Perverse sheaves 2

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II perverse sheaves : support and co-support condition.

All homology and cohomology groups have $\mathbb{Q}$-coefficients

sheaves.

$D^b_c(X, \mathbb{Q})$ is a triangulated category

$$\mathcal{U} \underset{\text{open}}{\overset{j}{\hookrightarrow}} X \overset{i}{\hookleftarrow} \underset{\text{closed}}{Z}$$

$\mathcal{U}$ in general Zariski open in X

$j^{-1} = j^!$ $\quad R i_* = R i_!$

$$Rj_! j^{-1} \mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \longrightarrow R i_* i^{-1} \mathcal{F}^\cdot \xrightarrow{+1}$$

adjunction formula.

$$R i_* i^! \mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \longrightarrow R j_* j^{-1} \mathcal{F}^\cdot \xrightarrow{+1}$$

These induce long exact sequences:

$$\cdots \longrightarrow H^i(X, Rj_! j^{-1}\mathcal{F}^\cdot) \longrightarrow \mathcal{H}^i(X, \mathcal{F}^\cdot) \longrightarrow H^i(Z, i^{-1}\mathcal{F}^\cdot) \longrightarrow \cdots$$

relative long exact sequence for (X, Z)

$$\cdots \longrightarrow H^i(X, Ri_* i^! \mathcal{F}^\cdot) \longrightarrow H^i(X, \mathcal{F}^\cdot) \longrightarrow H^i(\mathcal{U}, j^{-1}\mathcal{F}^\cdot) \longrightarrow \cdots$$

relative long e. s. for $(X, \mathcal{U})$

$$\cdots \longrightarrow H^i_c(\mathcal{U}, j^{-1}\mathcal{F}^\cdot) \longrightarrow H^i_c(X, \mathcal{F}^\cdot) \longrightarrow \mathcal{H}^i_c(Z, i^{-1}\mathcal{F}^\cdot) \longrightarrow \mathcal{H}^{i+1}_c(\mathcal{U}, j^{-1}\mathcal{F}^\cdot) \longrightarrow \cdots$$

if $\mathcal{F}^\cdot = \mathbb{Q}_X$ $\quad \cdots \to \mathcal{H}^i_c(\mathcal{U}) \to \mathcal{H}^i_c(X) \to \mathcal{H}^i_c(Z) \to \mathcal{H}^{i+1}_c(\mathcal{U}) \to \cdots$

Motivation

Weak Lefschetz THM Let X be a projective variety s.t. $X \subseteq \mathbb{P}^N$

Take any hyperplane H in $\mathbb{P}^N$ s.t. $\mathcal{U} = X \setminus H$ is smooth.

Then we have $\mathcal{H}^i(X) \simeq H^i(X \cap H)$ for $i \le n-2$

induced by inclusion $X \cap H \hookrightarrow X$

an injective map. $H^{n-1}(X) \hookrightarrow H^{n-1}(X \cap H)$

proof : $\cdots \longrightarrow H^i_c(\mathcal{U}) \longrightarrow H^i(X) \longrightarrow H^i(X \cap H) \longrightarrow \mathcal{H}^{i+1}_c(\mathcal{U}) \to \cdots$

The claim follows if $\mathcal{H}^i_c(\mathcal{U}) = 0$ for $i < n$

References. The topology of complex projective varieties after S. Lefschetz. (Lamotke)

Example $V \subset \mathbb{CP}^n$ is a hypersurface.

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Example: $V \subseteq \mathbb{CP}^n$ is a hypersurface.

Then $H^i(V) \simeq H^i(\mathbb{CP}^n)$ for $i \leq n-2$ since $\mathcal{U} = \mathbb{CP}^n \setminus V$ is affine and smooth.

$H^{n-1}(\mathbb{CP}^n) \hookrightarrow H^{n-1}(V)$

THM Any affine variety $\mathcal{U}$ with $\dim_{\mathbb{C}} \mathcal{U} = n$ has the homotopy type of a real n-dim CW complex. $\Rightarrow$ $H^i(\mathcal{U}) = 0$ for $i > n$

$\Updownarrow$ (if $\mathcal{U}$ is smooth)

$H_c^{2n-i}(\mathcal{U}) = 0$ for $i < n$

In fact, we have more $H^i(\mathcal{U}, \mathcal{L}) = 0$ for $i > n$ and any local system $\mathcal{L}$ on $\mathcal{U}$.

(since $H^i(\mathcal{U}, \mathcal{L})$ are homotopy invariants)

Example $\mathbb{C}^n \setminus \{0\}$ is not affine $(n \geq 2)$, since $H^{2n-1}(\mathbb{C}^n \setminus \{0\}, \mathbb{Q}) = \mathbb{Q}$.

