So w.l.o.g. we assume $Y = X$. and X is irreducible.

$\exists$ Whitney stratification $X = \amalg X_\alpha$ s.t. P is constructible with respect to it.

Say $\mathcal{U}$ is the only stratum which is Zariski dense in X.

Support Condition $\dim \mathrm{Supp}\, \mathcal{H}^q(P^\cdot) \le -q$.

$\dim X = n$. If $\mathcal{H}^i(P)|_{\mathcal{U}} \neq 0$. then $\dim \mathrm{Supp}\, \mathcal{H}^i(P) = \dim X = n \Rightarrow n \le -i$
$i \le -n$

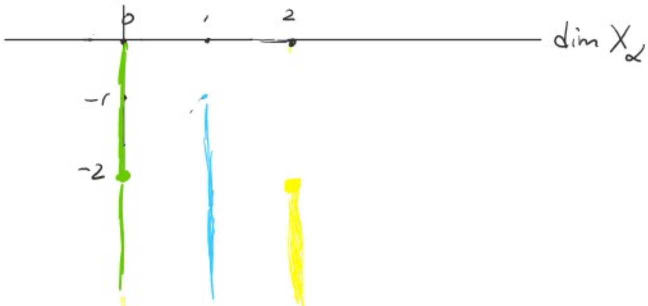
For any stratum X_α $\mathcal{H}^i(P)|_{X_\alpha}$ is a local system

If $\mathcal{H}^i(P)|_{X_\alpha} \neq 0$ then $\dim \mathrm{Supp}\, \mathcal{H}^i(P) \ge \dim X_\alpha \Rightarrow \dim X_\alpha \le -i$.
$i \le -\dim X_\alpha$

So support condition for $P^\cdot$ $\iff$ $\mathcal{H}^i(P)_x = 0$ for $i > -\dim X_\alpha$ and $x \in X_\alpha$

Example

the picture looks like this



X_0 = •

X_1 = (blue curve)

$\mathcal{U} = X_2$ = yellow area.

Cosupport Condition

Say $\mathcal{U}$ is the only stratum s.t. $\mathcal{U}$ is Zariski open in X

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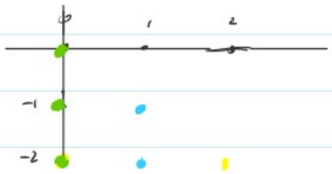
Note that $\mathcal{U}$ is smooth and $\mathcal{H}^i(P)|_{\mathcal{U}}$ is a local system for any i.

$D_X P$ is also local system on $\mathcal{U}$.

$\mathcal{H}^i(D_X P)|_{\mathcal{U}} = (\mathcal{H}^{-i}(P)|_{\mathcal{U}})^\vee$ (dual as local system)

$\Rightarrow$ $P|_{\mathcal{U}} = L[n]$ for some local system L on $\mathcal{U}$ with $\dim \mathcal{U} = n$.

Real picture for perverse sheaves.



Exercise Prove that for $P \in \mathrm{Perv}(X)$ $\mathcal{H}^i(P)_x = 0$ for $i < -\dim X$ and any point $x \in X$.

Def $P \in \mathrm{Perv}(X)$ is called simple if it has no non-trivial sub-perverse sheaves

semi-simple if direct sum of simple perverse sheaves.

Example X is a point $\mathbb{Q}_X^k$ is perverse $\mathbb{Q}_X$ is simple perverse.

Example X is smooth $\mathrm{Loc}(X) \subseteq \mathrm{Perv}(X)$

the category of local system on X

A local system is simple if its rep is irreducible.

X irreducible and $P \in \mathrm{Perv}(X)$ s.t. $\mathrm{Supp}(P) = X$ $\dim X = n$

$\exists$ $\mathcal{U} \subseteq X$ Zariski open and $\mathcal{U}$ smooth s.t. $P|_{\mathcal{U}} = \mathcal{L}[n]$ for some local system $\mathcal{L}$ on $\mathcal{U}$.

So P is some kind of extension of $\mathcal{L}[n]$.

