Perverse sheaves 1

2021年12月6日 14:05

Perverse sheaves and decomposition theorem

References De Cataldo, Migliorini — The decomposition theorem and the topology of algebraic maps. Bull. AMS. 2009

De Cataldo. Perverse sheaves and the topology of algebraic varieties.

Maxim. Intersection Homology & Perverse sheaves. GTM
Chapter 8 and 9.

Beilinson, Berstein, Deligne — Faisceaux pervers (BBD), 1982

logic always from smooth (spaces or maps) to singular.

---

algebraic topology. algebraic geometry. rep theory. combinatorics.

I Constructible complex

Motivation: Poincare duality.

$\mathbb{R}^n$

$$H^i(\mathbb{R}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i=0 \\ 0 & i\neq 0 \end{cases}$$

$$H^i_c(\mathbb{R}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i=n \\ 0 & i\neq n \end{cases}$$

Poincare Lemma.

cohomology with compact support can be obtained by singular cochain with compact support

e.g. $H^i_c(\mathbb{R}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i=1 \\ 0 & i\neq 1 \end{cases}$

$\mathbb{R}$

Poincare Lemma + MV sequence + induction $\Longrightarrow$ Poincare duality.

THM (PD) Let X be a smooth oriented closed manifold with dim n.
Then we have $H^i(X, \mathbb{Q}) \simeq H^{n-i}(X, \mathbb{Q})^\vee$

proof: find a "good cover".

From now on, we assume X is a locally closed quasi-proj variety of complex dim n.

Constant sheaf $\mathbb{Q}_X$ $\quad H^i(X, \mathbb{Q}) = H^i(X, \mathbb{Q}_X)$

$H^i(X, \mathbb{Q}) \simeq H^{2n-i}(X, \mathbb{Q})^\vee$ if X is smooth projective

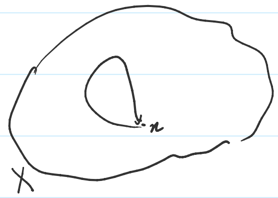
Constant sheaf $\mathbb{Q}_X$ $\qquad H^i(X, \mathbb{Q}) = H^i(X, \mathbb{Q}_X)$

PD $\quad H^i(X, \mathbb{Q}_X) \simeq H^{2n-i}(X, \mathbb{Q}_X)^\vee$ if X is smooth projective.

Local system: locally constant sheaf with finite rank.

$X \qquad \forall x \in X \qquad U_x$ a small enough neighborhood (analytic topology)

we have $\quad \mathcal{L}|_{U_x} \simeq \mathbb{Q}^k|_{U_x}$



$$\pi_1(X) \longrightarrow \mathrm{Aut}(\mathbb{Q}^k) \simeq GL_k(\mathbb{Q})$$

$$\{\text{local system on } X\}/_{\text{iso}} \xrightarrow{1:1} \{\pi_1(X) \longrightarrow GL_k(\mathbb{Q})\}/_{\text{conj}}$$

PD $\quad H^i(X, \mathcal{L}) \simeq (H^{2n-i}(X, \mathcal{L}^\vee))^\vee$ if X is smooth projective.

($\mathcal{L}^\vee$ is defined by the dual representation)

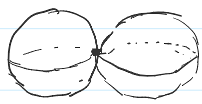
All local system on X forms an abelian category $\mathrm{Loc}(X, \mathbb{Q})$

What if X is singular? $\qquad$ PD fails

e.g. $\quad X = \{[x.y.z] \in \mathbb{CP}^2 \mid x \cdot y = 0\} \simeq \{x=0\} \vee \{y=0\} \simeq \mathbb{P}^1 \vee \mathbb{P}^1$

$\simeq S^2 \vee S^2$

$$H^i(X, \mathbb{Q}) = \begin{cases} \mathbb{Q}^2 & i=2 \\ \mathbb{Q} & i=0 \\ 0 & \text{else} \end{cases}$$



why? the local neighborhood around $[0.0.1]$ is not like $\mathbb{R}^2$.

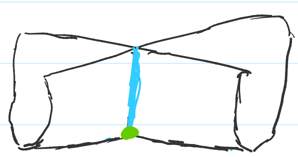
So we need to have a good understanding of the neighborhood first.

In particular, we should give a partition of X by the neighborhood for every point.

Def A stratification of X is a finite disjoint union decomposition $X = \bigsqcup X_\alpha$, where X_α is a smooth locally closed sub var.