$\mathbb{C}^n \setminus \mathbb{C}^m$ with $(n-m \geq 2)$ is not affine. since $H^{2n-2m-1}(\mathbb{C}^n \setminus \mathbb{C}^m, \mathbb{Q}) = \mathbb{Q}$.

$\mathbb{C}^n \setminus (\mathbb{C}^{n-1} \cup \{pt\})$ with $n \geq 2$ is not affine. $pt \notin \mathbb{C}^{n-1}$ $S^1 \vee S^{2n-1}$

Remark affine variety due to Karchyauskas

Stein manifold (due to Hamm) closed complex mfd in complex vector space

THM. Let $\mathcal{U}$ be an affine variety with $\dim_{\mathbb{C}} \mathcal{U} = n$. $\mathcal{F} \in Sh_c(\mathcal{U}, \mathbb{Q})$.

then we have $H^i(\mathcal{U}, \mathcal{F}) = 0$ for any $i > n$

References Nori Constructible sheaves (arxiv)

(using Noether normalization and induction on dim

What about complex of constructible sheaves?

$\mathcal{F}^\bullet \in D_c^b(\mathcal{U}, \mathbb{Q})$ $\mathcal{U}$ affine variety with $\dim_{\mathbb{C}} \mathcal{U} = n$

How can we have $H^i(\mathcal{U}, \mathcal{F}^\bullet) = 0$ for any $i > n$?

spectral sequences $E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow H^{p+q}(\mathcal{U}, \mathcal{F}^\bullet)$

since $\mathcal{F}^\bullet \in D_c^b(\mathcal{U}, \mathbb{Q})$ $\mathcal{H}^q(\mathcal{F}^\bullet)$ is a constructible sheaf on $\mathcal{U}$.

since $\mathcal{F}^\cdot \in D^b_c(\mathcal{U}, \mathbb{Q})$ $\quad \mathcal{H}^q(\mathcal{F}^\cdot)$ is a constructible sheaf on $\mathcal{U}$.

$$H^p(\operatorname{Supp}(\mathcal{H}^q(\mathcal{F}^\cdot)), \mathcal{H}^q(\mathcal{F}^\cdot)) = 0 \quad \text{for } \underline{p+q>n} \qquad p > n-q$$

so we would like to have $\dim \operatorname{Supp} \mathcal{H}^q(\mathcal{F}) \leq n-q$.

($\operatorname{Supp} \mathcal{H}^q(\mathcal{F}^\cdot)$ is closed in $\mathcal{U}$. hence also an affine variety).

Shift everything by n. and Reformulate the results

For $\mathcal{F}^\cdot \in D^b_c(X, \mathbb{Q})$ if $\dim \operatorname{Supp} \mathcal{H}^q(\mathcal{F}^\cdot) \leq -q$

then for any $\mathcal{U} \subseteq X$ affine variety

then $\mathcal{H}^i(\mathcal{U}, \mathcal{F}|_{\mathcal{U}}) = 0$ for $i > 0$.

$${}^pD^{\leq 0}(X) = \{ \mathcal{F}^\cdot \in D^b_c(X) \text{ s.t. } \underbrace{\dim \operatorname{Supp} \mathcal{H}^q(\mathcal{F}^\cdot)}_{\text{support condition}} \leq -q \}$$

$${}^pD^{\geq 0}(X) = \{ \mathcal{F}^\cdot \in D^b_c(X) \text{ s.t. } D_X\mathcal{F}^\cdot \text{ satisfies support condition} \}$$

$$D_X({}^pD^{\leq 0}(X)) = {}^pD^{\leq 0}(X)$$

$\underline{\operatorname{Perv}(X)} = {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$ is a full sub-category of $D^b_c(X)$

category of perverse sheaves.

Example X smooth variety with $\dim_{\mathbb{C}} X = n$

then $L[n]$ is perverse for any local system L on X

Example $Y \hookrightarrow X$ (closed embedding) $\quad Y$ is smooth

$L_Y[\dim Y]$ is perverse for any local system L_Y on Y.

Miracle: ${}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ is a t-structure.

Something about t-structure.

Def Let $D^{\leq 0}$, $D^{\geq 0}$ be strictly full sub categories of $D^b_c(X)$. Put $D^{\leq n} = D^{\leq 0}[-n]$, $D^{\geq n} = D^{\geq 0}[-n]$.