There are natural choices for extension of $\mathcal{L}[n]$.

$j : \mathcal{U} \hookrightarrow X$ $j_! \mathcal{L}[n]$

$Rj_* \mathcal{L}[n]$

Problems: 1 in general $j_! \mathcal{L}[n]$ and $Rj_* \mathcal{L}[n]$ are not perverse.

example: $j : \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}^n$ with $n \geq 2$.

By duality. $D_X(j_! \mathbb{Q}_{\mathcal{U}}[n]) = Rj_* \mathbb{Q}_{\mathcal{U}}[n]$

So we only need to check one of them.

$$\mathcal{H}^i(Rj_* \mathbb{Q}_{\mathcal{U}}[n])_{(0)} = H^{i+n}(\mathbb{C}^n \setminus \{0\}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = -n, n-1 \\ 0 & i \neq -n, n-1 \end{cases}$$

to resolve this issue. we take $\mathcal{U}$ smaller s.t. $Z = X \setminus \mathcal{U}$ is a Cartier divisor

to resolve this issue. we take $\mathcal{U}$ smaller s.t. $Z = X \setminus \mathcal{U}$ is a Cartier divisor in X.
(by refining the stratification)

Claim: $j_! \mathcal{L}[n]$ and $Rj_* \mathcal{L}[n]$ are perverse.

<u>idea of proof</u> By duality only need to check support condition for $Rj_* \mathcal{L}[n]$

for any $x \in Z$, $\mathcal{H}^i(Rj_* \mathcal{L}[n])_x = H^i(\mathcal{U}_x, \mathcal{L}[n]) = H^{i+n}(\mathcal{U}_x, \mathcal{L}|_{\mathcal{U}_x})$

$\mathcal{U}_x = B_x \cap \mathcal{U}$. $= 0$ for $i > 0$.

(since $\mathcal{U}_x$ is a Stein manifold. or by Milnor fibration)

2. $j_! \mathcal{L}[n]$ $Rj_* \mathcal{L}[n]$ are too large as perverse sheaves.

$n = 1$ $\quad \mathbb{C} \setminus \{0\} \xhookrightarrow{j} \mathbb{C} \hookleftarrow \{0\}$

$\quad \mathcal{U} \quad X \quad Z$

$$j_! \mathbb{Q}_\mathcal{U}[1] \longrightarrow \mathbb{Q}_X[1] \longrightarrow Ri_* \mathbb{Q}_{\{0\}}[1] \xrightarrow{+1}$$

$$0 \to Ri_* \mathbb{Q}_{\{0\}} \longrightarrow j_! \mathbb{Q}_\mathcal{U}[1] \longrightarrow \mathbb{Q}_X[1] \longrightarrow 0$$

by duality

$$0 \to \mathbb{Q}_X[1] \longrightarrow Rj_* \mathbb{Q}_\mathcal{U}[1] \longrightarrow Ri_* \mathbb{Q}_{\{0\}} \longrightarrow 0$$

} short exact sequences of perverse sheaves.

3. $j_! \mathcal{L}[n]$, $Rj_* \mathcal{L}[n]$ are not preserved by D_X.

Is there a preferred way to extend $\mathcal{L}[n]$?

$$j_! \mathcal{L}[n] \longrightarrow Rj_* \mathcal{L}[n] \longrightarrow Ri_* i^{-1} Rj_* \mathcal{L}[n] \xrightarrow{+1}$$

$$0 \to {}^p\mathcal{H}^{-1}(Ri_* i^{-1} Rj_* \mathcal{L}[n]) \longrightarrow j_! \mathcal{L}[n] \longrightarrow Rj_* \mathcal{L}[n] \longrightarrow {}^p\mathcal{H}^0(Ri_* i^{-1} Rj_* \mathcal{L}[n]) \to 0 \quad (*)$$

<u>Def</u>: intermediate extension of $\mathcal{L}[n]$

$j_{!*} \mathcal{L}[n] = \text{Image}(j_! \mathcal{L}[n] \longrightarrow Rj_* \mathcal{L}[n])$ in $Perv(X)$

when $j_! \mathcal{L}[n]$ and $Rj_* \mathcal{L}[n]$ are perverse.