"A naive idea" $\quad X \supset \mathrm{Sing}(X) \supset \mathrm{Sing}(\mathrm{Sing}(X))$

e.g. Whitney umbrella. $\quad x^2 = y^2 z$ in $\mathbb{C}^3$



blue point $\quad \times \quad \times \quad$ (a disk in $\mathbb{C}$)

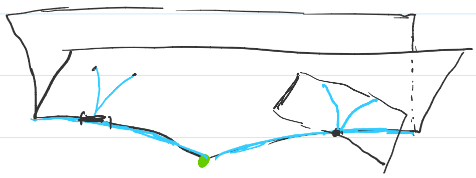
singularity part $\qquad$ manifold part

blue point $\times$ $\times$ (a disk in $\mathbb{C}$)

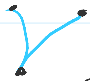
singularity part   manifold part

Def Whitney stratification is a "good" stratification such that the neighborhood is "constant" along every stratum X_α.

$\forall\, x \in X_\alpha \qquad B_x = (X_\alpha \cap B_x) \times C L_x.$



L_x



$= \text{cone point}$ L_x

$\parallel$

$[0,1] \times L_x \,/\, \{0\} \times L_x$

THM For a complex quasi-projective variety Whitney stratification always exists.

Remark pseudo-manifold   Intersection Cohomology

For any $x \in X_\alpha \qquad U_x \simeq CL_x \times B_x \qquad B_x$ is a small Ball in X_α

reduce the construction of intersection cohomology to CL_x. (cone of L_x)

by computation. by L_x. Note that L_x is also a pseudo-manifold.

proof done by induction on dim

Constructible sheaf. $\mathcal{F}$ is a sheaf on X s.t. there exists a stratification s.t. $\mathcal{F}|_{X_\alpha}$ is a local system $\forall \alpha$.

These sheaves form a abelian category $Sh_c(X, \mathbb{Q})$

$D^b_c(X, \mathbb{Q})$ bounded derived category of constructible complexes.

$\mathcal{F}^\cdot \in D^b_c(X, \mathbb{Q})$ is a complex of $\mathbb{Q}$-sheaves on X s.t.

- $\mathcal{H}^i(\mathcal{F}^\cdot) = 0$ except for finitely many i.
- for $\mathcal{H}^i(\mathcal{F}^\cdot) \neq 0$. $\mathcal{H}^i(\mathcal{F}^\cdot) \in Sh_c(X, \mathbb{Q})$

$D^b_c(X, \mathbb{Q})$ is stable with respect to six functors.

$X \xrightarrow{f} Y$ alg map Rf_* $Rf_!$ $f^!$ f^{-1} $\_ \otimes \_$ $R\mathcal{H}om(\ ,\ )$

for $\mathcal{F}^\cdot \in D^b_c(X, \mathbb{Q})$ and any point $y \in Y$ $\qquad$ $Rf_*, Rf_! : D^b_c(X) \to D^b_c(Y)$

$$H^i(Rf_* \mathcal{F}^\cdot)_y = H^i(f^{-1}(B_y), \mathcal{F}^\cdot)$$

B_y small enough neighborhood of $y \in Y$

$$\mathcal{H}^i(Rf_! \mathcal{F}^\cdot)_y = H^i_c(f^{-1}(y), \mathcal{F}^\cdot)$$

Facts : 1 $\quad H^i(X, \mathcal{F}^\cdot) = H^i(Y, Rf_* \mathcal{F}^\cdot)$

2: If f is a proper map, then $Rf_* = Rf_!$

$a : X \longrightarrow pt$ $\qquad$ $H^i(Ra_* \mathcal{F}^\cdot) = H^i(X, \mathcal{F}^\cdot)$

$\qquad\qquad\qquad\qquad$ $H^i(Ra_! \mathcal{F}^\cdot) = H^i_c(X, \mathcal{F}^\cdot)$

3 $\quad$ For any inclusion map $j : U \hookrightarrow X$. $Rj_! \mathcal{F}^\cdot$ is extension by 0 for $\mathcal{F}^\cdot \in D^b_c(U)$.

for any $\mathcal{F}^\cdot \in D^b_c(Y, \mathbb{Q})$ and any point $x \in X$. $\qquad$ $f^{-1}, f^! : D^b_c(Y) \to D^b_c(X)$

$$(f^{-1}\mathcal{F}^\cdot)_x = \mathcal{F}^\cdot_{f(x)}$$

$f^!$ is right adjoint to $Rf_!$

$$Rf_* R\mathcal{H}om(K^\cdot, f^! \mathcal{F}^\cdot) \simeq R\mathcal{H}om(Rf_! K^\cdot, \mathcal{F}^\cdot)$$

<u>THM</u> (Ehresmann's Fibration THM) $\quad$ Let $f : X \longrightarrow Y$ be a proper submersion between two smooth varieties. $\quad$ Then f is a locally trivial fibration, i.e.