The pair $(D^{\leq 0}, D^{\geq 0})$ is called a t-structure if

1. $D^{\leq -1} \subset D^{\leq 0}$ $\quad D^{\geq 1} \subset D^{\geq 0}$

2. $\mathrm{Hom}_{D^b_c}(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = 0$ for $\mathcal{F}^\cdot \in D^{\leq 0}$, $\mathcal{G}^\cdot \in D^{\geq 1}$

3. $\forall \mathcal{F}^\cdot \in D^b_c$ $\exists$ distinguished triangle $\tau_{\leq 0}\mathcal{F}^\cdot \to \mathcal{F}^\cdot \to \tau_{\geq 1}\mathcal{F}^\cdot \xrightarrow{+1}$

where $\tau_{\leq 0}\mathcal{F}^\cdot \in D^{\leq 0}$, $\tau_{\geq 1}\mathcal{F}^\cdot \in D^{\geq 1}$

where $\tau_{\leq 0} : D^b_c \to D^{\leq 0}$

$\tau_{\geq 1} : D^b_c \to D^{\geq 1}$

Two properties: $D^{\leq 0} \cap D^{\geq 0}$ is an abelian category.

$\tau_{\leq 0}\, \tau_{\geq 0}$ is a cohomological functor.

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the trivial t-structure.

$\mathcal{F}^\cdot \in D^b_c(X, \mathbb{Q})$ $\quad \cdots \to \mathcal{F}^{i-1} \xrightarrow{d^{i-1}} \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \to \cdots$

We define $\tau_{\leq i}\mathcal{F}^\cdot$ as: $\cdots \to \mathcal{F}^{i-1} \xrightarrow{d^{i-1}} \ker d^i \to 0 \to 0 \cdots$.

$\tau_{\geq i}\mathcal{F}^\cdot$ as $\cdots \to 0 \to \operatorname{coker} d^i \to \mathcal{F}^{i+1} \to \mathcal{F}^{i+2} \to \cdots$

$$\tau_{\leq 0}\mathcal{F}^\cdot \to \mathcal{F}^\cdot \to \tau_{\geq 1}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$D^{\leq 0}(X) = \{ \mathcal{F}^\cdot \in D^b_c(X, \mathbb{Q}) \mid \mathcal{H}^j(\mathcal{F}^\cdot) = 0 \text{ for } j > 0 \}$$

$$D^{\geq 0}(X) = \{ \mathcal{F}^\cdot \in D^b_c(X, \mathbb{Q}) \mid \mathcal{H}^j(\mathcal{F}^\cdot) = 0 \text{ for } j < 0 \}$$

$$D^{\leq 0}(X) \cap D^{\geq 0}(X) = Sh_c(X)$$

$\mathcal{H}^0 = \tau_{\geq 0}\tau_{\leq 0}$ is a cohomological functor.

But D_X does not behave well w.r.t. $Sh_c(X)$.

Formal consequences: $\mathrm{Perv}(X)$ is an abelian category.

$\exists$ a cohomological functor ${}^p\mathcal{H}^0$. i.e.

for a distinguished triangle $\mathcal{M}^\cdot \to \mathcal{F}^\cdot \to \mathcal{N}^\cdot \xrightarrow{(+1)}$

$$\to {}^p\mathcal{H}^i(\mathcal{M}^\cdot) \to {}^p\mathcal{H}^i(\mathcal{F}^\cdot) \to {}^p\mathcal{H}^i(\mathcal{N}^\cdot) \to {}^p\mathcal{H}^{i+1}(\mathcal{M}^\cdot) \to \cdots$$

Remark: the notion of a "sub"-perverse sheaf is weird.

Remark the notion of a "sub"-perverse sheaf is weird.

Example $X = \mathbb{C}$ $\quad j: \mathcal{U} = \mathbb{C} \setminus \{0\} \hookrightarrow X \xleftarrow{i} \{0\}$

$Rj_! \mathbb{Q}_{\mathcal{U}}[1]$ is perverse.

Check why $D_X(j_! \mathbb{Q}_{\mathcal{U}}[1]) = Rj_* \mathbb{Q}_{\mathcal{U}}[1]$

$H^0(Rj_* \mathbb{Q}_{\mathcal{U}}[1]) = \mathbb{Q}_{\{0\}}$ so $\mathcal{H}^0(Rj_* \mathbb{Q}_{\mathcal{U}}[1])$ s

$$j_! \mathbb{Q}_{\mathcal{U}}[1] \to \mathbb{Q}_X[1] \to \underbrace{Ri_* \mathbb{Q}_{\{0\}}[1]} \xrightarrow{[+1]}$$

$$0 \to Ri_* \mathbb{Q}_{\{0\}} \longrightarrow j_! \mathbb{Q}_{\mathcal{U}}[1] \longrightarrow \mathbb{Q}_X[1] \longrightarrow 0$$

short exact sequence in $\mathrm{Perv}(X)$

So In general if you want to compute Kernel of a map between perverse sheaves.

$$P \longrightarrow Q \longrightarrow R \xrightarrow{[+1]}$$

$$0 \to {}^p\mathcal{H}^{-1}(R) \to P \to Q \to {}^p\mathcal{H}^0(R) \to 0.$$

$\boxed{\text{Perverse is a "local" notion.}}$ support condition by checking stalk.
cosupport condition. $\mathcal{H}^i_c(B_x, P)$

Perverse sheaves is not a sheaf.