$j_{!*} \mathcal{L}[n] = \text{Image}({}^p\mathcal{H}^0(j_! \mathcal{L}[n]) \longrightarrow {}^p\mathcal{H}^0(Rj_* \mathcal{L}[n]))$ in $Perv(X)$

<u>Remark</u> $H^i(X, j_{!*} \mathcal{L}[n]) = \mathbb{H}^{i+n}(X, \mathcal{L})$ the intersection cohomology of X with local system $\mathcal{L}$ on $\mathcal{U}$.

Property $\mathbb{D} \cdot j_{!*} = j_{!*} \mathbb{D}$ "from $(*)$"

<u>Example</u> $j : \mathcal{U} = \mathbb{C} \setminus \{0\} \longrightarrow X = \mathbb{C}$

$$0 \to Ri_* \mathbb{Q}_{\{0\}} \to j_! \mathbb{Q}_\mathcal{U}[1] \to \mathbb{Q}_X[1] \to 0$$

$$0 \to \mathbb{Q}_X[1] \to Rj_* \mathbb{Q}_\mathcal{U}[1] \to Ri_* \mathbb{Q}_{\{0\}} \to 0$$

} $\Rightarrow$ $j_{!*} \mathbb{Q}_\mathcal{U}[1] = \mathbb{Q}_X[1]$

$$\left.\begin{array}{l} 0 \to R i_*^! \mathbb{Q}_{\{0\}} \to j_! \mathbb{Q}_U[1] \to \omega_X[1] \to 0 \\ 0 \to \mathbb{Q}_X[1] \to Rj_* \mathbb{Q}_U[1] \to Ri_* \mathbb{Q}_{\{0\}} \to 0 \end{array}\right\} \Rightarrow \quad j_{!*} \mathbb{Q}_U[1] = \mathbb{Q}_X[1]$$

Deligne's construction

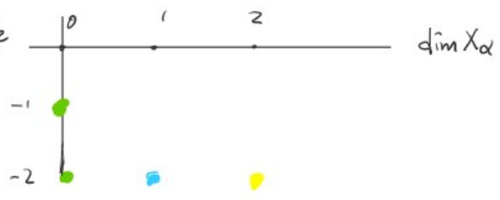
Whitney stratification $X = \coprod X_\alpha$ w.r.t. $\mathcal{L}$.

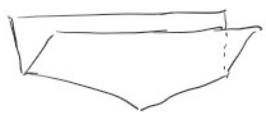
Set U_i = union of codim $\le i$ strata.

$$U = U_0 \xhookrightarrow{j_1} U_1 \xhookrightarrow{j_2} U_2 \hookrightarrow \cdots \xhookrightarrow{j_n} U_n = X$$

$$j_{!*}\mathcal{L}[n] = \tau_{\le -1} Rj_{n*} \cdots\cdots \tau_{\le -n+1} Rj_{2*} \; \tau_{\le -n} Rj_{1*} \mathcal{L}[n]$$