$\forall\, y \in Y$ $\quad \exists$ small enough neighborhood B_y s.t. $f^{-1}(B_y) \simeq B_y \times f^{-1}(y)$ diffeomorphism.

In particular.

$$H^i(Rf_* \mathbb{Q}_X) \text{ is a local system on } Y.$$

<u>THM</u> (Thom's first isotopy Lemma)

Let $f : X \longrightarrow Y$ be algebraic map between alg varieties. Then there exist Whitney stratifications of $X = \coprod X_\alpha$ and $Y = \coprod Y_\beta$ s.t.

for any X_α and Y_β s.t. $f(X_\alpha) \subseteq Y_\beta$. $\quad f|_{X_\alpha} : X_\alpha \longrightarrow Y_\beta$ is a submersion.

<u>This explains why Rf_* $Rf_!$ being stable for D^b_c.</u>

This explains why Rf_* $Rf_!$ being stable for D^b_c.

(every quasi-projective is homotopy equivalent to a finite CW complex)

Dualizing functor. $a : X \to pt$ Set $\omega_X = a^! \mathbb{Q}_{pt}$

$$D_X := R\mathcal{H}om(- , \omega_X) \quad : \quad D^b_c(X) \longrightarrow D^b_c(X)$$

for $\mathcal{F}^\cdot \in D^b_c(X)$ $D_X \mathcal{F}^\cdot$ is called the Verdia duality of $\mathcal{F}^\cdot$

property of D_X

① $D_X \circ D_X = id$ $\qquad X \xrightarrow{f} Y$

② $D_Y \cdot Rf_* = Rf_! \, D_X$

③ $D_X f^! = f^{-1} D_Y$

$$Ra_* R\mathcal{H}om(\mathcal{F}^\cdot , a^! \mathbb{Q}_{pt}) \simeq R\mathcal{H}om(Ra_! \mathcal{F}^\cdot ; \mathbb{Q}_{pt})$$

$$H^i(X , D_X\mathcal{F}^\cdot) \simeq \left(H^{-i}_c(X , \mathcal{F}^\cdot)\right)^\vee \qquad \text{Generalization of P.D.}$$

for any point $x \in X$ $\qquad \mathcal{H}^i(D_X \mathcal{F}^\cdot)_x = \left(H^{-i}_c(B_x , \mathcal{F}^\cdot|_{B_x})\right)^\vee$

Example If X is smooth. $\qquad \mathcal{H}^i(D_X \mathbb{Q}_X)_x = (H^{-i}_c(B_x))^\vee$

$$= \begin{cases} \mathbb{Q} & i = -2n \\ 0 & \end{cases}$$

$$\Rightarrow \quad D_X \mathbb{Q}_X = \mathbb{Q}_X[2n]$$

($\mathcal{F}^\cdot[k]$ means shift $\mathcal{F}^\cdot$ to the left k steps)

$$H^i(X , \mathbb{Q}_X) = H^{2n-i}_c(X , \mathbb{Q}_X)$$

To have symmetry. $\qquad D_X(\mathbb{Q}_X[n]) = \mathbb{Q}_X[n]$

$$H^i(X , \mathbb{Q}_X[n]) = H^{-i}_c(X , \mathbb{Q}_X[n])$$

Same results work for local system on X.

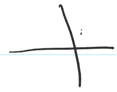
If X is singular. $D_X(\mathbb{Q}_X)$ is not a sheaf in general.

Example $\qquad X = \{ [x , y , z] \in \mathbb{P}^2 \mid x \cdot y = 0 \}$

If $x \neq [0, 0, 1]$ then the stalk of $D_X(\mathbb{Q}_X)$ is clear.

Example $X = \{[x.y.z] \in \mathbb{P}^2 \mid x \cdot y = 0\}$

If $x \neq [0.0.1]$ then the stalk of $\mathbb{D}_X(\mathbb{Q}_X)$ is clear.

$x = [0.0.1]$ $\quad \mathcal{H}^i(\mathbb{D}_X \mathbb{Q}_X)_x = \mathcal{H}_c^{-i}(B_x)^\vee$  in $\mathbb{C}^2$

$$\cdots \longrightarrow H_c^i(B_x) \longrightarrow H^i(X) \longrightarrow H^i(2\text{ pts}) \longrightarrow H_c^{i+1}(B_x) \longrightarrow \cdots$$

$$0 \longrightarrow \underset{\substack{\| \\ 0}}{H_c^0(B_x)} \longrightarrow \mathbb{Q} \hookrightarrow \mathbb{Q}^2 \longrightarrow \underset{\substack{\| \\ \mathbb{Q}}}{H_c^1(B_x)} \longrightarrow 0 \qquad H_c^2(B_x) = H^2(X) = \mathbb{Q}^2$$