By Definition. $\mathrm{Perv}(X)$ is stable by D_X, $\quad D_X^2 = \mathrm{id}$
i.e. $D_X P \in \mathrm{Perv}(X)$ for any $P \in \mathrm{Perv}(X)$

PD $\quad H^i(X, P) \simeq H^{-i}(X, D_X P)^\vee$ for any $P \in \mathrm{Perv}(X)$.

One last thing. $\mathcal{F}^\cdot \in {}^pD^{\le 0}(X) \iff$ for any affine Zariski open $\mathcal{U} \subseteq X$
$H^i(\mathcal{U}, \mathcal{F}|_{\mathcal{U}}) = 0$ for any $i > 0$.

<u>Artin vanishing theorem</u>

proof: $\Rightarrow$ obvious.

$\Leftarrow$ Fix a Whitney stratification of $\mathcal{U}$ w.r.t. $\mathcal{F}^\cdot$

Take a generic hyperplane H w.r.s. $\mathcal{U}$

$$H \cap \mathcal{U} \xhookrightarrow{i} \mathcal{U} \xhookleftarrow{j} \mathcal{U} \setminus H = \mathcal{U}'$$

Note that $i^! = i^{-1}[-2]$ since H is generic.

$\mathcal{U}'$ is also affine. set $\mathcal{F}' = \mathcal{F}^\cdot|_{\mathcal{U}'}$

$\mathcal{U}'$ is also affine. set $\mathcal{F}' = \mathcal{F}^\cdot|_{\mathcal{U}'}$

$$\cdots \to \mathcal{H}^{i-2}(H\cap\mathcal{U}, \mathcal{F}^\cdot|_{H\cap\mathcal{U}}) \longrightarrow \mathcal{H}^i(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{H}^i(\mathcal{U}', \mathcal{F}') \longrightarrow \mathcal{H}^{i-1}(H\cap\mathcal{U}, \mathcal{F}^\cdot|_{H\cap\mathcal{U}}) \to$$

The assumption implies that $\mathcal{H}^i(H\cap\mathcal{U}, \mathcal{F}^\cdot|_{H\cap\mathcal{U}}) = 0$ for $i > -1$.

Continue this process (since $H\cap\mathcal{U}$ is also affine)

$\Rightarrow$ For a generic codim k section H

$$H^i(H\cap\mathcal{U}, \mathcal{F}^\cdot|_{H\cap\mathcal{U}}) = 0 \quad \text{for } i > -k.$$

When codim $H = \dim X$, $H \cap X$ are just finitely many points

$\Rightarrow$ for any $x \in H \cap X$, $\mathcal{H}^i(\mathcal{F}^\cdot_x) = 0$ for $i > -n$.

The proof goes by induction.

Let S be a stratum in the stratification.

The above argument proves the support condition for S when $\dim S = \dim X$.

Let us assume that the claims holds for $\dim S > k$.

Now assume $\dim S = k$. Take a generic slice H s.t. $H\cap S$ are just finitely many points (codim $H = k$)

$$H\cap(\mathcal{U}\setminus S) \xrightarrow{j} H\cap\mathcal{U} \xleftarrow{i} H\cap S$$

$$\cdots \to \underbrace{\mathcal{H}^i(H\cap\mathcal{U}, j_!\mathcal{F}^\cdot|_{H\cap(\mathcal{U}\setminus S)})}_{=0 \text{ for } i>-k} \to \underbrace{\mathcal{H}^i(H\cap\mathcal{U}, \mathcal{F}^\cdot|_{H\cap\mathcal{U}})}_{=0 \text{ for } i>-k} \longrightarrow \bigoplus_{x\in H\cap S} \mathcal{H}^i(\mathcal{F}_x) \to \cdots$$

$\longrightarrow$ since $H\cap\mathcal{U}$ is affine and H is generic

since $j_!\mathcal{F}^\cdot|_{H\cap(\mathcal{U}\setminus S)} \in {}^pD^{\le -k}(H\cap\mathcal{U})$ by induction.

$\square$.