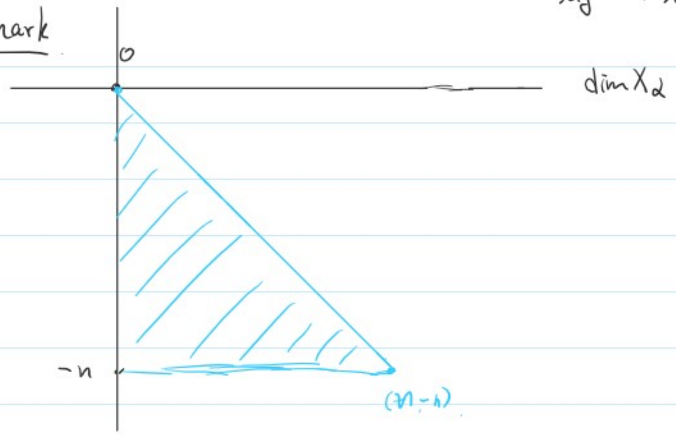
Example

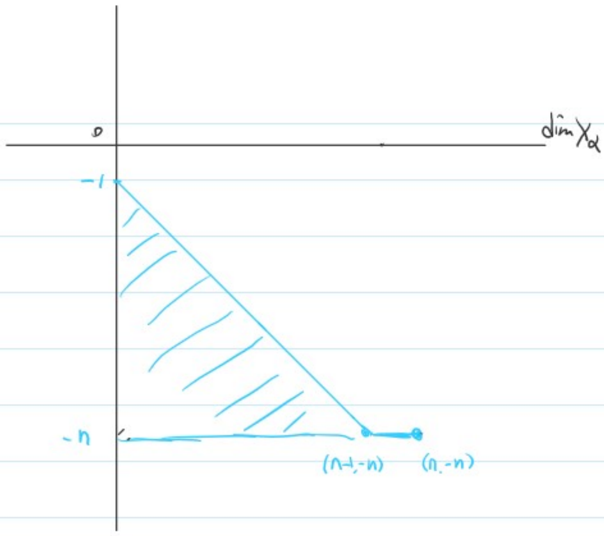




Remark

say $\dim X = n$





Remark Say $U \hookrightarrow U' \xhookrightarrow{j'} X$. Then $j_{!*}\mathcal{L}[n] = j'_{!*}\mathcal{L}'[n]$

$\mathcal{L} = \mathcal{L}'|_U$

$\Rightarrow$ If X is smooth. $j_{!*}\mathbb{Q}_U[n] = \mathbb{Q}_X[n]$.

Exercise Check this by Deligne's construction.

Back to our example.

$U = \mathbb{C}\setminus\{0\} \xhookrightarrow{j} \mathbb{C} = X$

$\mathcal{L}$ a local system on U.

$$\mathcal{H}^i(j_{!*}\mathcal{L}[1])_{\{0\}} = \begin{cases} H^0(U, \mathcal{L}) & i = -1 \\ 0 & i \ne -1 \end{cases}$$

In particular. if $\mathcal{L}$ is a rank one local system. $\pi_1(U) \longrightarrow \lambda \in \mathbb{C}^*$ $\lambda \ne 1$

$$j_!\mathcal{L}[1] = Rj_*\mathcal{L}[1] = j_{!*}\mathcal{L}[1]$$

Example Curve Case. $U \hookrightarrow X$ both curves. for any $x \in X \setminus U$.

$$(j_{!*}\mathcal{L}[1])_x = \begin{cases} \mathcal{H}^0(\mathcal{U}_x, \mathcal{L}|_{\mathcal{U}_x}) & i=-1 \\ 0 & i\neq -1. \end{cases} \qquad j_{!*}\mathcal{L}[1] = j_*\mathcal{L}[1]$$

<u>Property of $j_{!*}$</u> $\quad j_{!*}$ preserves injective maps in Perv sheaves category
surjective maps.

But $j_{!*}$ is not an exact functor.

<u>Example</u> $\quad \mathcal{U} = \mathbb{C}\setminus\{0\} \xrightarrow{j} \mathbb{C} = X$

$\mathcal{L}$ is a rank 2 local system. $\quad \pi_1(\mathcal{U}) \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$0 \to \mathbb{Q}_{\mathcal{U}}[1] \longrightarrow \mathcal{L}[1] \longrightarrow \mathbb{Q}_{\mathcal{U}}[1] \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Q}_X[1] \longrightarrow j_{!*}\mathcal{L}[1] \longrightarrow \mathbb{Q}_X[1] \longrightarrow 0 \quad \text{is not exact.}$$

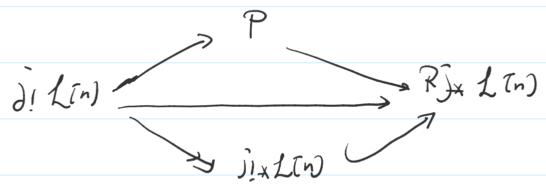
$$\mathcal{H}^i(j_{!*}\mathcal{L}[1])_{\{0\}} = \mathcal{H}^{i+1}(\mathcal{U}, \mathcal{L}) = \begin{cases} \mathbb{Q} & i=-1, 0 \\ 0 & i\neq -1, 0. \end{cases}$$

Exercise: Show that $j_{!*}\mathcal{L}[1]$ has length 3.

<u>THM 1 (BBD)</u> $\quad$ If $\mathcal{L}$ is a simple local system on smooth $\mathcal{U}$,
then $j_{!*}\mathcal{L}[n]$ is a simple perverse sheaves
Moreover, every simple perverse sheaves on X appear in this way.

<u>Exercise</u> $\quad$ Prove Theorem 1. $\quad$ Hint: Exercise 2.7.9 in de Cataldo's paper
"Perverse sheaves and the topology of algebraic varieties"

First, $\mathcal{L}$ has to be a simple perverse sheaves.



THM 2 (BBD) $\quad$ Perv(X) is an Noetherian and Artian abelian category.
Moreover, every perverse sheaf P admits a filtration
$$P_1 \subseteq P_2 \subseteq \cdots \subseteq P_m = P \quad \text{s.t. } P_i/P_{i-1} \text{ is simple.}$$

m is called the length of P.

<u>Remark</u> $\quad$ It's very common to reduce the proof to simple perverse sheaves by induction on length.

<u>Example</u> $\quad \mathcal{U} = \mathbb{C}\setminus\{0\} \xrightarrow{j} \mathbb{C} = X$

- $j_!\mathbb{Q}_{\mathcal{U}}[1]$ has length 2.
- $\mathcal{L}$ is a rank one local system which maps $\pi_1(\mathcal{U})$ to $\lambda \in \mathbb{C}^*$, $\lambda \neq 1$.

- $j_! \mathbb{Q}_u(1)$ has length 2.
- $\mathcal{L}_\lambda$ is a rank one local system which maps $\pi_1(U)$ to $\lambda \in \mathbb{C}^*$ $\lambda \neq 1$

then $j_! \mathcal{L}(1)$ has length 1.

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proof of THM 2 based on THM 1: (Every perverse sheaves has finite length)

Suppose $\mathrm{Supp}(P) = X$.

Induction on $\dim X$. $\dim X = 0$ easy to check.

Say the claim holds for $\dim \leq n-1$. Suppose $\dim X = n$.

$$U \xrightarrow{j} X \xleftarrow{i} Z \qquad \dim Z \leq n-1 \quad \text{s.t. } Z \text{ is a cartier divisor}$$

$$j_! P|_U \longrightarrow P \longrightarrow Ri_* i^* P$$

$$0 \to {}^p\mathcal{H}^{-1}(Ri_* i^* P) \longrightarrow j_!(P|_U) \longrightarrow P \longrightarrow {}^p\mathcal{H}^0(Ri_* i^* P) \longrightarrow 0$$

by induction. they have finite length. (they are supported on Z)

So we only need to check $j_! P|_U$ where $\mathcal{L}[n] = P|_U$

By definition of $j_{!*}$ $\quad 0 \longrightarrow K \longrightarrow j_! \mathcal{L}(n) \longrightarrow j_{!*} \mathcal{L}[n] \longrightarrow 0$

K has support on Z.

So it is enough to prove it for $j_{!*}\mathcal{L}[n]$.

If $\mathcal{L}$ is simple. $j_{!*}\mathcal{L}(n)$ is simple by THM 1

If $\mathcal{L}$ is not simple. there exists short exact sequence of local systems

$$0 \to \mathcal{L}_1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

$\Rightarrow \quad 0 \longrightarrow j_{!*}\mathcal{L}_1(n) \longrightarrow j_{!*}\mathcal{L}(n) \longrightarrow j_{!*}\mathcal{L}_2(n) \longrightarrow 0$ as a complex.

(although it is not exact. the perverse cohomology in the middle has support on Z)

Then proof is done by the fact $\mathcal{L}$ has finite rank.